Abstract

This paper considers the long-run distribution of capital holdings in a model with heterogeneous agents, complete asset markets, and progressive taxation. We are able to prove that the model implies negative correlations between capital and labor income regardless of parameter values; typically we find that this correlation is very close to $-1$ quantitatively. In addition, holders of capital face the lowest marginal tax rates in the population. These results suggest that the assumption of complete markets may be troublesome for political economy considerations.

1. Introduction

This paper studies the long-run distribution of wealth in a model with three features: heterogeneous consumers, progressive income taxation, and complete asset markets. There have been several papers that demonstrate the strange properties of long-run wealth distributions in models with complete markets. Becker (1980) shows that an environment with heterogeneous discount factors leads to a collapse of the wealth distribution – only the most patient household has positive wealth in the long run. Wang (1996) and Coen-Pirani (2004) explore the wealth distribution in the presence of heterogeneous coefficients of risk aversion, finding that it again collapses. Other papers – such
as Chatterjee (1994) and Krusell and Ríos-Rull (1999) – show that the distribution of wealth with complete markets has no mobility.

Most relevant for this study is Sarte (1997), who demonstrates that progressive income taxation can prevent the wealth distribution from collapsing. The key reason that the wealth distribution may not collapse in the steady state with progressive taxation can be understood by looking at the steady state Euler equation

\[ 1 = \beta_i \left( 1 + \left( 1 - \tau' (y_i) \right) (r - \delta) \right). \]

If taxes are flat, so that \( \tau' (y_i) \) is the same for all individuals, the RHS contains only one individual-specific variable, the discount factor, while the LHS is constant. Therefore, if \( \beta \) is not constant this equation can only hold for one individual, and it turns out that individual has the largest \( \beta \). With progressive taxation the distribution may not collapse because variations in \( y_i \) can undo the effects of \( \beta_i \) and permit this equation to hold with equality for more than one type. The resulting distribution of wealth is determinate as well, since labor supply is inelastic.

Our first contribution is to explore how this argument must be modified when labor supply is elastic. The long-run wealth distribution can have wealth held by more than one type of agent and is determinate; any agent that is unconstrained in assets, however, must face the same tax rate. Under the additional assumption that the tax rate function is marginal-rate progressive (i.e., the marginal tax rate is monotone increasing), we can show that the unconstrained agents face the lowest tax rate in the population; by monotonicity, they must all therefore have the lowest income in the population. If there are agents in the economy who are asset-constrained, they have higher income than unconstrained agents do, leading to low levels of wealth coinciding with relatively high income; quantitatively we obtain correlations around \(-0.62\). This result stands in stark contrast to the empirical correlation of 0.6 (from Budría Rodríguez et al. 2002), and it holds for any time-separable utility function with homogeneous discount factors.\(^1\)

The long-run distribution can have three groups – nonworkers who own wealth, workers who hold assets above the lower limit, and workers who are asset-constrained – ranked according to their idiosyncratic productivity. The nonworkers have the lowest productivity levels and hold the most wealth, while oddly the highest productivity workers are asset-constrained. Both of these groups can be empty for some parameter values. All members of the unconstrained working group

\(^1\)Specifically, the utility function can be nonseparable in consumption and leisure. Furthermore, the result holds for heterogeneous risk aversion and preference for leisure as well as a case of endogenous discount factors as in Uzawa (1968).
(this set is never empty) have assets above the lower bound so they must face the same tax rates as the nonworkers, meaning they must have the same income. Within the unconstrained working group, as productivity rises households choose to work more (leading to higher labor income) and therefore reduce their asset income one-for-one, leading to a perfect negative correlation between the two types of income for this group. Since the other two groups have one only one source of income, the overall correlation in the economy is negative; we typically find correlations that are smaller than −0.9, which is not consistent with the data (Budría Rodríguez et al. 2002 do not explicitly report this value, but given the correlations between income sources and (earnings, income, wealth) reported in their Table 4 it must be positive).

2. The Model

The model economy is populated by three groups of agents: households, firms, and the government.

2.1. Households

The economy consists of a unit continuum of households. Households differ with respect to their labor productivity $\varepsilon_i$; each type has measure $\psi_i$. Preferences for a type $i \in I$ household give rise to a time-separable lifetime utility function

$$U_i = \sum_{t=0}^{\infty} \beta^t u(c_i, 1 - h_{it})$$

where $\beta \in (0, 1)$. We assume that $u(c, 1 - h)$ is strictly increasing in $c$ and $1 - h$ and strictly concave, as well as continuously-differentiable as many times as needed; $u$ need not be separable.

The household maximizes this function while facing the period-$t$ budget constraint

$$c_{it} + a_{i,t+1} \leq y_{it} + a_{it} - \tau(y_{it}) \quad (2.1)$$

where

$$y_{it} = (r_t - \delta)a_{it} + w_t h_{it}\varepsilon_{it} \quad (2.2)$$

is household $i$’s period-$t$ income and $\tau(y_{it})$ is the period-$t$ tax burden. $a_{it}$ is the combined stock of physical capital and government debt owned by the household at the beginning of period $t$. 
Households are permitted to borrow up to an exogenous borrowing limit,

\[ a_{i,t+1} \geq a^b. \]  

(2.3)

Since our economy will not feature uncertainty, there is no demand for state-contingent assets and we therefore ignore them.

