THE MOTIVIC $\mathrm{Ext}_{\mathcal{A}(2)}$ OVER $\mathbb{C}$

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Abstract. We compute the motivic analogue to Mahowald's classical computation of $\mathrm{Ext}_{\mathcal{A}(2)}$, working over the field $\mathbb{C}$.

1. Set-Up

Voevodsky has shown that working over $\mathbb{C}$, the motivic dual Steenrod algebra has a form very similar to the classical story [**?**, **?**]. Recall that the motivic cohomology of a point working over $\mathbb{C}$ is given by

$$\mathbb{M}_2 = H^{*,*}(*) = \mathbb{F}_2[t].$$

Voevodsky's result is that $\mathcal{A}_*^{Mot}$ is a Hopf algebra over $\mathbb{M}_2$ given by

$$\mathcal{A}_*^{Mot} = \mathbb{M}_2[\xi_1, \dots][\tau_0, \dots]/(\tau_i^2 = t\xi_{i+1}),$$

where the coproducts on the classes ξ_i and τ_i are the classical coproducts, and where the bidegree of each is (something like)

$$|\xi_i| = (2^{i+1} - 2, 2^i) \text{ and } |\tau_i| = (2^{i+1} - 1, 2^i).$$

Remark. *At an odd prime, the result is formally identical. Here, as an algebra, $\mathcal{A}_*^{Mot}$ is the classical Steenrod algebra base changed to $\mathbb{M}_p$. However, if we recall the Dyer-Lashof action, we recover the analogue of the relation $\tau_i^2 = t\xi_{i+1}$:*

$$\beta Q^{?}(\tau_i) = \langle \underbrace{\tau_i, \dots, \tau_i}_{} p \rangle = t^{p-1}\xi_{i+1}.$$

This follows from degree considerations and the requirement that if we set $t = 1$, then we recover the classical story.

There is a canonical sub-Hopf algebra given by

$$P = \mathbb{M}_2[\xi_1, \dots].$$

This is isomorphic to the double of the classical Steenrod algebra, base-changed to $\mathbb{M}_2$. This gives us a short exact sequence of Hopf algebras over $\mathbb{M}_2$

$$1 \to P \to \mathcal{A}_*^{Mot} \to E_{\mathbb{M}_2}(\tau_0, \dots) \to 1,$$

and there is an associated Cartan-Eilenberg spectral sequence:

$$E_2 = \mathrm{Ext}_P\left(\mathbb{M}_2, \mathrm{Ext}_{E_{\mathbb{M}_2}(\tau_0,\dots)}(\mathbb{M}_2, \mathbb{M}_2)\right) \Longrightarrow \mathrm{Ext}_{\mathcal{A}^{Mot}}(\mathbb{M}_2, \mathbb{M}_2).$$

The inner Ext group can be easily computed:

$$\mathrm{Ext}_{E_{\mathbb{M}_2}(\tau_0,\dots)}(\mathbb{M}_2, \mathbb{M}_2) = \mathbb{M}_2[v_0, \dots],$$

where in the cobar complex $v_i = [\tau_i]$, but the Hopf algebra P does not trivially coact on this ring. However, we can apply a May style filtration to this algebra so that in the associated graded the action is trivial.

Theorem 1.1. *There is a spectral sequence of algebras converging to the Cartan-Eilenberg E_2 term with*

$$E_1 = \mathrm{Ext}_{D\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)[t][v_0, \dots],$$

where $D\mathcal{A}$ is the double of the classical Steenrod algebra.

In the computable cases, this spectral sequence is the essential computation; the Cartan-Eilenberg spectral sequence appears to collapse (just as it does in the classical case at an odd prime [**?**]).

The Ext term appear on the E_1 page of this spectral sequence is the classical Adams E_2 term for the homotopy groups of the sphere, regraded. While not apparent from the formulation, we can also remember at each stage the additional $\mathbb{G}_m$ degrees. Moveover, since P is a sub-Hopf algebra of $\mathcal{A}^{Mot}$, $\mathrm{Ext}_{\mathcal{A}^{Mot}}(\mathbb{M}_2, \mathbb{M}_2)$ is an algebra over $\mathrm{Ext}_P(\mathbb{M}_2, \mathbb{M}_2)$. This is the edge homomorphism in the Cartan-Eilenberg spectral sequence.

