Look for places where the tangent plane is horizontal.

$\leftrightarrow$ equation of the plane is of the form $z = c = 0$.

The equation of the tangent plane is

$$z - f(a,b) = \frac{\partial f}{\partial x}(a,b) \cdot (x-a) + \frac{\partial f}{\partial y}(a,b) \cdot (y-b)$$

$\Rightarrow$ tangent plane is horizontal iff

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0.$$

**Def** $(a,b)$ is a **critical point** if $\nabla f(a,b) = \mathbf{0}$.

**Ex:**

$f(x,y) = x^3 - 3x + y^3 - 3y$

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3 = 0 \Rightarrow y = \pm 1$$

Critical points:

$(1,1), (1,-1), (-1,1), (-1,-1)$.

Since the gradient points in the direction of steepest ascent, at
critical points, there is no such direction.

**Def** A function $f$ has a **local max** at $(a,b)$ if

$$f(a,b) \geq f(c,d)$$

for all $(c,d)$ close to $(a,b)$.

A function $f$ has a **local min** at $(a,b)$ if

$$f(a,b) \leq f(c,d)$$

for all $(c,d)$ close to $(a,b)$.

Maxima & minima occur at critical points.

**Ex**

$f(x,y) = xy$

$$\frac{\partial f}{\partial x} = y = 0 \Rightarrow y = 0$$

$$\frac{\partial f}{\partial y} = x = 0 \Rightarrow x = 0$$

There is one critical point: $(0,0)$
This point is not a maximum or minimum. It looks like a saddle.

In the 1-var case, have the second derivative test:

\[
\text{If } f'(a) = 0 \Rightarrow \begin{cases} f''(a) > 0 \Rightarrow \text{min} \\ f''(a) < 0 \Rightarrow \text{max} \end{cases}
\]

In the 2-var case, have more directions to worry about:

\[
\frac{\partial^2 f}{\partial x^2}(a,b) \begin{cases} > 0 & \text{curve is concave up in } x\text{-direction} \\ < 0 & \text{curve is concave down in } x\text{-direction} \end{cases}
\]

\[
\frac{\partial^2 f}{\partial y^2}(a,b) \begin{cases} > 0 & \text{curve is concave up in } y\text{-direction} \\ < 0 & \text{curve is concave down in } y\text{-direction} \end{cases}
\]

Let \[D = \left( \frac{\partial^2 f}{\partial x^2} \right) \cdot \left( \frac{\partial^2 f}{\partial y^2} \right) - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2\]

Then if \((a,b)\) is a critical point then at \((a,b)\), \(f\) has a 

- **maximum** if \(D > 0\), \(\frac{\partial^2 f}{\partial x^2}(a,b) < 0\)
- **minimum** if \(D > 0\), \(\frac{\partial^2 f}{\partial x^2}(a,b) > 0\)
- **saddle point** if \(D < 0\)

\[- \left( \frac{\partial^2 f}{\partial x \partial y} \right)\] is always less than or equal to zero,

if \(D > 0\), then \(\left( \frac{\partial^2 f}{\partial x^2} \right) \cdot \left( \frac{\partial^2 f}{\partial y^2} \right) > 0\).

Thus if \(D > 0\), \(\frac{\partial^2 f}{\partial x^2} \neq \frac{\partial^2 f}{\partial y^2}\) have the same sign.

\[\Rightarrow x\text{-curves } \& \ y\text{-curves have same concavity.}\]

**Can check sign of either** \(\frac{\partial f}{\partial x^2}\) or \(\frac{\partial f}{\partial y^2}\).

**Ex:** \(f(x,y) = xy\). Saw critical point at \((0,0)\).

\[
\frac{\partial^2 f}{\partial x^2} = 0 = \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = 1, \quad \text{so } D = -1.
\]

This is a saddle.
Ex: \( f(x,y) = x^3 - 3xy + y^3 \)  
\[ D = 36xy \]

<table>
<thead>
<tr>
<th>C.p.</th>
<th>( D )</th>
<th>( \frac{df}{dx^2} )</th>
<th>max? min? saddle?</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1,1))</td>
<td>36 &gt; 0</td>
<td>6 &gt; 0</td>
<td>min</td>
</tr>
<tr>
<td>((1,-1))</td>
<td>-36 &lt; 0</td>
<td></td>
<td>saddle</td>
</tr>
<tr>
<td>((-1,1))</td>
<td>-36 &lt; 0</td>
<td></td>
<td>saddle</td>
</tr>
<tr>
<td>((-1,-1))</td>
<td>36 &gt; 0</td>
<td>-6 &lt; 0</td>
<td>max</td>
</tr>
</tbody>
</table>

Ex: \( f(x,y) = x^3 - 3xy + y^3 \)

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 3x^2 - 3y = 0 \\
\frac{\partial f}{\partial y} &= 3y^2 - 3x = 0
\end{align*}
\]

must be true simultaneously:

\[
\begin{align*}
x &= y \\
y &= x
\end{align*}
\]

2 c.p.: \((0,0)\)  
\((1,1)\)

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= 6x \\
\frac{\partial^2 f}{\partial x \partial y} &= -3 \\
\frac{\partial^2 f}{\partial y^2} &= 6y
\end{align*}
\]

\[
\begin{align*}
D &= (6x)(6y) - (-3)^2 = 36xy - 9
\end{align*}
\]

<table>
<thead>
<tr>
<th>C.p.</th>
<th>( D )</th>
<th>( \frac{df}{dx^2} )</th>
<th>?</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0))</td>
<td>-9 &lt; 0</td>
<td></td>
<td>saddle</td>
</tr>
<tr>
<td>((1,1))</td>
<td>27 &gt; 0</td>
<td>6 &gt; 0</td>
<td>min</td>
</tr>
</tbody>
</table>