Differentiating the lifetime utility function of each type with respect to \( a_{t+1} \) and \( h_t \) yields the following efficiency conditions for all \( i, t \):

\[
\beta \left( \frac{\partial U_i}{\partial c_{i,t+1}} \right) \left[ \left( 1 - \frac{\partial \tau(y_{i,t+1})}{\partial y_{i,t+1}} \right) (r_{t+1} - \delta) + 1 \right] \leq \frac{\partial U_i}{\partial c_{i,t}} \tag{2.4}
\]

\[
\frac{\partial U_i}{\partial c_{i,t}} \left( 1 - \frac{\partial \tau(y_{i,t})}{\partial y_{i,t}} \right) w_{i\epsilon} + \frac{\partial U_i}{\partial h_{i,t}} \leq 0 \tag{2.5}
\]

\[
(r_t - \delta)a_{i,t} + wh_{i\epsilon} + a_{i,t} - \tau(y_{i,t}) - a_{i,t+1} - c_{i,t} = 0 \tag{2.6}
\]

where the inequality in (2.4) or (2.5) is an equality if \( a_{i,t+1} > a^b \) or \( h_{i,t} > 0 \) respectively. The possibility of households hitting corners turns out to have some implications for the results.

### 2.2. Production and Government

A stand-in firm rents labor and capital from the households and produces output according to a constant-returns-to-scale production function \( F(K_t, N_t) \). We make the assumptions that \( F \) is continuously differentiable, strictly increasing, strictly concave, obeys the Inada conditions, and satisfies \( F(K,0) = F(0,N) = 0 \). Factor prices are determined within a competitive market and thus each factor is paid its marginal product:

\[
w_t = F_N(K_t, N_t) \tag{2.7}
\]

\[
r_t = F_K(K_t, N_t). \tag{2.8}
\]

Government activity is represented by a fixed level of expenditure \( \bar{G} \) that is financed through tax revenues and government debt, \( D \). The government budget constraint is therefore

\[
D_{t+1} - D_t + \int_I \tau(y_{i,t}) \, di = \bar{G} + (r_t - \delta) D_t, \tag{2.9}
\]

where the net return to holding government debt must equal the return to physical capital to rule
out arbitrage.

2.3. Equilibrium Conditions

In equilibrium the labor, asset, and goods markets must clear:

\[ K_t + D_t = \int a_i \psi_i di \]  
(2.10)

\[ N_t = \int \varepsilon_i h_{it} \psi_i di \]  
(2.11)

\[ \int c_{it} \psi_i di + K_{t+1} + C = F (K_t, N_t) + (1 - \delta) K_t. \]  
(2.12)

We can define a competitive equilibrium in the usual way: households maximize, firms maximize, the government budget constraint is satisfied, and markets clear.

3. Steady State

In a steady state, equations (2.4)-(2.6) reduce to

\[ \beta \left[ \left(1 - \frac{\partial \tau (y_i)}{\partial y_i} \right) (r - \delta) + 1 \right] \leq 1 \quad \text{w/ equality iff } a_i > a^b \]  
(3.1)

\[ \frac{\partial U_i}{\partial c_i} \left(1 - \frac{\partial \tau (y_i)}{\partial y_i} \right) w_\varepsilon_i + \frac{\partial U_i}{\partial h_i} \leq 0 \quad \text{w/ equality iff } h_i > 0 \]  
(3.2)

\[ (r - \delta) a_i + w h_i \varepsilon_i - \tau (y_i) - c_i = 0 \]  
(3.3)

We require an assumption about the tax function in order to proceed.

**Assumption 3.1.** \( \tau \) is uniformly continuous and \( \frac{\partial \tau (y)}{\partial y} \) is strictly bounded away from 1 \( \forall y \geq 0 \).

This assumption allows us to prove our first proposition.

**Proposition 1.** Under the restrictions on \( \tau \) stated in Assumption, in any steady state characterized by a distribution of asset holdings with \( a_i > a^b \) and \( a_{i'} > a^b \) for \( i, i' \in I \), the following two conditions must be true: (i) \( \frac{\partial \tau (y)}{\partial y} \big|_{y=y_i} = \frac{\partial \tau (y)}{\partial y} \big|_{y=y_{i'}} = m^* \); (ii) \( m^* \leq \frac{\partial \tau (y)}{\partial y} \big|_{y=y_{i''}} \forall i'' \in I, i'' \neq i, i' \). Furthermore, it need not be the case that \( i = i' \) so that the asset distribution may not degenerate.

We relegate the proof to the appendix. Proposition 1 provides the conditions under which an economy with complete markets and progressive taxation can have a long-run wealth distribution
that does not collapse. Essentially, there can be multiple types holding assets only if they all face the same tax rate and this tax rate is the lowest one in the population. It is immediate that a monotone increasing marginal tax function implies wealth is held by the households with the lowest income level.

**Corollary 1.** If \( \frac{\partial^2 \tau(y)}{\partial y^2} > 0 \) then \( a_i > a^b \) and \( a_j > a^b \) implies \( y_k = y_j = \min_{i \in I} y_i \).

