

## 1. INTRODUCTION

My research focuses on computational algebraic topology. I study both the general, algebraic techniques and the specific applications of these to spaces and spectra of interest. I currently have three major research projects:

- (1) Work out basic computations and tools in the homotopy of Hopkins-Miller higher real  $K$ -theory spectra (joint with Hopkins and Ravenel).
- (2) Simplify and streamline computations of topological cyclic homology from topological Hochschild homology (joint with Angeltveit and Lawson).
- (3) Carry out fundamental computations in motivic homotopy over  $\mathbb{R}$  and  $\mathbb{C}$  (joint with Dugger and Isaksen).

All three of these projects sit at the interface of algebraic topology, algebraic geometry, and number theory.

Work of Hopkins and Miller introduced the theory of elliptic curves to algebraic topology, producing the spectrum of topological modular forms:  $tmf$ . The classical theory of elliptic curves informs many of the results about  $tmf$ , from close connection between the theory of modular forms and the homotopy of  $tmf$  to the families of spectra built out of  $tmf$  by considering curves with level structures. Recent work of Behrens and Lawson extends the Hopkins-Miller program to more complicated families of abelian varieties, associating spectra to certain classes of Shimura varieties. Just as with  $tmf$ , these tie together classical questions in number theory and algebraic topology, and the computation of the homotopy of these spectra relies on understanding the automorphic forms associated to the underlying Shimura variety. Behrens and Lawson have also shown that the higher real  $K$ -theories from my first project play an important role in the local structure of their theories, and better understanding computationally this connection is one of the future projects described below.

Algebraic  $K$ -theory contains arithmetic information about rings,  $C^*$ -algebras, and, in the topological context, ring spectra. For example, for a commutative ring  $R$ ,  $K^0(R)$  contains information about the finitely generated projective  $R$ -modules, while  $K^1(R)$  is a generalization of the group of multiplicative units of  $R$ . The primary homotopy theoretic approach to computing the algebraic  $K$ -theory of a ring is to find the topological cyclic homology of the ring. Even these computations are difficult, failing even in basic cases like  $\mathbb{Z}/p^2$ . The goal of the second project is ultimately to better understand how to compute the algebraic  $K$ -groups using homotopy theoretic machinery.

Motivic homotopy provides techniques to compute algebraic  $K$ -theory from an algebraic geometry perspective. However, just as with algebraic  $K$ -theory itself, computations in motivic homotopy theory are quite difficult. My third project is to provide a foundation of computations of motivic homotopy groups of spheres working over the real and complex numbers. In these cases, there are close connections between the algebraic geometry from the motivic picture and classical homotopy. Here the flow of information is bidirectional, allowing us to use classical results from stable homotopy to deduce similar results motivically and providing for us an motivic description of mysterious, classical phenomena.

**1.1. Notation.** At this point, my exposition will become more technical. I will therefore fix some notation. Unless otherwise specified, all computations are done localized at some fixed prime  $p$ , and when describing current and future research,  $q$  will denote  $p-1$ . In what follows,  $F_n$  will denote a fixed height  $n$  formal group over a perfect field  $k$  containing  $\mathbb{F}_{p^n}$ . Let  $\mathbb{G}_n$  denote the Morava stabilizer group, the automorphisms of  $F_n$  over  $k$ . Finally, I will use  $E_n$  to denote the  $E_\infty$ -ring spectrum associated to the data  $(F_n, k)$  by the Hopkins-Miller theorem, and if  $G$  is a finite subgroup of  $\mathbb{G}_n$ , then  $EO_n(G)$  will denote  $E_n^{hG}$ .