This allows us to derive several immediate and easy corollaries about the structure of $\mathrm{Ext}_{\mathcal{A}^{Mot}}$.

Corollary 1.2. *The classes h_i, represented in the cobar complex by $[\xi_1^{2^i}]$, satisfy the relations*

$$h_{i-1}h_i = 0 \text{ and } h_i^3 = h_{i-1}^2 h_{i+1}$$

for $i > 0$.

The class $\eta = h_0$ is *not* nilpotent in Ext. This is a rather striking difference from the classical story. However, it acts nilpotently on the higher Ext groups.

Corollary 1.3. *For all $x \in \mathrm{Ext}_{D\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$, there is a k such that $\eta^k x = 0$.*

This almost gives the same result for the motivic Ext. The classes v_i are at first blush η torsion free. If $p(\overrightarrow{v})$ is a homogeneous polynomial in the v_i such that $\eta d_*(p(\overrightarrow{v})) = 0$, then it is possible that $\eta p(\overrightarrow{v})$ supports an infinite η tower. We conjecture the following

Conjecture 1. *With the exception of η times polynomials in v_1 and v_2, all classes in $\mathrm{Ext}_{\mathcal{A}^{Mot}}(\mathbb{M}_2, \mathbb{M}_2)$ are η torsion.*

There are a number of polynomials in this family. In particular, the classes $P^i\eta = v_1^{4i}\eta$ all survive, as do classes like $c_0 =$.

1.1. **Finite Quotients.** Just as in the classical case, we define Hopf algebra quotients of $\mathcal{A}_*^{Mot}$ which are finite over $\mathbb{M}_2$.

Definition 1.4. *Let $\mathcal{A}^{Mot}(n)_*$ denote the quotient*

$$\mathbb{M}_2[\xi_1, \dots, \xi_n][\tau_0, \dots, \tau_n]/(\xi_i^{2^{n-i}}, \tau_i^2 - t\xi_{i+1}).$$

This is a free $\mathbb{M}_2$ module of rank 2^{n+2}, and if we set $t = 1$, then we recover the classical $\mathcal{A}(n)_*$. This is the dual to the subalgebra of the Motivic Steenrod algebra generated by $Sq^1, \dots, Sq^{2^n}$.

The decomposition of $\mathcal{A}^{Mot}$ into the "polynomial part" P and an "exterior part" applies equally well here. For $\mathcal{A}^{Mot}(n)$, let P_n denote the corresponding [truncated] polynomial part.

Theorem 1.5. *We have a short exact sequence of Hopf algebras*

$$1 \to P_n \to \mathcal{A}^{Mot}(n)_* \to E_{\mathbb{M}_2}(\tau_0, \ldots, \tau_n) \to 1,$$

where P_n is the double to the classical Hopf algebra $\mathcal{A}(n-1)$, base changed to $\mathbb{M}_2$.

This gives Cartan-Eilenberg spectral sequences in these cases as well, and there are associated May spectral sequences to compute these E_2 terms. We will run this for $n = 1$. However, since this is a nice, finite Hopf algebra over $\mathbb{M}_2$, we can also apply other filtrations, giving rise to different spectral sequences converging to Ext. We will run an example of this for $n = 2$.

2. Example: $\mathrm{Ext}_{\mathcal{A}^{Mot}(1)}$

For $n = 1$, Theorem 1.5 shows that there is a spectral sequence computing with Cartan-Eilenberg E_2 term with E_1 term given by

$$\mathrm{Ext}_{E(\xi_1)}(\mathbb{F}_2, \mathbb{F}_2)[t][v_0, v_1].$$

In this filtration, everything in the Ext portion is in filtration 0. We also place v_0 in filtration 0, and if v_1 is in filtration 1, then there are no non-trivial coactions.

This E_1 term is very easy to calculate, since it is just Ext over an exterior algebra adjoin polynomial generators:

$$E_1 = \mathbb{F}_2[t][\eta, v_0, v_1].$$

The class v_0 is a permanent cycle, since we know that τ_0 is primitive in $\mathcal{A}^{Mot}_*$. The class τ_1 is not primitive: the reduced coproduct is $\tau_0 \otimes \xi_1$. This produces for us a d_1 differential of the form

$$d_1(v_1) = v_0\eta.$$

On this Cartan-Eilenberg spectral sequence, there are Steenrod operations which can be used to compute higher differentials. This is exactly the same as in the classical case. We need only recall that since t is in cohomological degree 0, the only non-vanishing operation is $Sq^0(t) = t^2$.