Given our assumptions about \( u, \varepsilon, \) and \( \tau \), it can be shown that the resulting decision rules \( h(a_i, \varepsilon_i) \) and \( a'(a_i, \varepsilon_i) \) are continuous in \( \varepsilon_i \) and that the latter is a weakly decreasing function in \( \varepsilon_i \). A proof of this fact is provided in the appendix. The resulting distribution of hours and assets across types can be characterized by a pair \((\varepsilon_w, \varepsilon_c)\) which partitions the population into three groups: non-workers \( I_n \), unconstrained workers \( I_u \), and borrowing-constrained workers \( I_{bc} \). We are able to characterize these sets in terms of cutoff values and order those cutoffs as shown in Proposition 2; the proof is contained in the appendix.

**Proposition 2.** The sets \( \{I_n, I_u, I_{bc}\} \) obey \( I_n = \{i \in I : \varepsilon_i \leq \varepsilon_w\} \), \( I_u = \{i \in I : \varepsilon_w < \varepsilon_i < \varepsilon_c\} \), and \( I_{bc} = \{i \in I : \varepsilon_c \leq \varepsilon_i\} \) if \( \frac{\partial^2 \tau(y)}{\partial y^2} > 0 \).

The proof of this proposition also establishes two interesting facts. First, for all \((i, i') \in I_n\), \( h_i = 0 \) and \( a_i = a_{i'} > a^b \). Second, for all \( i \in I_u \) \( h \) is weakly increasing and \( a \) is strictly decreasing in \( \varepsilon \). The oddity of this result is that it implies assets will be held by the relatively income-poor if marginal tax rates are increasing.

The following corollary is immediate: labor and capital income must be negatively correlated.

**Corollary 2.** The correlation between \((r - \delta) a_i \) and \( w \varepsilon_i h_i \) is negative if \( \frac{\partial^2 \tau(y)}{\partial y^2} > 0 \).

**Proof.** This result follows from three facts: labor income is constant at zero over \( I_n \), capital income is constant over \( I_{bc} \) and nonpositive, and the correlation between \( a_i \) and \( h_i \varepsilon_i \) is \(-1\) over \( I_u \).

A second corollary is also immediate: income and wealth are negatively correlated if \( I_{bc} = \emptyset \).

**Corollary 3.** The correlation between \( a_i \) and \( y_i \) is negative if \( \frac{\partial^2 \tau(y)}{\partial y^2} > 0 \) and \( I_{bc} = \emptyset \).

**Proof.** By Proposition 1, \( \forall i \in I_{bc} \) \( \tau'(y_i) > \tau'(y_j) \) for all \( j \notin I_{bc} \). By monotonicity of \( \tau'(y) \), \( y_i > y_j \). But \( a_i < a_j \) so the correlation is negative.
If $\mathcal{I}_{bc} = \emptyset$ then income is equal across all households, so the correlation coefficient would not be well-defined; the covariance would be zero, however. If households are asset-constrained their income is "too high" and their assets are very low, leading to the negative correlation in the population.

Why does the model produce such a strange distribution of income and wealth? It turns out that the key feature is the requirement that agents who hold asset levels above the debt limit must have the same income level; one can see this easily using the steady state Euler equation. An individual with higher $\varepsilon$ finds leisure relatively costly, so asset income is reduced and labor income is increased, but the tradeoff is one for one. Only when the debt limit is reached does income diverge from the minimum; those households have higher income than the rest, but obviously have lower wealth. Increasing $\varepsilon$ within $\mathcal{I}_{bc}$ translates to higher labor income, since leisure is getting more expensive, but the individual cannot reduce asset income further.

We suspect that our result is connected to the general suboptimality of capital taxation in Ramsey models, as in Chamley (1986). In our economy, capital taxation is inefficient but inescapable. With a single type of individual, there is nothing the economy can do to shift the tax burden on capital. With heterogeneous agents, however, capital can be reallocated to minimize the burden, which is exactly what happens in our economy: capital is held by agents with low income and correspondingly low marginal tax rates.\(^2\) Naturally, the ones with the most assets should be those with the lowest productivity; having these agents work little (to keep their income low) is socially the least wasteful. As their productivity rises, however, it becomes increasingly costly to have these agents not supplying labor, so they begin to work more; to keep the burden on capital low, these agents simultaneously shed their assets.

4. Quantitative Results

To illustrate the quantitative significance of our theorems we calibrate our economy and compute some examples. The examples are useful to understand how parameter changes affect the size of the three groups of households; for example, the measure of the set $\mathcal{I}_{bc}$ drives the negative correlation between income and wealth, and we would like to have some idea how large this group is. Furthermore, our theorems do not apply to the case of heterogeneous preferences, so we use numerical evidence that suggests which kinds of preference heterogeneity can fix the problem.

\(^2\)With a flat tax reallocation does not affect the tax burden on capital.
To reasonably approximate a continuous distribution of productivity types, we set \( I = \{1, \ldots, 100\} \) and choose the values and measures of each type to match the distribution of US wages from Floden and Lindé (2001). That is, we assume \( \log(\varepsilon) \sim N\left(0, \sigma_{\psi}^2\right) \) and we choose \( \sigma_{\psi}^2 = 0.1175 \) to match the observed permanent differences. We space our grid points for each type evenly over a range that covers three standard deviations in each direction.