2. PREVIOUS RESEARCH

**2.1. The  $eo_{p-1}$  homology of  $B\Sigma_p$ .** The  $H\mathbb{F}_p$  based Adams spectral sequence in the category of  $eo_{p-1}$ -modules has a simple  $E_2$ -term, computable in terms of a Hopf algebra due to Henriques and me. For  $p = 2$ , this recovers the classical “change of rings” result for computing  $ko$ -homology, and for  $p = 3$ , this provides a technique for computing 3-local  $tmf$ -homology. For larger primes, this presupposes the existence of such a spectrum. I used this spectral sequence to compute the  $p$ -local  $eo_{p-1}$  homology of the classifying space  $B\Sigma_p$ . The problem is analogous in many ways to Mahowald’s computation of the  $ko$ -homology of  $\mathbb{R}P^\infty$ , and my computational approach mirrors Mahowald’s by applying judicious choices of filtration to the homology of  $B\Sigma_p$ .

**2.2. The topological Hochschild homology of  $\ell$  and  $ko$ .**  $THH$  is the starting point for computing the algebraic  $K$ -theory of structured ring spectra. McClure and Staffeldt computed the  $V(0)$ -homology of  $THH(\ell)$ , using as a starting point the easier computation, essentially due to Bökstedt, of the  $V(1)$ -homology. Using recent advances in the theory of structured ring spectra, Angeltveit, Lawson, and I recast their computations in terms of topological Hochschild homology with coefficients in a bimodule and extend their results to  $p$ -local statements about the homotopy of  $THH(\ell)$ . Since  $ku$  is an  $E_\infty$   $ko$ -algebra, we were able to apply similar methods to also compute the 2-local homotopy of  $THH(ko)$ .

As an application of this computation, using a result of Blumberg-Cohen-Schlichtkrull concerning the  $THH$  of Thom spectra, we have found a conceptually easier proof that  $ku$  and  $ko$  are not Thom spectra. If we assume that these are the Thom spectra of a triple loop map, then there is not finite complex whose  $ko$  or  $ku$ -homology is the homotopy of the first few  $ko$  or  $ku$ -cells of  $THH$ . This non-existence can be shown directly with the Adams spectral sequence.

**2.3. The 5-local homotopy of  $eo_4$ .** Using the obvious connective version of the Hopf algebroid found by Hopkins-Gorbounov-Mahowald, I computed the Adams-Novikov  $E_2$ -term for the homotopy groups of  $EO_4$  and its conjectural connective cover  $eo_4$ . The method was similar to that employed by Bauer in his description of the homotopy of  $tmf$ , as I used a series of Bockstein spectral sequences to build up the Adams-Novikov  $E_2$ -term from simpler, classically known cohomology computations.

Using work of Hopkins and Miller, I also computed the Adams-Novikov differentials and solve the various multiplicative extension problems for  $eo_4$ , for  $(eo_4)/5$ , and for  $(eo_4)/(5, v_1)$ . In particular, one sees immediately a result of Hopkins that everything in the image of  $J$  in the homotopy groups of spheres maps to zero under the Hurewicz homomorphism. Moreover, one seems quite explicitly Gross-Hopkins and Mahowald-Rezk dualities in the homotopy, again mirroring the situation with  $tmf$  at the prime 3.

This computation has the final application of producing full homotopy ring of  $eo_4[\Delta^{-1}]$  and of  $EO_4$ . In particular, we find as a ring the zero line of the Adams-Novikov spectral sequence for  $EO_4$ , resolving a difficult problem in invariant theory. These methods apply to all of the spectra  $EO_{p-1}$ , though not in a sufficiently practical way to provide a complete, prime independent, description. Pending the computation of an analogous Hopf algebroid, these methods should yield insight also into the Adams-Novikov zero line for  $EO_{f(p-1)*}$ .

