Research Statement

Lei Zhao

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My research lies in an area of mathematics known as representation theory, a tool of mathematics that provides a precise way to describe the symmetries in nature. Symmetry is one of the most fundamental properties in mathematics and physics. A sphere demonstrates a simple symmetry: turn it by any amount and it appears unchanged. Underlying any symmetrical shape is a mathematical object called a group. Representation theory is a way of studying the groups and their infinitesimal counterparts such as Lie (super)algebras by expressing them in terms of matrices. It is very powerful since it reduces problems in abstract algebraic formalism to problems in algebras of matrices, which are more concrete and easier to manipulate. Representation theory depends upon the type of coefficients over which the matrices are defined. The most important cases are complex numbers and finite fields (or their algebraic closures).

The representation theory over finite fields and their closures are also known as modular representation theory. The modular representations of Lie algebras have been developed over the years with intimate connections to algebraic groups. Modular representation theory of Lie algebras differs from representation theory over complex numbers in a essential way: not only do the classical methods no longer work, also most of the fundamental results obtained from complex representations cannot be transferred to this "modular" setting.

In search for a unified theory of everything, physicists have proposed String Theory as a candidate theory. Supersymmetry adds another invisible dimension (an odd dimension as opposed to the ordinary dimension called even) to such considerations and the study of Lie superalgebras and supergroups is crucial to understanding the supersymmetry. The complex representation theory of Lie superalgebras has been studied quite intensively and has played a large role in quantum theory and the theory of elementary particles.

However, the modular counterpart for Lie superalgebras has yet to be developed. My research aims to initiate and develop systematically the modular representation theory of Lie superalgebras.

Let us be more specific here. A representation of a Lie superalgebra is a systematic way of expressing elements in a Lie superalgebra as square matrices of a given dimension. Our methodology for studying representations has largely taken what philosophers would call an "atomistic" view of the structure of representations. That is, we try to determine how the representation is built from its simplest units (called irreducible representations), and then determine more precisely what these simplest units look like.

We started with looking at the irreducible representations. We first observe that the representations of a Lie superalgebra are parameterized naturally by so-called p-characters χ in the dual space of the even part of the Lie superalgebra. This allows us to associate the representations of the Lie superalgebra to those of a family associative superalgebras U_χ which are called reduced enveloping algebras.

We next observed that all irreducible representations of a Lie superalgebra are finite-dimensional. The next question come to us quite naturally is what the dimensions of these irreducible representations would be. Based on explicit computations of examples and knowledge of the non-super case, we made a conjecture (which will be called super KW property) which states that the dimension of any irreducible representation of a Lie superalgebra is divisible by d_χ, where d_χ is an integer that is determined by the geometry of the coadjoint orbit of χ of the corresponding supergroup. This conjecture generalizes and includes as a special case a celebrated conjecture of Kac-Weisfeiler for non-super case, which was proved by A. Premet in 1995. Premet's work, published on a top journal, was regarded as a breakthrough in the field.

The first main result in my dissertation was that we established the super KW property for basic classical Lie superalgebras, the most important class of Lie superalgebras. Our approach took full advantage of a combination of techniques developed by Premet and Skryabin, which allowed us to establish the Super KW Property for basic Lie superalgebras with nilpotent p-characters bypassing completely sophisticated geometric methods, which is not available for Lie superalgebras. In addition, we established a Morita equivalence to reduce general p-characters to nilpotent ones, adapting a classical work of Friedlander-Parshall to the superalgebra setup. At several places we have to find ways to overcome new implications and difficulties which are not presented in the non-super setup. The proof leads naturally to the notion of finite W-superalgebras, whose representation theory together with that of their complex counterpart deserve systematic study. This work will appear in a paper (joint with my advisor) at Proc. London Math. Soc. (2009).

Another important class of Lie superalgebras besides basic classical ones is the queer Lie superalgebra, which can be viewed as a true super-analogue of the general linear Lie algebra (the Lie algebra of all $n \times n$-matrices). We developed the modular representation theory of queer Lie superalgebra in a systematic way. In particular, we gave a semisimplicity criterion for reduced enveloping algebras U_χ with semisimple p-characters χ and established the super KW property for nilpotent p-characters. We also worked out completely the representation theory for the rank 2 queer Lie superalgebra. A paper on this has been written (joint with my advisor) and submitted.

Due to the importance of the matter, I gave another new proof for super KW property for basic classical Lie superalgebras which uses a different idea of families of associative superalgebras. The proof extends an earlier idea of Premet-Skryabin and it has been written in a preprint (2008).

In a recently completed preprint (2009), I established an equivalence of categories which reduces the "typical" representation theory of the so-called type I Lie superalgebras to that of its even part, which has been extensively studied in the last 3 decades. Such an equivalence allows us to "transfer" known results of reductive Lie algebras/groups, such as progresses towards the famous Lusztig conjecture, to the super setting.

The non-typical representation theory for Lie superalgebras is a pure super phenomenon and will be pursued further in my dissertation. It will offer new surprises and insights for future understanding of the super world.