We set the period utility function to

\[
u(c_{it}, 1 - h_{it}) = \log(c_{it}) + \frac{B}{1 - \sigma}(1 - h_{it})^{1 - \sigma}
\]

with \( B \geq 0 \) and \( \sigma \geq 0 \) and the production function to

\[
F(K_t, N_t) = K_t^{\alpha}N_t^{1 - \alpha}
\]

with \( \alpha \in [0,1] \). The steady state equations become

\[
\beta \left[ \left( 1 - \frac{\partial \tau(y_i)}{\partial y_i} \right) (r - \delta) + 1 \right] \leq 1 \quad \text{w/ equality iff } a_i > 0 \quad \text{(4.1)}
\]

\[
\frac{1}{c_i} \left( 1 - \frac{\partial \tau(y_i)}{\partial y_i} \right) w\varepsilon_i - B (1 - h_i)^{-\sigma} \leq 0 \quad \text{w/ equality iff } h_i > 0 \quad \text{(4.2)}
\]

\[
(r - \delta)a_i + wh_i\varepsilon_i - \tau(y_i) - c_i = 0 \quad \text{(4.3)}
\]

where

\[
\tau(y_i) = \nu_0 \left( y - (y^{-\nu_1} + \nu_2)^{-\frac{1}{\nu_1}} \right)
\]

\[
\frac{\partial \tau(y_i)}{\partial y_i} = \nu_0 \left[ 1 - y_i^{-\nu_1 - 1} \left( y_i^{-\nu_1 + \nu_2} \right)^{-\frac{1}{\nu_1} - 1} \right] > 0,
\]

which is the tax function estimated by Gouveia and Strauss (1994). The appendix details how we solve this system of inequalities.

\( \{\beta, B, G, D, \delta, \alpha\} \) are calibrated to match some relevant moments from US data: \( K = 3.0Y \), \( H = 0.33, \ G = 0.2Y, \ D = 0.5Y, \ I = 0.15Y, \) and \( rK = 0.36Y \). Following the estimates in Gouveia and Strauss (1994) we set \( \nu_0 = 0.258 \) and \( \nu_1 = 0.768 \), while \( \nu_2 \) is set to the value that clears the government’s budget constraint. \( \sigma \) is initially chosen to be 2.0 and the borrowing limit is set to \(-1.5\). The baseline values are displayed in Table 1; in the robustness section we will consider
alternative values, particularly for $\sigma$ and $a^b$.

Figure 1 shows the distribution of capital income in the steady state of our model under the baseline parametrization; this distribution is the same as the distribution of wealth because all savings earns the same pre-tax return. The first thing to notice is that households with low productivity, and therefore low effective wages, hold a majority share of the assets in the economy, while those households with high productivity are borrowing constrained. Moderate productivity households find it optimal both to work and maintain assets strictly above the borrowing limit. As shown above, all households that are not borrowing constrained must face the lowest marginal tax rate, and thus must have the same lowest level of income in the economy. As $\varepsilon$ rises, labor income for the same number of hours worked rises. These households have two avenues by which to compensate for their higher effective wage: work less or save less.

Examining the distribution of hours across types (shown in Figure 2), the relationship between work and savings becomes clear. Figures 1 and 2 display the partition of the economy along $\varepsilon$ with vertical lines. The large-dashed vertical line shows the highest value of $\varepsilon$, $\varepsilon_w$, for which the non-negativity constraint on hours is binding, while the small-dashed line shows the lowest value of $\varepsilon$, $\varepsilon_c$, for which the borrowing constraint does not bind. We see that for the baseline case, a fraction of low $\varepsilon$ types choose not to work; that is, $I_n$ is not empty.\(^3\) Therefore, households with $\varepsilon$ sufficiently close to $\varepsilon_{\min}$ find it optimal to match the work and savings decisions of the lowest productivity households. On the margin for low values of $\varepsilon$ adjusting hours is less costly (in utility terms) than adjusting savings because the effective wage is so small. As $\varepsilon$ rises however, the opportunity cost of leisure also increases and so it becomes more efficient to adjust total income by reducing savings and increasing labor supply. For $\varepsilon_w < \varepsilon < \varepsilon_c$ we see that asset holdings are declining and hours are increasing until households reach the borrowing constraint. These high productivity ($\varepsilon \geq \varepsilon_c$) households can only reduce hours worked to adjust income; however the opportunity cost of leisure for these households is quite high. As a result, they choose not to set hours to the level necessary to match the income of the lower productivity households, but rather they choose to face higher marginal tax rates. Thus, these high productivity households have greater income and greater consumption than the other two groups. Figure 3 shows the marginal tax rate faced by each type of agent in equilibrium; note that the low $\varepsilon$ types face the lowest marginal tax rate and the effective tax rate only increases for $\varepsilon \geq \varepsilon_c$.

Figure 4 shows the distribution of labor income; the distribution of capital income is the same.

