A UNIVERSAL OPERATOR

JAMES ROVNYAK

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Linear operators arise in many problems in analysis and its applications. The modern approach is a century old and due to David Hilbert. Hilbert's great inspiration was that there is a powerful analogy between the linear operators of analysis and linear algebra. However, one must generalize the finite-dimensional theory to an infinite-dimensional setting. Classical results of matrix theory remain a very good starting point. In this lecture, I will try to show in one example how this process works. I assume familiarity with linear algebra but will review the basic ideas of complex vector spaces so as to keep the prerequisites minimal.

1. Inner product spaces

A vector space is a purely algebraic object. If we use the field $\mathbf{C}$ of complex numbers, then a **vector space** $\mathcal{V}$ is a set of objects $\mathbf{x}$, $\mathbf{y}, \dots$ called vectors satisfying these axioms:

(A) For every pair of vectors $\mathbf{x}$ and $\mathbf{y}$ there there is a vector $\mathbf{x}+\mathbf{y}$ called the sum of $\mathbf{x}$ and $\mathbf{y}$, such that:

- $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$;
- $(\mathbf{x}+\mathbf{y})+\mathbf{x} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$;
- there is a unique vector 0 such that $\mathbf{x}+0 = \mathbf{x}$ for every $\mathbf{x}$;
- for every vector $\mathbf{x}$ there is a unique $-\mathbf{x}$ such that $\mathbf{x}+(-\mathbf{x}) = 0$.

(B) For every vector $\mathbf{x}$ and complex number α there is a vector $\alpha\,\mathbf{x}$ such that:

- $\alpha(\beta\,\mathbf{x}) = (\alpha\beta)\,\mathbf{x}$;
- $1\,\mathbf{x} = \mathbf{x}$ for every vector $\mathbf{x}$.

(C) The distributive laws hold:

- $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\,\mathbf{x}+\beta\,\mathbf{y}$;
- $(\alpha+\beta)\,\mathbf{x} = \alpha\,\mathbf{x}+\beta\,\mathbf{y}$.

Example. Let $\mathbf{C}^n$ be the set of n-dimensional column vectors of complex numbers. We add vectors and multiply by scalars coordinatewise: if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

then

$$\mathbf{x}+\mathbf{y}=\begin{bmatrix}x_1+y_1\\ \vdots\\ x_n+y_n\end{bmatrix},\qquad \alpha\,\mathbf{x}=\begin{bmatrix}\alpha x_1\\ \vdots\\ \alpha x_n\end{bmatrix}.$$

The vector space axioms describe the algebraic properties of vector spaces. In analysis we also need notions of length and distance that allow us to speak of limits and convergence.

Motivation: The dot product of vectors $\mathbf{a}=(a_1,a_2)$ and $\mathbf{b}=(b_1,b_2)$ in R^2 is defined by

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2\,.$$

Using the dot product we can define notions of length, distance, and even orthogonality. The length of $\mathbf{a}=(a_1,a_2)$ is

$$\|\mathbf{a}\|^2=\mathbf{a}\cdot\mathbf{a}=|a_1|^2+|a_2|^2.$$

The distance between $\mathbf{a}$ and $\mathbf{b}$ is

$$\operatorname{dist}(\mathbf{a},\mathbf{b})=\|\mathbf{a}-\mathbf{b}\|.$$

From calculus, we know that the angle θ between $\mathbf{a}$ and $\mathbf{b}$ satisfies

$$\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta,$$

and therefore $\mathbf{a}\perp\mathbf{b}$ if and only if $\mathbf{a}\cdot\mathbf{b}=0$.

We extract the idea of the dot product in axioms and adapt it to complex vector spaces.