Proposition 2.1. *There is a d_2 differential of the form*

$$d_2(v_1^2) = t\eta^3.$$

Proof. We apply the squaring operation Sq^1:

$$d_*(v_1^2) = d_*(Sq^1 v_1) = Sq^1 d_1(v_1) = Sq^1(v_0\eta).$$

Using the Cartan formula, we see that

$$Sq^1(v_0\eta) = Sq^0(v_0)Sq^1(\eta) + Sq^1(v_0)Sq^0(\eta) = (t\eta)\eta^2,$$

since on a cobar representative, Sq^0 acts by squaring each term and $v_0 = [\tau_0]$ then yields $[\tau_0^2] = [t\xi_1] = t\eta$. □

At this point, and for degree reasons, the spectral sequence collapses. We have the following $E_3 = E_\infty$ page.

Theorem 2.2. *As an algebra over $\mathbb{M}_2$,*

$$\mathrm{Ext}_{\mathcal{A}^{Mot}(1)}(\mathbb{M}_2, \mathbb{M}_2) = \mathbb{M}_2[v_0, \eta, [v_0v_1^2], v_1^4]/(v_0\eta, t\eta^3, \eta[v_0v_1^2], [v_0v_1^2]^2 - \eta^2 v_1^4).$$

Just as in the classical case, there are Massey product formulations for the classes involving v_1:

$$[v_0v_1^2] = \langle 2, \eta, t\eta^2\rangle \text{ and } v_1^4 = \langle \eta, t\eta^2, \eta, t\eta^2\rangle.$$

3. Example the Second: $\mathrm{Ext}_{\mathcal{A}^{Mot}(2)}$

For variety, we compute this via a slightly different method. We filter the dual Steenrod algebra by assigning to $\xi_i^{2^j}$ and τ_i the filtration $2i-1$. The associated graded is a primitively generated Hopf algebra on exterior classes corresponding to the classes $\xi_i^{2^j}$ and the classes τ_i. When we compute Ext over this, we get the E_1 term of the spectral sequence:

$$E_1 = \mathbb{F}_2[h_0, h_1, h_2, h_{20}, h_{21}, h_{30}],$$

where in the cobar complex, h_{i+1} corresponds to $\xi_1^{2^i}$, h_{21} corresponds to ξ_2, h_0 corresponds to τ_0, and h_{i0} corresponds to τ_i. This is a quadruply graded spectral sequence, graded by (s,t,u,v), where s is the cohomologic degree, t is the internal degree, u is the $\mathbb{G}_m$ degree, and v is the filtration degree. In this set-up, the d_r differentials changes degrees as $(s,t,u,v)\mapsto(s+1,t,u,v-r)$. The quad-degrees of the generators are $|h_i| = (1, 2^i, , 1)$ and $|h_{ij}| = (1, 2^j(2^i-1), , 2i-1)$. In most cases, we will suppress the filtration degree and $\mathbb{G}_m$ degree and refer only to the Adams style bidegree of the elements $(t-s, s)$.

3.1. The E_1 and E_2 Pages.

The d_1 differential comes immediately from the actual coproducts:

$$d_1(h_{20}) = h_0h_1, \quad d_1(h_{21}) = h_1h_2, \quad d_1(h_{30}) = h_0h_{21} + h_2h_{21}.$$

Taken together, these differentials imply that there is a new cycle

$$x_7 = h_1h_{30} + h_{20}h_{21} = \langle h_0, h_1, h_2, h_1\rangle = \langle h_2, h_1, h_0, h_1\rangle.$$

We also have Massey product descriptions for h_{2i}^2:

$$b_{20} = h_{20}^2 = \langle h_0, h_1, h_0, h_1\rangle, \quad b_{21} = h_{21}^2 = \langle h_2, h_1, h_2, h_1\rangle.$$

The Massey product formulations imply in turn two relations

$$h_0x_7 = h_2b_{20}, \quad h_2x_7 = h_0b_{21}.$$

These relations also follow from computing the differentials on $h_{20}h_{30}$ and $h_{21}h_{30}$. We pause here to remark that the brackets described in b_{20} and b_{21} will stop being well defined by E_3 or E_4. The brackets are of a high filtration, but we know that $h_2^2 = \langle h_1, h_2, h_1\rangle$, while $th_1^2 = \langle h_0, h_1, h_0\rangle$.

