Lecture 22 - Green's Theorem

**Common Picture:**

**Today:** Special Case of Stoke's Theorem: Green's Theorem

**Def.** A **simple closed curve** is a closed curve that doesn't intersect itself.

In other words, if \( \vec{r}(t) \) traces out the curve, \( a \leq t \leq b \), then
\[
\vec{r}(a) = \vec{r}(b),
\]
and for \( a < t_1 < t_2 < b \),
\[
\vec{r}(t_1) \neq \vec{r}(t_2).
\]

A simple closed curve bounds a region, and we say the curve is **positively oriented** if the region is always to the left of the curve. \( \leftarrow \) The curve is oriented counterclockwise.

**Thm (Green's Theorem).** Let \( C \) be a positively oriented, simple closed curve, enclosing a region \( D \). Then
\[
\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

We will often write \( C = \partial D \), so this is essentially the middle box in our diagram.

**Ex.** \( C = \text{square with vertices} (\pm 1, \pm 1) \), \( \vec{F} = \langle e^{\tan x} - y, x + e^y \rangle \)

Then
\[
\int_C P \, dx + Q \, dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \iint_D 1 - 1 \, dA = \boxed{8}
\]
Thus we can use double integrals to evaluate line integrals. Can also do reverse.

If $C$ is the boundary of a region $D$, then

$$\text{Area}(D) = \oint_C \left( -\frac{1}{a} y \, dx + x \, dy \right) = \int_C x \, dy = \int_C y \, dx$$

**Example:** Let $C$ be the ellipse $x = 2 \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$, and $D$ the solid ellipse. Then

$$\text{Area} \quad D = \oint_C \left( -\frac{1}{2} y \, dx + x \, dy \right) = \int_0^{2\pi} \left( -\sin t \cdot \frac{1}{2} \cos t - \sin t \cdot \frac{1}{2} \sin t \right) \, dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi.$$

We'll prove Green's Theorem for a type II region. We will actually show that $\iint_D Q \, dy = \iint_D \frac{\partial Q}{\partial x} \, dA$.

$$D = \left\{ (x, y) \in \mathbb{R}^2 : a(y) \leq x \leq b(y) \right\}$$

$$\int_D \frac{\partial Q}{\partial x} \, dA = \int_c^d \int_{a(y)}^{b(y)} \frac{\partial Q}{\partial x} \, dx \, dy = \int_c^d Q(x, y) \left[ x = b(y) \right] \, dy = \int_c^d Q(b(y), y) \, dy - \int_c^d Q(a(y), y) \, dy$$

(1)

Now look at $\iint_D Q \, dy$: Can break $C$ into $C_1, C_2, C_3, C_4$:

$$\int_C Q \, dy = \int_{C_1} Q \, dy + \int_{C_2} Q \, dy + \int_{C_3} Q \, dy + \int_{C_4} Q \, dy$$

On $C_2, C_4$, $y$ is constant $\Rightarrow dy = 0 \Rightarrow \int Q \, dy = 0$

On $C_1$: Can param. with respect to $y$:  

$$\int_C Q \, dy = \int_0^1 Q \left( 2 \cos t, \sin t \right) \, dt$$
\[ x = b(y), \quad c \leq y \leq d \Rightarrow \int_C Q \, dy = \int_c^d Q(b(y), y) \, dy \]

On \( C_3 \): \( C_3 \) is of the same form; we just run it backwards:
\[
\int_{C_3} Q \, dy = -\int_c^d Q(a(y), y) \, dy
\]

So \( \int_C Q \, dy = \int_c^d Q(b(y), y) \, dy - \int_c^d Q(a(y), y) \, dy = \int_D \frac{\partial Q}{\partial x} \, dA \)

by (6)

Similar arguments work for type I regions. Why does this work?

1. Can break up regions into type I or II regions:

\[ D \quad \xrightarrow{c} \quad C \]

Thus \( \partial D_1 = C_1 + C_3 \)

\( \partial D_2 = C_2 - C_3 \)

So \( \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \int_C P \, dx + Q \, dy \)

\[
\begin{align*}
\iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA &= \int_{C_1} P \, dx + Q \, dy + \int_{C_3} P \, dx + Q \, dy \\
\iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA &= \int_{C_2} P \, dx + Q \, dy - \int_{C_3} P \, dx + Q \, dy
\end{align*}
\]

As a consequence, we can apply Green's theorem to regions with multiple boundary components.

**Ex:** \( \int_C - \frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy \) is the same for any path around \((0,0)\).

Let \( C_1 \) be a small circle around \((0,0)\). Then
\[
\begin{align*}
0 &= \iint_D \left( \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right) \right) \, dA \\
&= \int_C - \frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy - \int_{C_1} - \frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy
\end{align*}
\]
Computing around $C_1$ is easy: $x = \cos t$, $y = \sin t$

\[ \int_{C_1} -\frac{y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy = \int_0^{2\pi} 1 \, dt = 2\pi. \]