THE HOMOLOGY OF THE THH OF THE MOTIVIC EILENBERG-MACLANE SPECTRA $H\mathbb{F}_p$ AND $H\mathbb{Z}$ OVER AN ALGEBRAICALLY CLOSED FIELD OF CHARACTERISTIC ZERO

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Abstract. We run a putative Bökstedt spectral sequence to compute motivically the mod p homology of $THH(H\mathbb{F}_p)$ and $THH(H\mathbb{Z})$ over an algebraically closed field of characteristic zero.

1. Introduction

Bökstedt found a spectral sequence of the form

$$E_2 = HH_*(H_*(R;\mathbb{F}_p)) \Longrightarrow H_*(THH(R);\mathbb{F}_p).$$

This spectral sequence has been run in the classical case for many cases of R by Bökstedt, Angeltveit-Rognes, McClure-Staffeldt, and many others. In this note, we assume that Bökstedt's construction applies in the motivic setting. In particular, we assume that there is a good symmetric monoidal category of E_∞ or A_∞ motivic spectra, and that the usual notion of THH applies. We do not address directly the issue of convergence (though it seems that for degree reasons, the Bökstedt spectral sequence converges here).

The key input for our computations is due to Voevodsky: the homology of the Eilenberg-MacLane spectra $H\mathbb{F}_p$ and $H\mathbb{Z}$. Since we are working over an algebraically closed field of characteristic zero, the dual Steenrod algebra is a Hopf algebra over $\mathbb{M}_p = \mathbb{F}_p[t]$. Unless otherwise stated, all algebra functors like $E(-)$, the exterior algebra functor, and $\Gamma(-)$, the divided powers Hopf algebra functor, and all tensor products are taken over $\mathbb{M}_p$.

Theorem 1. *At an odd prime,*

$$\mathcal{A}_*^{Mot} = \mathbb{M}_p[\xi_1,\dots] \otimes E(\tau_0,\dots),$$

where $|\xi_i| = (2p^i - 2, p^i - 1)$ *and* $|\tau_i| = (2p^i - 1, p^i - 1)$. *The coproducts are the classical coproducts.*

At $p = 2$*,*

$$\mathcal{A}_*^{Mot} = \mathbb{M}_2[\xi_1,\dots][\tau_0,\dots]/(\tau_i^2 - t\xi_{i+1}),$$

where again $|\xi_i| = (2^{i+1} - 2, 2^i - 1)$, $|\tau_i| = (2^{i+1} - 1, 2^i - 1)$*, and the coproducts are the classical ones.*

The motivic Bockstein again detects multiplication by p, so we conclude that the homology of $H\mathbb{Z}$ is similar to the classical case.

Proposition 1. *As an* $\mathcal{A}_*^{Mot}$*-comodule,*

$$H_*(H\mathbb{Z};\mathbb{F}_p) = \mathcal{A}_*^{Mot} \Box_{E(\tau_0)} \mathbb{M}_p.$$

There is a classical relationship between the classes τ_i and ξ_{i+1}:

$$\beta Q^0(\tau_i) = \langle \tau_i, \dots, \tau_i \rangle = \xi_{i+1}.$$

The $\mathbb{G}_m$-degree of these two sides differs, so we conclude that the motivic analogue is

$$\beta Q^0(\tau_i) = \langle \tau_i, \dots, \tau_i \rangle = t^{p-1}\xi_{i+1}.$$

In particular, this implies

$$Q^0(\tau_i) = t^{p-1}\tau_{i+1}.$$

These formulas will be used to derive the differentials in the Bökstedt spectral sequence and to derive multiplicative extensions, just as in the classical case.

We lastly recall some formulas for Tor. Let $\Gamma(x)$ denote the divided powers Hopf algebra on a class x. We then have

$$Tor^{E(x)}(\mathbb{F}_p, \mathbb{F}_p) = \Gamma([\sigma x]) \text{ and } Tor^{\mathbb{F}_p[x]}(\mathbb{F}_p, \mathbb{F}_p) = E([\sigma x]),$$

where $[\sigma x]$ represents the class in Tor_1 coming from x in the cobar complex.

2. Mod p Case

2.1. Homology.

Theorem 2. *As an $\mathcal{A}_*^{Mot}$-comodule algebra, we have*

$$H_*(THH(H\mathbb{F}_p); \mathbb{F}_p) = \mathcal{A}_*^{Mot} \otimes E(\lambda_1, \dots) \otimes \mathbb{M}_p[\mu_0, \dots]/(\mu_i^p - t^{p-1}\mu_{i+1}, t^{p-1}\lambda_i = \lambda_{i-1}\mu_{i-1}^{p-1}),$$

where $\lambda_i = [\sigma\xi_i]$ and $\mu_i = [\sigma\tau_i]$.