Definition 1.1. *We call a complex vector space $\mathcal{V}$ an* **inner product space** *if for every pair of vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *there is a complex number* $\langle\,\mathbf{x},\,\mathbf{y}\rangle$ *called the inner product of* $\mathbf{x}$ *and* $\mathbf{y}$ *such that:*

(i) $\langle\alpha\,\mathbf{x}+\beta\,\mathbf{y},\,\mathbf{z}\rangle=\alpha\,\langle\,\mathbf{x},\,\mathbf{z}\rangle+\beta\,\langle\,\mathbf{y},\,\mathbf{z}\rangle$;
(ii) $\langle\,\mathbf{x},\,\mathbf{y}\rangle=\overline{\langle\,\mathbf{y},\,\mathbf{x}\rangle}$;
(iii) *for every* $\mathbf{x}$, $\langle\,\mathbf{x},\,\mathbf{x}\rangle\geq 0$ *with equality only for* $\mathbf{x}=0$.

The **length** *of a vector* $\mathbf{x}$ *is defined by*

$$\|\,\mathbf{x}\|^2=\langle\,\mathbf{x},\,\mathbf{x}\rangle\,.$$

The **distance** *between vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *is defined to be*

$$\operatorname{dist}(\,\mathbf{x},\,\mathbf{y})=\|\,\mathbf{x}-\,\mathbf{y}\|.$$

We call $\mathbf{x}$ *and* $\mathbf{y}$ **orthogonal** *if* $\langle\,\mathbf{x},\,\mathbf{y}\rangle=0$.

Pythagorean Theorem. *If* $\mathbf{x}\perp\,\mathbf{y}$*, then*

$$\|\,\mathbf{x}+\,\mathbf{y}\|^2=\|\,\mathbf{x}\|^2+\|\,\mathbf{y}\|^2.$$

Proof. In fact,

$$\begin{aligned}\|\,\mathbf{x}+\mathbf{y}\|^2 &= \langle\,\mathbf{x}+\mathbf{y},\,\mathbf{x}+\mathbf{y}\rangle\\ &= \langle\,\mathbf{x},\,\mathbf{x}\rangle+\langle\,\mathbf{x},\,\mathbf{y}\rangle+\langle\,\mathbf{y},\,\mathbf{x}\rangle+\langle\,\mathbf{y},\,\mathbf{y}\rangle\\ &= \langle\,\mathbf{x},\,\mathbf{x}\rangle+\langle\,\mathbf{y},\,\mathbf{y}\rangle\\ &= \|\,\mathbf{x}\|^2+\|\,\mathbf{y}\|^2.\end{aligned}$$

$\square$

Example. An inner product on $\mathbf{C}^n$ is defined by

$$\langle\,\mathbf{x},\,\mathbf{y}\rangle = x_1\bar{y}_1+\cdots+x_n\bar{y}_n,$$

where

$$\mathbf{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix},\qquad \mathbf{y}=\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}.$$

A vector space $\mathcal{V}$ is called finite-dimensional if it has a finite basis, that is, a set of vectors $\mathbf{b}_1,\ldots,\mathbf{b}_n$ such that every $\mathbf{x}$ in $\mathcal{V}$ can be written

$$\mathbf{x}=\alpha_1\mathbf{b}_1+\cdots+\alpha_n\mathbf{b}_n$$

In an inner product space, a basis $\mathbf{u}_1,\ldots,\,\mathbf{u}_n$ can be chosen with additional properties:

$$\begin{aligned}\mathbf{u}_j \perp \mathbf{u}_k,&\qquad j\neq k,\\ \|\,\mathbf{u}_j\|=1,&\qquad j=1,\ldots,n.\end{aligned}$$

A basis having such properties is called an **orthonormal basis**.

Example. A basis in $\mathbf{C}^n$ is given by

$$\mathbf{b}_1=\begin{bmatrix}1\\0\\0\\ \vdots\\0\end{bmatrix},\quad \mathbf{b}_2=\begin{bmatrix}1\\1\\0\\ \vdots\\0\end{bmatrix},\ldots,\mathbf{b}_n=\begin{bmatrix}1\\1\\1\\ \vdots\\1\end{bmatrix}.$$

An orthonormal basis in $\mathbf{C}^n$ is given by

$$\mathbf{u}_1=\begin{bmatrix}1\\0\\0\\ \vdots\\0\end{bmatrix},\quad \mathbf{u}_2=\begin{bmatrix}0\\1\\0\\ \vdots\\0\end{bmatrix},\ldots,\,\mathbf{u}_n=\begin{bmatrix}0\\0\\0\\ \vdots\\1\end{bmatrix}.$$