**2.4. The existence of a  $v_2^{32}$  self map on  $S^0/(2, v_1^4)$ .** One of the early successes of  $tmf$  was the determination of a minimal  $v_2$ -self map on the generalized Smith-Toda complex  $M(1, 4)$ , the 4 cell complex which is the cone on both 2 and  $v_1^4$ . The Periodicity Theorem of Devinatz-Hopkins-Smith shows that some power of  $v_2$  occurs as a self-map on this finite spectrum, and Hopkins and Mahowald showed that  $v_2^{32}$  is the smallest surviving power of  $v_2$  in the Adams spectral sequence for  $M(1, 4)$ . Behrens and I completed the argument,

filling in details about the necessary modifications to the Adams spectral sequence, about vanishing lines in the Adams  $E_2$ -term for Brown-Gitler modules, and about the algebraic  $tmf$ -resolution of a spectrum.

### 3. CURRENT RESEARCH

**3.1. The action of finite subgroups of  $\mathbb{G}_n$  on  $E_{n*}$ .** Hopkins conjectured that for finite subgroups  $G$  of  $\mathbb{G}_n$ ,  $E_{qf*}$  is equivariantly isomorphic to a much more readily describable  $G$ -algebra. This is a natural extension of Hopkins and Miller’s early work on the subject, where they showed using formal group techniques that this is true for  $f = 1$ . Recent work with Hopkins and Ravenel provides a solution to the problem, using the theory of formal  $A$ -modules (where  $A$  is  $\mathbb{Z}_p[\zeta]$ ), the theory of crystals, and elementary Tate cohomology. Elementary obstruction theory allows us to reduce to the case of showing that Hopkins’ conjecture is true for  $G = \mathbb{Z}/p$ , and I will sketch our argument in this case.

Formal  $A$ -modules have a deformation theory similar to that of formal groups, and there is a natural  $\mathbb{Z}/p$ -equivariant forgetful map from  $E_{qf*}$  to the ring  $E_f$  representing deformations of  $F_n$  in formal  $A$ -modules. As a  $\mathbb{Z}/p$ -module,  $E_f$  is particularly simple: everything in degree  $(-2k)$  is in the  $\zeta^k$  eigenspace. This ring also admits a natural, surjective map from  $R$ , the symmetric algebra on  $f$  copies of  $\bar{\rho}$  (appropriately localized and completed), where  $\bar{\rho}$  is the torsion-free quotient of the regular representation by the trivial representation. An equivalent form of Hopkins’ conjecture is that the Tate cohomology of  $R$  is the same as the Tate cohomology of  $E_{fq*}$ .

Here the theory of crystals provides a key step. While it is not *a priori* clear that we can equivariantly lift the formal  $A$ -module over  $E_f$  to one over the  $R$ , using crystals we can easily produce a lift over a well-behaved quotient. This is actually sufficient: the kernel of the natural surjective map from  $E_{fq*}$  to this quotient has computable Tate cohomology. Combining this with the long exact sequence in Tate cohomology associated to this short exact sequence of modules yields the result.

**3.2. Computation of the homotopy groups of  $EO_{f(p-1)}(\mathbb{Z}/p)$ .** Our description of the group action gives the Adams-Novikov  $E_2$ -term for the homotopy of  $EO_{fq}$ . The computation of the Adams-Novikov differentials is more involved, drawing on both the  $E_\infty$ -structure of  $EO_{fq}$  and on Ravenel’s “method of infinite descent” for computing homotopy groups.

Since  $E_{fq}$  is an  $E_\infty$  ring spectrum, given any map  $u: S^{2k} \rightarrow E$ , we can form a  $\mathbb{Z}/p$ -equivariant map

$$Nu = u \dots g^{p-1}(u): S^{2kp} \rightarrow E.$$

As an equivariant spectrum the source of this map is  $S^{k\rho}$ , where  $\rho$  is the complex regular representation. Since this map is equivariant, it induces a map of homotopy fixed point spectra and of homotopy fixed point spectral sequences. The source of these maps can be identified with the Spanier-Whitehead dual of a Thom spectrum over  $B\mathbb{Z}/p$ , together with its cellular filtration, and the differentials are related to attaching maps. Naturality then allows us to conclude a great number of differentials in the corresponding homotopy fixed point spectral sequence for  $EO_{fq}$ . This was Hopkins and Miller’s original argument for height  $p - 1$ . Standard power operation techniques tell us the remaining differentials, once we are able to identify the target. Ravenel’s “method of descent” allows us to explicitly identify the targets.

