A MAY SPECTRAL SEQUENCE FOR $\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$

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1. Introduction

Hopkins and Mahowald have shown that $\mathrm{Ext}_{\mathcal{A}(2)}$ provides the Adams E_2-term for computing the tmf-homology of a space or spectrum [1]. While this computation has been done numerous times, we present a simple version using the May spectral sequence. We make no claims of originality and carry out the computation only to illustrate what goes into it and Massey product structures.

2. The Filtration

We will actually compute $\mathrm{Ext}_{\mathcal{A}(2)_*}$ in the category of comodules over the coalgebra $\mathcal{A}(2)_*$. This Hopf algebra is a quotient of the dual Steenrod algebra. As an algebra, we have

$$\mathcal{A}(2)_* = \mathbb{F}_2[\xi_1, \xi_2, \xi_3]/(\xi_1^8, \xi_2^4, \xi_3^2),$$

and the coproducts are determined by

$$\psi(\xi_i) = \sum_{j+k=i} \xi_j \otimes \xi_k^{2^j}.$$

Our filtration follows that of Ravenel [2]. We filter this algebra by assigning to $\xi_i^{2^j}$ the filtration $2i-1$. The associated graded is a primitively generated Hopf algebra on exterior classes corresponding to the classes $\xi_i^{2^j}$. When we compute Ext over this, we get the E_1 term of the spectral sequence:

$$E_1 = \mathbb{F}_2[h_0, h_1, h_2, h_{20}, h_{21}, h_{30}],$$

where in the cobar complex, h_i corresponds to $\xi_1^{2^i}$ and h_{ij} corresponds to $\xi_i^{2^j}$. This is a tri-graded spectral sequence, graded by (s,t,u), where s is the cohomologic degree, t is the internal degree, and u is the filtration degree. In this set-up, the d_r differentials changes degrees as $(s,t,u) \mapsto (s+1, t, u-r)$. The tridegrees of the generators are $|h_i| = (1, 2^i, 1)$ and $|h_{ij}| = (1, 2^j(2^i-1), 2i-1)$. In most cases, we will suppress the filtration degree and refer only to the Adams style bidegree of the elements $(t-s, s)$.

3. The E_1 and E_2 Pages

The d_1 differential comes immediately from the actual coproducts:

$$d_1(h_{20}) = h_0h_1, \quad d_1(h_{21}) = h_1h_2, \quad d_1(h_{30}) = h_0h_{21} + h_2h_{21}.$$

Taken together, these differentials imply that there is a new cycle

$$x_7 = h_1h_{30} + h_{20}h_{21} = \langle h_0, h_1, h_2, h_1\rangle = \langle h_2, h_1, h_0, h_1\rangle.$$

We also have Massey product descriptions for h_{2i}^2:

$$b_{20} = h_{20}^2 = \langle h_0, h_1, h_0, h_1\rangle, \quad b_{21} = h_{21}^2 = \langle h_2, h_1, h_2, h_1\rangle.$$

The Massey product formulations imply in turn two relations

$$h_0 x_7 = h_2 b_{20}, \quad h_2 x_7 = h_0 b_{21}.$$

Let $b_{30} = h_{30}^2$. The cycle representing x_7 shows the final extension:

$$x_7^2 = h_1^2 b_{30} + b_{20} b_{21}.$$

We have checked all of the generators, so there are no more d_1 differentials. This gives that E_2 is generated by classes h_0, h_1, h_2, b_{20}, b_{21}, b_{30} ,x_7 and subject to the relations

$$h_0 h_1 = h_1 h_2 = 0, h_0 x_7 = h_2 b_{20}, h_2 x_7 = h_0 b_{21}, x_7^2 = h_1^2 b_{30} + b_{20} b_{21}.$$

For degree reasons, there are no d_2 differentials, so this is also E_3. The degrees of the generators are $|h_i| = (2^i - 1, 1)$, $|x_7| = (7, 2)$, $|b_{20}| = (4, 2)$, $|b_{21}| = (10, 2)$, and $|b_{30}| = (12, 2)$.

4. The E_3 and E_4 Pages

The algebraic Steenrod squaring operations provide our two basic d_3 differentials:

$$d_3(b_{20}) = Sq^1\big(d_1(h_{20})\big) = h_1^3 + h_0^2 h_2, \quad d_3(b_{21}) = h_2^3.$$

From these and from the relations, we get the differentials on the remaining generators:

$$d_3(x_7) = h_0 h_2^2, \quad d_3(b_{30}) = h_1 b_{21}.$$

These give a plethora of new cycles. From the relations that h_1 kills h_0 and h_2, we see that $c_0 = h_1 x_7$, $a = h_0 b_{30}$, and $b = h_2 b_{30}$ are d_3 cycles. An algebraic check also shows that $e = x_7 b_{21}$ is a cycle, and we know that the squares of the generators $d = x_7^2$, $g = b_{21}^2$, $v_1^4 = b_{20}^2$, and b_{30}^2 are all cycles.

These satisfy various complicated relations coming from the algebra structure:

$$c_0^2 = h_1^2 d, a^2 = h_0^2 b_{30}^2, b^2 = h_2^2 b_{30}^2, h_0 c_0 = h_2 c_0 = c_0 a = c_0 b = 0,$$
$$ab = h_0 h_2 b_{30}^2, d^2 = v_1^4 g, h_2^2 v_1^4 = h_0^2 d, e^2 = dg, h_2^2 d = h_0^2 g.$$

Computing the differential on the pairwise products of the generators produces new relations:

$$d_3(b_{20} b_{30}) = h_1 d + h_0^2 b, \quad d_3(x_7 b_{30}) = h_0 h_2 b + h_1 e, \quad d_3(b_{21} b_{30}) = h_2^2 b + h_1 g.$$

The algebra structure of E_4 is therefore that given by the relations h_i and the generators c_0, $a, \ldots g$, v_1^4, and b_{30}^2, subject to all of these relations. For degree reasons, there is again no d_4 differentials.

5. The E_5 page and E_∞

For degree reasons, we see that there is but a single possible d_5 differential: $d_5(b_{30}^2) = h_2 g$. Using Steenrod squares, this is also very easy to verify. This leaves a large number of new cycles. We have classes $h_1 b_{30}^2$, $h_0 b_{30}^2$, $h_0 h_2 b_{30}^2$, $h_2^2 b_{30}^2$, and $c_0 b_{30}^2$. The d_5 differential produces only the relation that $h_2 g = 0$.

At this point, the spectral sequence collapses. For degree reasons, there are no more extensions, and the entire result is periodic over classes v_1^4 and $v_2^8 = b_{20}^4$, both of which act freely. To best describe the situation, we draw a picture, labeling the generators and their products.

References

1. Michael J. Hopkins and Mark Mahowald, *From elliptic curves to homotopy theory*, Available on the Hopf Archive, 1998.
2. Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press Inc., Orlando, FL, 1986.