\(^3\)For some parameterizations, mainly those with small $B$, this set may be empty.
as the distribution of wealth. The two sources of income are negatively correlated in the model, just as proven above. The incentive to keep income low induced by the progressive tax structure causes households to adjust savings and hours in opposite directions. It is this behavior that yields the strong negative correlation between labor income and asset income in the steady state. Notice that the only reason why we do not find perfect negative correlation between these two streams of income is that hours are bounded and assets are bounded from below and the measure of constrained agents is positive. If we set \( a^b \) instead to the natural debt limit – the lowest asset holding consistent with nonnegative consumption – the correlation gets closer to \(-1\); we eliminate \( I_{bc} \) entirely. If we also could eliminate \( I_n \), say by permitting negative hours, then the correlation would be exactly \(-1\).\(^4\)

4.1. Robustness

We have explored a wide range of parameter values for the economy around our chosen calibration; details of these experiments are available from the authors upon request. We find that most parameters have little to no quantitative effect on the correlations of interest, and all the movements are monotone except for \( B \), the relative weight of leisure. For all cases considered – changes in \( \alpha, \beta, \delta, G, B \), and \( \sigma \) – the correlation between labor and capital income always is less than \(-0.9\) and the correlation between wealth and income is always less than \(-0.58\).\(^5\) There are generally two effects at play when parameters change, and these effects work through the size of the sets \( \{I_n, I_u, I_{bc}\} \). On the one hand, decreasing \( I_u \) would tend to push the correlation away from \(-1\), since the correlation between labor and capital income is \(-1\) for this group; for the other groups the covariance is zero but the correlation is not defined. On the other hand, as \( I_u \) gets close to zero measure, the economy approaches a two-group state, and the correlation must therefore approach either 1 or \(-1\); in this case, since one group has zero labor income and the other has constant nonpositive capital income, the correlation must approach \(-1\).\(^6\)

If we permit certain kinds of heterogeneity in preferences our results remain, although our theorems do not apply to these economies so our evidence is numerical. For example, we let \( \sigma \)

\(^4\)There is a way to interpret the absence of a nonnegativity constraint for hours; see the appendix of Smith and Wang (2005).

\(^5\)For small values of \( B \) the correlation between capital and labor income is rising, but then falls as \( B \) gets large enough. These effects are caused by the changing sizes of the three groups, as discussed in the text. The effect is quantitatively negligible. When \( B = 0 \), so that labor supply is inelastic, the correlation is still below \(-0.9\).

\(^6\)For each of these experiments we permit the stock of government debt to adjust to clear the government budget constraint; this approach keeps us from having to disentangle the changes in the tax function from the direct effect of the parameter changes.
be drawn uniformly from three values, \{1.5, 2.0, 2.5\}, and these probabilities are independent of \( \varepsilon \). The resulting correlation between capital and labor income is \(-0.93\), basically where it is for the benchmark economy, and it rises in the range of \( \sigma \) (but not quantitatively very much); the resulting correlation between wealth and income is \(-0.59\), still very far from the empirical value. Permitting \( \sigma \) to be correlated with \( \varepsilon \) has some minor effect. If \( \sigma \) and \( \varepsilon \) are perfectly negatively correlated (there are 100 types and each has a pair \((\varepsilon, \sigma)\), with high \( \varepsilon \) types also having low \( \sigma \)), we can raise the correlation between capital and labor income to \(-0.75\) and the correlation between income and wealth is \(-0.56\). It does not appear that heterogeneity in labor supply elasticities is going to rescue the complete market economies, even if one is willing to permit it to be correlated with productivity in an extreme manner.

5. Conclusion

In this section we comment briefly on the relevance of our results. In particular, we focus our comments here on the necessary implications that income and wealth and labor and capital income are negatively correlated under complete markets. Both of these observations are starkly rejected by the data. This result is a strike against complete market models, for it does not hold in a model based on Aiyagari (1994). Our results would be of particular concern for political economy models of redistribution, notably Krusell and Ríos-Rull (1999) and Azzimonti, de Francisco, and Krusell (2005). In the data, high income individuals tend to be wealthy as well, and they earn a higher percentage of their income from assets than individuals with lower incomes do.\(^7\) That is, the high income individuals would tend to prefer labor income taxes to capital income taxes. But our results imply that the high income individuals would prefer capital taxes because they are asset-constrained. As shown in Domeij and Heathcote (2004), in an incomplete market economy the lower income prefer capital taxes, as one would expect they do in the data.

Reconciling a complete markets model with the data must introduce an additional individual-specific quantity into the steady state Euler equations; in the steady state, many forms of preference heterogeneity would vanish. In order for either the correlation between income and wealth or the

\(^7\)Budría Rodríguez et al. (2002) note that the fraction of income coming from labor and transfers varies over the wealth distribution from 98.2 percent in the lowest quintile to 60.0 in the highest and from 98.9 in the lowest 1 percent to 34.5 in the highest.
correlation between capital income and labor income to be positive in the steady state,

\[ \beta \left[ 1 - \frac{\partial \tau (y_i)}{\partial y_i} \right] (r - \delta) + 1 \leq 1 \]

must hold with equality for \( y_i \) greater than the minimum. All else being equal, increasing income drives the lefthand side away from unity because of the strong monotonicity of the marginal tax function. To make the LHS equal to 1 and match both correlations, some form of heterogeneity must be introduced which appears outside of income in the Euler equation but which is correlated with income; the only candidates are \( \beta \), \( r - \delta \), and the '1' term.

Having heterogenous \( r - \delta \) requires incompleteness in the market for assets. For example, if capital were firm-specific, so that it could not be traded in an economy-wide market, then returns across types need not be equated. But if there then also existed a bond market open to all agents, the return on the bond would be the same for all types, so that \( r - \delta \) must also be equalized. Essentially, unless there exists a subset of types who do not interact through asset markets with any type outside the subset, returns cannot be heterogeneous, and this segmentation amounts to market incompleteness. If production is constant returns to scale, an economy-wide labor market would also tie returns together, as \( r \) can be written as a function of \( w \). The '1' term represents the price of old capital; for this to be heterogeneous would again require market incompleteness.