The method of descent is built from the filtration of  $MU$  by the Thom spectra  $X(i)$  which appear in the proofs of the Nilpotence and Periodicity theorems. Ravenel has shown that just as  $MU$  splits into a wedge of copies of  $BP$ , the spectra  $X(i)$  split into a wedge of copies of spectra  $T(i)$  whose  $BP$  homology is  $BP_*[t_1, \dots, t_i] \subset BP_*BP$ . The spectra  $T(i)$  have a filtration by  $T(i - 1)$ -module spectra for which the associated graded is  $T(i - 1)[t_i]$ .

For  $i = 1$ , the attaching maps are exactly the classes we needed identified in the homotopy fixed point spectral sequence. Using similar techniques to the norm arguments (with the key fact that these spectra are not  $E_\infty$ ), we find short differentials in the homotopy fixed point spectral sequence for  $T(i)_*EO_{fq}$  and use these to inductively produce all of the desired longer differentials.

Analysis of the  $T(i)$ -homology of  $E_{fq}$  has two additional pay-offs. Firstly, for a range of values of  $f$  (on the order of  $f < p + 2$ ), we can use a cohomology version of the method of descent to understand the action of  $\mathbb{Z}/p$  on  $E_{fq}$ . The key fact, which follows easily from Devinatz and Hopkins' original work, is that as a  $\mathbb{Z}/p$ -module,  $E_{fq_*}T(f)$  is the symmetric algebra on  $f$  copies of the regular representation of  $\mathbb{Z}/p$ , localized by inverting an additive trace and then completed. For  $f$  in this range, simple degree arguments show that the cohomology descent spectral sequence collapses at each stage. This in particular quickly provides a description of the group action. Secondly, the equivariant description of  $E_{fq_*}T(f)$  extends in an obvious way to larger groups: if  $\mathbb{Z}/p^k$  is a subgroup of  $\mathbb{G}_n$ , then  $E_{fq_*}T(f)$  is the symmetric algebra on  $f/p^{k-1}$  copies of the regular representation of  $\mathbb{Z}/p^k$ , localized and completed. This fact should be helpful in some of the further applications.

#### 4. PROPOSED RESEARCH PROJECTS

**4.1. Computing the homotopy Groups of  $EO_{qf}(G)$ .** For groups with  $p$ -torsion subgroup larger than  $\mathbb{Z}/p$ , even computing the Adams-Novikov  $E_2$ -term is difficult. The chief problem is that for  $\mathbb{Z}/p^{k+1}$ , the full Adams-Novikov zero line for  $\mathbb{Z}/p^k$  occurs in higher cohomology. In particular, we have to understand the ring of invariants of the cyclic action of  $\mathbb{Z}/p^k$  on  $\mathbb{F}_p[x_1, \dots, x_{p^k}]$ . Techniques of invariant theory should resolve much of this, possibly allowing a complete description similar to that found for simple  $p$ -torsion. For the  $\mathbb{Z}/p^2$  case, cursory calculations suggest that the full algebra structure is understandable, and we have conjecturally identified elements of order  $p^2$  in the homotopy fixed point spectral sequence. In particular, we appear to see substantially more of the homotopy groups of spheres.

Using transfer arguments, one can show close connections between the Adams-Novikov spectral sequence for  $EO_n(\mathbb{Z}/p^k)$  and the one for  $EO_n(\mathbb{Z}/p^j)$  for  $j < k$ , and this allows for the identification of many classes and many permanent cycles. This also allows us to produce differentials on many classes to conclude that other classes are permanent cycles. The norm based geometric differentials used for the  $\mathbb{Z}/p$  action work equally well here, though the attaching maps in the underlying Thom spectra over  $B\mathbb{Z}/p^k$  are more difficult to understand.