Proof. The E_2 term of the Bökstedt spectral sequence is (via the usual change of rings argument)

$$\mathcal{A}_*^{Mot} \otimes E([\sigma\xi_1], \dots) \otimes \Gamma([\sigma\tau_0], \dots).$$

Angeltveit and Rognes' argument about the differentials in the Bökstedt spectral sequence applies equally well here, so we conclude that there is a d_{p-1} differential of the form

$$d_{p-1}([\sigma\tau_i]) = \beta Q^0([\sigma\tau_i]) = [\sigma\beta Q^0\tau_i] = t^{p-1}[\sigma\xi_{i+1}].$$

This is a spectral sequence of Hopf algebras over $\mathbb{M}_p$ (again by Angeltveit and Rognes' classical case), and we therefore conclude that the E_p page is the E_∞ page, since E_p is generated by elements of Tor degree at most $p-1$. We therefore have that

$$E_\infty = \mathcal{A}_*^{Mot} \otimes E([\sigma\xi_1], \dots)/(t^{p-1}[\sigma\xi_1]) \otimes \mathbb{M}_p[[\sigma\tau_0], \dots]/([\sigma\tau_i]^p).$$

At this point, we again appeal to the Dyer-Lashoff action and Angeltveit and Rognes' result:

$$[\sigma\tau_i]^p = Q^0([\sigma\tau_i]) = [\sigma Q^0\tau_i] = t^{p-1}[\sigma\tau_{i+1}].$$

We defer the proof of the final multiplicative relation (the hidden extension linking λ_i and elements of higher filtration) until after the integral computation. We remark that it is essentially a consequence of the surprising t divisibility of the power of μ_0. □

We remark that setting $t = 1$ reconstructs the classical computation. In fact, for degree reasons, this is the only possible E_∞ page which would work to give the classical homology when setting $t = 1$.

2.2. **Homotopy.** We now run the motivic Adams spectral sequence. Since the homology of free over $\mathcal{A}_*^{Mot}$, we conclude that the spectral sequence collapses.

Theorem 3. *As an algebra*

$$THH_*(H\mathbb{F}_p) = E(\lambda_1, \dots) \otimes \mathbb{M}_p[\mu_0, \dots]/(\mu_i^p - t^{p-1}\mu_{i+1}, t^{p-1}\lambda_1, t^{p-1}\lambda_i = \lambda_{i-1}\mu_{i-1}^{p-1}).$$

Since t does not act faithfully, we conclude that this does not split as a product of Eilenberg-MacLane spectra. This is again in *stark* contrast to the classical case, since this is a module over $H\mathbb{F}_p$.

3. Integral Case

3.1. **Homology.** Again, very little changes from the classical to the motivic case. The proof of the following is essentially identical to the case of $H\mathbb{F}_p$, so we omit it. We remark that just as in the classical case, we know that $[\sigma\xi_1]$ survives. Again, let λ_i denote the class $[\sigma\xi_i]$, and let μ_i denotes the class $[\sigma\tau_i]$.

Theorem 4. *As an $\mathcal{A}_*^{Mot}$-comodule algebra, we have*

$$\begin{aligned} &H_*(THH(H\mathbb{Z}); \mathbb{F}_p) = \\ &\quad \mathcal{A}_*^{Mot} \square_{E(\tau_0)} E(\lambda_1, \lambda_2, \dots) \otimes \mathbb{M}_p[\mu_1, \dots]/(\mu_i^p - t^{p-1}\mu_{i+1}, t^{p-1}\lambda_i = \lambda_{i-1}\mu_{i-1}^{p-1}). \end{aligned}$$

3.2. **Homotopy.** We here can either run the Adams spectral sequence, which, since the comodule structure on $H_*(THH(H\mathbb{Z}))$ is extended up from $E(\tau_0)$ is especially simple, or we can run the Bockstein spectral sequence from the homotopy of $THH(H\mathbb{F}_p)$. These approaches are essentially equivalent. The Adams E_2 term is given by

$$E(\lambda_1, \lambda_1\mu_1^{p-1}, \lambda_2, \dots) \otimes \mathbb{M}_p[\mu_2, \dots][v_0]/(v_0\lambda_1, \mu_i^p - t^{p-1}\mu_{i+1}, t^{p-1}\lambda_i = \lambda_{i-1}\mu_{i-1}^{p-1}),$$

where all classes with the exception of v_0 are in Adams filtration 0 and where v_0 is in Adams filtration 1. This is where the surprising extension between λ_i and decomposables appears. We know that after setting $t = 1$, we have a differential of the form

$$d_i(\mu_1^{p^i}) = v_0^i \lambda_1 \mu_1^{p^{i-1}(p-1)}.$$

This differential must be reflected in the motivic world. However, $\mu_1^{p^{i-1}(p-1)} = t^{p^{i-2}(p-1)}\mu_i$. This means that the target of this differential must be $t^{p^{i-2}(p-1)}$ divisible. A simple induction argument shows that this implies that $t^{p-1}\lambda_i = \lambda_{i-1}\mu_{i-1}^{p-1}$.

Corollary 1. *There are d_i differentials of the form*

$$d_i(\mu_i) = v_0^i \lambda_i.$$

This has the tremendous advantage of being more aesthetically pleasing.

Corollary 2. *As $\mathbb{Z}_{(p)}[t]$-modules,*

$$THH_k(H\mathbb{Z}_{(p)}) = \begin{cases} \mathbb{Z}/p^{\nu_p(k)+1}[t] & \textit{when } k = 2pj - 1, \\ 0 & \textit{otherwise.} \end{cases}$$

We can also record the $\mathbb{G}_m$ degrees of the generators. Write k as $2p^i(ap+b) - 1$, where $1 \leq b \leq p - 1$. Then the $\mathbb{G}_m$ degree of the generator is $(ap + b)(p^i - 1)$.