What if there were different progressive tax functions for capital and labor, so that the budget constraint was

\[ c_{it} + a_{it+1} \leq w_i h_{it} \varepsilon_i + (1 + r - \delta) a_{it} - \tau_t (w_i h_{it} \varepsilon_i) - \tau_k ((r - \delta) a_{it})? \]

The Euler equation would then be

\[ \beta \left[ 1 - \frac{\partial \tau ((r - \delta) a_{it})}{\partial ((r - \delta) a_{it})} \right] (r - \delta) + 1 \leq 1; \]

if \( \tau' (x) \) is monotone, then the wealth distribution is uniform and therefore capital income must have zero covariance with labor income. This argument extends to an arbitrary number of assets facing different progressive tax functions.

Of the three alternatives, heterogeneity in \( \beta \) is therefore the only one left standing. In order to get the Euler equation to hold with equality for more than one level of income, \( \beta_i \) must be positively correlated with \( y_i \). Given the positive correlation of \( y_i \) and \( a_i \) in the data, we have the natural result
that more patient agents need to accumulate more wealth. To also get the correlation between labor and capital income to be positive, $\beta_i$ must be positively correlated with $h_i \varepsilon_i$; as long as the wealth effect on leisure is not too strong, getting this correlation positive implies that $\beta_i$ and $\varepsilon_i$ are positively correlated. We can reverse engineer a distribution of $(\beta_i, \varepsilon_i, \sigma_i)$ given an observed distribution $(a_i, y_i, h_i)$ through the following steps. First, guess $r$ and compute $w = (1 - \alpha) \left( \frac{r}{\alpha} \right)^{\alpha - 1}$.

The market clearing $r$, which only depends on the distributions through the aggregates, solves the equation

$$r = \alpha K^{\alpha-1} N(r)^{1-\alpha}$$

where

$$N(r) = \frac{Y}{w(r)} - \frac{(r - \delta) K}{w(r)}$$

for aggregates $(K, Y)$. The values for $(\beta_i, \varepsilon_i, \sigma_i)$ can then be obtained from the Euler equations, the budget constraints, and the static labor-leisure conditions, respectively.

6. Appendix A: Theoretical Proofs

6.1. Proof of Proposition 1:

We prove Proposition 1 in a series of lemmata. Let $B \equiv \{i \in I : a_i > a_b\}$. Clearly, $B \subset I$.

Lemma 1. $B$ is non-empty.

Proof. Assume not. $B = \emptyset \Rightarrow a_i = a_b, \forall i \in I$. At the efficiency conditions for the household we must have

$$\left(1 - \frac{\partial\tau(y_i)}{\partial y_i}\right) (r - \delta) \leq \frac{1}{\beta} - 1.$$

The RHS is a finite constant. The LHS converges to $\infty$ under the assumptions on $F$ and $\tau$.  

Lemma 2. $\forall i \in B$, $\frac{\partial\tau(y_i)}{\partial y_i} = \hat{\tau}$ and $\hat{\tau} \leq \frac{\partial\tau(y_i)}{\partial y_i}, \forall i \notin B$.

Proof. To prove the first part, assume not. Then $\exists j, k \in B$ such that $a_j > a_b, a_k > a_b$, and $\frac{\partial\tau(y_j)}{\partial y_j} \neq \frac{\partial\tau(y_k)}{\partial y_k}$. Since $j, k \in B$ it follows that

$$\beta \left[\left(1 - \frac{\partial\tau(y_j)}{\partial y_j}\right) r + 1 - \delta\right] = \beta \left[\left(1 - \frac{\partial\tau(y_k)}{\partial y_k}\right) r + 1 - \delta\right]$$

We were unable to find solutions for $(\beta_i, \varepsilon_i, h_i)$ for homogeneous $\sigma$, because the problem is numerically ill-conditioned.
which requires
\[
\frac{\partial \tau(y_j)}{\partial y_j} = \frac{\partial \tau(y_k)}{\partial y_k} = \hat{\tau}.
\]

For the second part, again assume not. Then \(\exists i \in I/B\) such that \(\frac{\partial \tau(y_i)}{\partial y_i} < \hat{\tau}\). This leads to the following condition:
\[
\beta \left[ (1 - \frac{\partial \tau(y_i)}{\partial y_i}) r + 1 - \delta \right] \leq 1 = \beta [(1 - \hat{\tau}) r + 1 - \delta].
\]

or
\[
\left( 1 - \frac{\partial \tau(y_i)}{\partial y_i} \right) \leq (1 - \hat{\tau}).
\]

This requires
\[
\frac{\partial \tau(y_i)}{\partial y_i} \geq \hat{\tau}.
\]

\[\blacksquare\]