Let $b_{30} = h_{30}^2$. The cycle representing x_7 shows the final extension:

$$x_7^2 = h_1^2b_{30} + b_{20}b_{21}.$$

We have checked all of the generators, so there are no more d_1 differentials. This gives that E_2 is generated by classes h_0, h_1, h_2, b_{20}, b_{21}, b_{30}, x_7 and subject to the relations

$$h_0h_1 = h_1h_2 = 0, h_0x_7 = h_2b_{20}, h_2x_7 = h_0b_{21}, x_7^2 = h_1^2b_{30} + b_{20}b_{21}.$$

For degree reasons, there are no d_2 differentials, so this is also E_3. The degrees of the generators are $|h_i| = (2^i-1, 1)$, $|x_7| = (7, 2)$, $|b_{20}| = (4, 2)$, $|b_{21}| = (10, 2)$, and $|b_{30}| = (12, 2)$.

3.2. **The E_3 and E_4 Pages.** The algebraic Steenrod squaring operations provide our two basic d_3 differentials:

$$d_3(b_{20}) = Sq^1\big(d_1(h_{20})\big) = th_1^3 + h_0^2h_2, \quad d_3(b_{21}) = h_2^3.$$

From these and from the relations, we get the differentials on the remaining generators:

$$d_3(x_7) = h_0h_2^2, \quad d_3(b_{30}) = th_1b_{21}.$$

These give a plethora of new cycles. From the relations that h_1 kills h_0 and h_2, we see that $c_0 = h_1x_7$, $a = h_0b_{30}$, and $b = h_2b_{30}$ are d_3 cycles. An algebraic check also shows that $e = x_7b_{21}$ is a cycle, and we know that the squares of the generators $d = x_7^2$, $g = b_{21}^2$, $v_1^4 = b_{20}^2$, and b_{30}^2 are all cycles. The class h_1b_{21} survives as a simple t-torsion class, as do the classes h_1^k for $k \geq 4$. Considering the differential on x_7b_{20}, we see that $h_1^2c_0$ survives as a simple t-torsion class as well.

These satisfy various complicated relations coming from the algebra structure:

$$c_0^2 = h_1^2d, a^2 = h_0^2b_{30}^2, b^2 = h_2^2b_{30}^2, h_0c_0 = h_2c_0 = c_0a = c_0b = 0,$$
$$ab = h_0h_2b_{30}^2, d^2 = v_1^4g, h_2^2v_1^4 = h_0^2d, e^2 = dg, h_2^2d = h_0^2g.$$

Computing the differential on the pairwise products of the generators produces new relations:

$$d_3(b_{20}b_{30}) = th_1d + h_0^2b, \quad d_3(x_7b_{30}) = h_0h_2b + th_1e, \quad d_3(b_{21}b_{30}) = h_2^2b + th_1g.$$

The algebra structure of E_4 is therefore that given by the relations h_i and the generators c_0, $a, \ldots g$, v_1^4, and b_{30}^2, subject to all of these relations. For degree reasons, there is again no d_4 differentials.

3.3. **The E_5 page and E_∞.** Using Steenrod operations, it is easy to compute the differential on b_{30}^2:

$$d_5(Sq^2(b_{30})) = Sq^2(d_3(b_{30})) = Sq^2(th_1b_{21}) = Sq^0(t)Sq^0(h_1)Sq^2(b_{21}) = t^2h_2g.$$

This leaves a large number of new cycles. We have classes $h_1b_{30}^2$, $h_0b_{30}^2$, $h_0h_2b_{30}^2$, $h_2^2b_{30}^2$, and $c_0b_{30}^2$. The d_5 differential produces only the relation that $t^2h_2g = 0$. This leaves h_2g and th_2g.

This actually also produces t^3-torsion classes! The element h_1dg satisfies the relation

$$th_1dg = h_0h_2ag.$$

Classically, this annihilates it. In the motivic world, this truncates it at t^3.

At this point, the spectral sequence collapses. For degree reasons, there are no more extensions, and the entire result is periodic over classes v_1^4 and $v_2^8 = b_{20}^4$, both of which act freely.