**4.2. Non-existence of Smith-Toda complexes.** Hopkins, Ravenel, and I plan to use the computations with  $EO_{fq}$  to strengthen Nave's non-existence results for Smith-Toda complexes. Nave used Hopkins and Miller's computation of the differential in the Adams-Novikov spectral sequence for  $EO_q$  to show that the Smith-Toda complex  $V((p+1)/2)$  does not exist at  $p$ . Since  $EO_{fq}(G)$  serves in some sense as a better approximation to the sphere as  $f$  increases (and seems to do so in a quite strong sense as the  $p$ -torsion of  $G$  increases), there is some hope that mirroring Nave's methods will produce stricter non-existence results.

Careful analysis of the Adams-Novikov  $E_2$ -term allows us to related classes on the zero line to powers of the  $BP_*$  classes  $v_i$ . This in turn will allow us to compute various stages of an algebraic Atiyah-Hirzebruch spectral sequence computing the Adams-Novikov  $E_2$ -term for the  $EO_{fq}$ -homology of  $V(i)$ . Since these spectral sequences are still quite sparse, our hope is that purely algebraic methods will produce all of the differentials from the differentials for the homotopy of  $EO_{fq}$ , just as in my computation of the  $V(0)$  and  $V(1)$ -homologies of  $eo_4$ .

**4.3. Applications of geometric models of  $EO_n$ .** Behrens and Lawson have a program which focuses on the moduli stack of abelian varieties with nice properties. On certain classes of Shimura varieties, Lurie’s derived Artin representability produces a sheaf of  $E_\infty$  ring spectra with desirable properties mirroring the local properties of the Shimura varieties. In particular, the  $K(n)$ -localizations for appropriate values of  $n$  are closely related to  $EO_n(G)$  for various  $G$  related to the underlying abelian varieties. Behrens and Lawson have also produced an analogue to the image of  $J$  spectrum, generalizing Behrens’  $Q(2)$  spectrum for  $TMF$ .

Computing with Behrens and Lawson’s sheaf, in particular finding the homotopy groups of the global sections, requires knowing the coherent cohomology of the Shimura variety. In general, even  $H^0$  is not known. This makes computation almost impossible and forces consideration of other avenues. Behrens and Lawson have shown that understanding the action of finite subgroups of the Morava stabilizer group provides a way to understand the  $K(n)$ -local homotopy and then to try to understand some of the global structure.

Behrens and Lawson’s sheaf also has very nice applications to the existence of appropriate connective models for  $EO_n(G)$ . If one assumes certain, often mild conditions, then the Shimura stack on which the sheaf is defined is actually compact. While this in general will not force the global sections to be connective, Serre duality suggests that there will be a gap in the homotopy, similar to the gap present in the  $L_2$ -localization of  $tmf$ . This sort of gap would allow one to simply take the connective cover without losing much information, producing a nice model for  $eo_n(G)$ . Cursory computations at  $n = 4$  suggest that the resulting object has  $\pi_0$  strictly larger than  $\mathbb{Z}_p$ , making these slightly bigger than we might hope.

**4.4. The geometry of  $TR$ .** One of the primary approaches to computing the algebraic  $K$ -theory of an  $E_\infty$  ring spectrum is to work up the  $TR$ -tower of Hesselholt and Madsen, computing eventually the Bökstedt-Hsiang-Madsen  $TC$ . The key ingredients are computations of the homotopy groups of the homotopy fixed point spectrum  $THH(R)^{h\mathbb{Z}/p^k}$  and of the Tate spectrum  $THH(R)^{t\mathbb{Z}/p^k}$  for all  $k$ . These fit into fiber squares with the geometric fixed points and allow us to approximate the geometric fixed points with respect to the (non-closed) subgroup  $\mathbb{Z}/p^\infty$  of  $S^1$ . Many of the arguments are subtle and difficult to understand, relying on (sometimes miraculous) theorems which establish strong co-connectivity results that allow us to conclude the behavior of the general case from a few introductory cases. Even in simple cases like  $R = H\mathbb{Z}/p^2$ , these methods have yet to yield results. Work with Angeltveit and Lawson seeks to use unstable homotopy, equivariant homotopy, and geometry to better understand the constituent homotopy fixed point and Tate spectral sequences.