**Lemma 3.** \#B may exceed 1, so that more than one type of household may not be borrowing-constrained.

**Proof.** Consider \(j, k \in I\) and without loss of generality let \(j \in B\). Further, let it be such that \(h_j = h_k = 0\). From the budget constraint we then have \(y_j = (r - \delta)a_j\) and \(y_k = (r - \delta)a_k\). If \(a_k < 0\), then \(y_k < 0\) and \(c_k < 0\). Therefore \(a_k > 0 \geq a^b\). From above we know that
\[
\frac{\partial \tau(y_j)}{\partial y_j} = \frac{\partial \tau(y_k)}{\partial y_k}
\]
y \(j = y_k\) and therefore \(a_j = a_k\). It is not necessary that \(h_j = h_k = 0\). Assume instead that \(h_j > 0\) and \(h_k > 0\). If
\[
\frac{MUL_j}{MUL_k} = \frac{MUC_j \varepsilon_j}{MUC_k \varepsilon_k}
\]
then \(a_j = a_k\). Since \(j\) is not borrowing constrained, \(\frac{\partial \tau(y_j)}{\partial y_j} = \hat{\tau}\). In order for \(a_j = a_k\), \(\frac{\partial \tau(y_k)}{\partial y_k} = \hat{\tau}\) also. The static conditions for leisure are
\[
MUC_j w(1 - \hat{\tau}) \varepsilon_j = MUL_j
\]
\[
MUC_k w \left( 1 - \frac{\partial \tau(y_k)}{\partial y_k} \right) \varepsilon_k = MUL_j.
\]
If (*) holds then
\[
\frac{1 - \frac{\partial \tau(y_k)}{\partial y_k}}{1 - \hat{\tau}} = 1,
\]
and therefore \(\frac{\partial \tau(y_k)}{\partial y_k} = \hat{\tau}\). Similar logic shows that in the case \(h_j = 0\) and \(h_k > 0\), a sufficient condition for \(a_j = a_k\) is
\[
MUL_j \geq \frac{MUC_j \varepsilon_j}{MUC_k \varepsilon_k} MUL_k.
\]
This condition then implies
\[
\frac{(1 - \hat{\tau})}{1 - \frac{\partial \tau(y_k)}{\partial y_k}} \leq 1.
\]
or
\[
(1 - \hat{\tau}) \leq \left(1 - \frac{\partial \tau(y_k)}{\partial y_k}\right).
\]
By definition \(\hat{\tau}\) is the lowest marginal tax rate faced by any household and so \(\hat{\tau} = \frac{\partial \tau(y_k)}{\partial y_k}\).

**6.2. Proof of Proposition 2**

We prove this proposition in a series of lemmata.

**Lemma 4.** Let \(I_u \equiv \{i \in I : h_i > 0\} \cup \{i \in I : a_i > a^b\}\). \(\forall i, j \in I_u, \) if \(\varepsilon_i > \varepsilon_j\), then \(h_i > h_j\) and \(a_i < a_j\).

**Proof.** Given \(i, j\), define the notation \(\Delta x = x_i - x_j\). \(i, j \in I_u\) implies the following two conditions:
\[
wh_i \varepsilon_i + (r - \delta) a_i = wh_j \varepsilon_j + (r - \delta) a_j,
\]
\[
-MUC_k w \left(1 - \frac{\partial \tau}{\partial y_k}\right) \varepsilon_k + MUL_k = 0 \quad k = \{i, j\}
\]

Note that the \(a_i, a_j > a^b\) necessitates that \(y_i = y_j\) by the strong monotonicity of the marginal tax function. In the steady state, \(c_i = y_i - \tau(y_i)\) for all \(i\). Thus, \(c_i = c_j\). Taking these into account and equating the second condition across \(i\) and \(j\), yields the following:
\[
\frac{MUL_i}{MUL_j} = \frac{\varepsilon_i}{\varepsilon_j}
\]

Given that \(\varepsilon_i > \varepsilon_j\), the \(MUL_i > MUL_j\). Thus, by strict concavity of \(U\) in \(l, l_i < l_j\) or \(h_i > h_j\).
Equating and rearranging the first condition, yields

\[ w(h_i \varepsilon_i - h_j \varepsilon_j) = -(r - \delta) \Delta a \] (*)

\[ \left( \frac{h_i \varepsilon_i}{h_j \varepsilon_j} - 1 \right) h_j \varepsilon_j = -\frac{(r - \delta)}{w} \Delta a \]

From above

\[ \left( \frac{h_i \varepsilon_i}{h_j \varepsilon_j} - 1 \right) > 0 \]

Thus the left-hand side of * must be positive. \( \tau_m^* < 1 \) and \( \beta < 1 \) is a sufficient condition to insure that \( (r - \delta) > 0 \), and thus \( \Delta a < 0 \) (i.e. \( a_i < a_j \)).

Lemma 5. Consider \( i, j \) such that \( h_i = 0 \) and \( h_j = 0 \). Then \( a_i = a_j \).

Proof. First, given that the \( \lim_{c \to 0^+} U_c = -\infty \), any types that do not have labor income must have capital income. Therefore \( a_i \) and \( a_j \) are strictly positive. By Proposition 1 \( y_i = y_j \). Then it is immediate that

\[ w(h_i \varepsilon_i - h_j \varepsilon_j) = -(r - \delta) \Delta a. \]

Therefore, \( \Delta a = 0 \).

Lemma 6. Consider \( i \in I_n \) and \( j \in I_u \). Then \( a_j < a_i \) and \( \varepsilon_i > \varepsilon_j \).

Proof. Theorem of the Maximum implies \( a(a_i, \varepsilon_i) \) is continuous in \( \varepsilon_i \) which rules out any other case.

A similar argument establishes the following lemma:

Lemma 7. Consider \( i \in I_{bc} \) and \( j \in I_u \). \( a_j > a_i \) and \( \varepsilon_i > \varepsilon_j \).

These results collectively prove that the sets must be ordered as stated in Proposition 2.

7. Appendix B: Computation

Computing the steady state allocation is straightforward, once we have converted the system of Kuhn-Tucker inequalities into a system of equations. A steady state allocation is a solution to the
system of inequalities

\[
\beta \left[ \left(1 - \frac{\partial \tau(y_i)}{\partial y_i} \right) \left( \alpha \left( \sum_i a_i \psi_i - D \right)^{\alpha - 1} \left( \sum_i \varepsilon_i h_i \psi_i \right)^{1-\alpha} - \delta \right) + 1 \right] - 1 + \lambda_i = 0
\]

\[
\frac{1}{c_i} \left(1 - \frac{\partial \tau(y_i)}{\partial y_i} \right) (1-\alpha) \left( \sum_i a_i \psi_i - D \right)^{\alpha} \left( \sum_i \varepsilon_i h_i \psi_i \right)^{-\alpha} \varepsilon_i - B (1-h_i)^{-\sigma} + \phi_i = 0
\]

\[
y_i - \tau(y_i) - c_i = 0
\]

\[
y_i - \left( \alpha \left( \sum_i a_i \psi_i - D \right)^{\alpha - 1} \left( \sum_i \varepsilon_i h_i \psi_i \right)^{1-\alpha} - \delta \right) a_i - (1-\alpha) \left( \sum_i a_i \psi_i - D \right)^{\alpha} \left( \sum_i \varepsilon_i h_i \psi_i \right)^{-\alpha} \varepsilon_i h_i = 0
\]

\[
\sum_i \tau(y_i) \psi_i - G - \left( \alpha \left( \sum_i a_i \psi_i - D \right)^{\alpha - 1} \left( \sum_i \varepsilon_i h_i \psi_i \right)^{1-\alpha} - \delta \right) D = 0
\]

\[
\lambda_i \geq 0
\]

\[
\phi_i \geq 0
\]

\[
a_i \geq 0
\]

\[
h_i \geq 0
\]

\[
\lambda_i a_i = 0
\]

\[
\phi_i h_i = 0
\]

where \( \lambda \) and \( \phi \) are the multipliers on the asset limit and the nonnegativity of hours constraints; the market for goods holds by Walras’ law. Following Garcia and Zangwill (1981) we introduce the functions

\[
\lambda_i^+ = (\max \{0, \lambda_i\})^k
\]

\[
\lambda_i^- = (\max \{0, -\lambda_i\})^k
\]

\[
\phi_i^+ = (\max \{0, \phi_i\})^k
\]

\[
\phi_i^- = (\max \{0, -\phi_i\})^k
\]

for some integer \( k > 1 \). Note that

\[
\lambda_i^+ \lambda_i^- = 0
\]

\[
\phi_i^+ \phi_i^- = 0.
\]
The system of inequalities can then be written as

\[\beta \left[ \left( 1 - \frac{\partial \tau(y_i)}{\partial y_i} \right) \left( \alpha \left( \sum_i a_i \psi_i - D \right)^{\alpha-1} \left( \sum_i \varepsilon_i h_i \psi_i \right)^{1-\alpha} - \delta \right) \right] + 1 - 1 + \lambda_i^+ = 0\]

\[\frac{1}{c_i} \left( 1 - \frac{\partial \tau(y_i)}{\partial y_i} \right) (1 - \alpha) \left( \sum_i a_i \psi_i - D \right)^{\alpha} \left( \sum_i \varepsilon_i h_i \psi_i \right)^{-\alpha} \varepsilon_i - B (1 - h_i)^{-\sigma} + \phi_i^+ = 0\]

\[y_i - \tau(y_i) - c_i = 0\]

\[y_i - \left( \alpha \left( \sum_i a_i \psi_i - D \right)^{\alpha-1} \left( \sum_i \varepsilon_i h_i \psi_i \right)^{1-\alpha} - \delta \right) a_i - (1 - \alpha) \left( \sum_i a_i \psi_i - D \right)^{\alpha} \left( \sum_i \varepsilon_i h_i \psi_i \right)^{-\alpha} \varepsilon_i h_i = 0\]

\[\sum_i \tau(y_i) \psi_i - D - \left( \alpha \left( \sum_i a_i \psi_i - D \right)^{\alpha-1} \left( \sum_i \varepsilon_i h_i \psi_i \right)^{1-\alpha} - \delta \right) D = 0\]

\[\lambda^+_i - a_i + \varphi^+_i = 0\]

\[\phi^+_i - h_i = 0.\]

We then use a nonlinear equation solver to obtain the solution \(\{c_i, h_i, a_i, y_i, \lambda_i, \phi_i\}_{i=1}^{l}, D\) to this system.
References


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<th>$\beta$</th>
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<th>$D$</th>
<th>$\nu_2$</th>
<th>$\sigma$</th>
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</tbody>
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Figure 1

Distribution of Capital Income

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---

\( \epsilon \)

\( \epsilon_w \)

\( \epsilon_c \)

asset income

0 0.5 1 1.5 2 2.5 3

-1 -0.5 0 0.5 1 1.5 2 2.5 3
Figure 2

Hours Worked by Type

$h$
Figure 3

Effective Marginal Tax

\[ \epsilon \]

\[ \tau \]
Figure 4
Distribution of Labor Income

- $\varepsilon_w$
- $\varepsilon_c$