If  $R$  is an  $E_\infty$  ring spectrum, then  $THH(R)$  is an  $E_\infty$   $R$ -algebra. The methods Hopkins, Ravenel, and I employ to compute the differentials for  $EO_{fq}$  are universal: given any  $E_\infty$  ring spectrum on which  $\mathbb{Z}/p^k$  acts via  $E_\infty$  self-maps, we can produce differentials from a norm argument. In particular, if  $\beta$  is the periodicity generator in  $H^2(\mathbb{Z}/p^k; R_0)$  coming from the unit, then the norm argument produces a family of differentials which are determined by the attaching maps in  $B\mathbb{Z}/p^k$ . Appropriately interpreted, this allows us to reproduce many of the classical differentials for  $R = H\mathbb{F}_p$  or  $H\mathbb{Z}_p$ . There is therefore strong evidence that applying the same techniques will give differentials in more complicated settings.

The second approach is to analyze the geometric fixed point spectrum directly. The map to the homotopy fixed points is a kind of power operation construction, and approaching the direct geometry of the geometric fixed points (the ultimate object of study) should make more transparent some of the results which heretofore are mysterious.

**4.5. Computations in motivic stable homotopy.** Recent work of Dugger and Isaksen has introduced me to the motivic Steenrod algebra and motivic Adams spectral sequence. At this point, my interest is intensely exploratory, with three main foci.

Building on my experience from my other projects, I have conjecturally computed the homotopy of  $THH(H\mathbb{F}_p)$  and  $THH(\mathbb{Z}_p)$ , working over an algebraically closed field of characteristic zero. The computation relied on several fairly large suppositions and yielded surprising, though quite aesthetic, answers, and from these, we reproduce the classical computations of Bökstedt through standard “motivic-to-classical” legerdemain. My first proposal is to understand to what extent the Bökstedt spectral sequence works in the motivic context and to understand its convergence.

With Dugger, I also computed the Adams  $E_2$ -terms for motivic analogues of  $tmf$  at 2 over  $\mathbb{C}$  and of  $ko$  at 2 over  $\mathbb{R}$  and  $\mathbb{C}$ . These also yield surprising and sometimes shocking complicated results. I am interested in understanding the Adams differentials in this context. Working over  $\mathbb{C}$ , there is a natural comparison map to the classical case. However, convergence is not clear in this context, and over  $\mathbb{R}$ , the situation becomes significantly more murky. Understanding this (and if there are motivic spectra whose homotopy is computed by these spectral sequences) is the second goal.

Using standard techniques of stable homotopy and Voevodsky’s description of the dual Steenrod algebra over  $\mathbb{R}$ , I found a Bockstein spectral sequence which computes the Adams  $E_2$ -term over  $\mathbb{R}$  from the Adams  $E_2$ -term over  $\mathbb{C}$ . This seems to be some sort of descent spectral sequence associated to the Galois cover  $Spec(\mathbb{C}) \rightarrow Spec(\mathbb{R})$ . I used this to quickly compute the aforementioned Adams  $E_2$ -term for  $ko$  over  $\mathbb{R}$ . Dugger has applied this method to compute  $\pi_{0,0}$  and  $\pi_{-1,-1}$  of the  $H\mathbb{F}_2$ -nilpotent completion of the sphere, verifying in these cases the close connection with  $\mathbb{Z}/2$ -equivariant homotopy theory. Dugger, Isaksen, and I plan to further explore this approach, computing more homotopy groups of spheres and trying to understand  $\mathbb{Z}/2$ -equivariant phenomena from a motivic perspective.