We can use bases to show that every finite-dimensional inner product space $\mathcal{V}$ is isomorphic with $\mathbf{C}^n$: there is a one-to-one mapping

$$U:\mathcal{V}\to\mathbf{C}^n$$

which preserves all structure:

$$U(\alpha\,\mathbf{x}+\beta\,\mathbf{y})=\alpha\,U\,\mathbf{x}+\beta\,U\,\mathbf{y},$$
$$\langle U\,\mathbf{x},U\,\mathbf{y}\rangle=\langle\,\mathbf{x},\,\mathbf{y}\rangle\,.$$

2. MATRICES AND OPERATORS

The equivalence of a finite-dimensional inner product space $\mathcal{V}$ and the concrete space $\mathbf{C}^n$ has many advantages in the study of matrices and linear transformations. It gives us two views of the same object.

Let us first consider $n\times n$ matrices,

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}.$$

Matrices have some of the characteristics of ordinary numbers. For example, we can add and multiply matrices. Every $n\times n$ matrix A induces a mapping of vectors

$$T\colon\mathbf{C}^n\to\mathbf{C}^n,$$

which is determined by writing $T\,\mathbf{x}=\;\mathbf{y}$ if

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix}.$$

Concrete examples of matrices are also easy to construct:

$$U_n=\begin{bmatrix} 0 & 0 & \cdots & 0 & 1\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0\end{bmatrix},\qquad S_n=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}.$$

Matrices act like numbers in a different way. We call a complex number λ an **eigenvalue** of A if there is a vector $\mathbf{x}\neq 0$ in $\mathbf{C}^n$ such that

$$A\,\mathbf{x}=\lambda\,\mathbf{x}.$$

Thus in the direction $\mathbf{x}$, A acts like the number λ.

A **linear transformation** T on a vector space $\mathcal{V}$ is a mapping

$$T\colon\mathcal{V}\to\mathcal{V}$$

such that

$$T(\alpha\,\mathbf{x}+\beta\,\mathbf{y})=\alpha T\,\mathbf{x}+\beta T\,\mathbf{y}.$$

If $\mathcal{V}$ is an inner product space, the term **operator** is often used to mean the same thing as a linear transformation.

We identify classes of operators depending on how they interact with the notions of length and distance.

An operator $T\colon \mathcal{V} \to \mathcal{V}$ on an inner product space is

- **isometric** if $\|T\,\mathbf{x}\| = \|\,\mathbf{x}\|$ for every x in $\mathcal{V}$,
- **contractive** if $\|T\,\mathbf{x}\| \leq \|\,\mathbf{x}\|$ for every x in $\mathcal{V}$,
- **bounded** if $\|T\,\mathbf{x}\| \leq M\|\,\mathbf{x}\|$ for every x in $\mathcal{V}$ and some constant M.

Examples in $\mathbf{C}^n$:

(1) The circulant matrix

$$U_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

induces an isometric operator on $\mathbf{C}^n$.

(2) The shift

$$S_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

induces a contraction operator on $\mathbf{C}^n$.

(2) Every matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

induces a bounded operator on $\mathbf{C}^n$.

3. Invariant subspaces

A **subspace** of an inner product space $\mathcal{V}$ is a nonempty subset $\mathcal{M}$ such that

$$\alpha\,\mathbf{x} + \beta\,\mathbf{y} \in \mathcal{M}$$

for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{M}$ and all numbers α and β. The subspace $\mathcal{M}$ is itself an inner product space, because the axioms for an inner product hold for $\mathcal{M}$ as particular cases of the axioms for $\mathcal{V}$.

A subspace $\mathcal{M}$ of an inner product space $\mathcal{V}$ is said to be **invariant** under an operator $T\colon \mathcal{V} \to \mathcal{V}$ if

$$T\mathcal{M} \subseteq \mathcal{M}.$$

In linear algebra, we study linear transformations by means of their invariant subspaces, whether the results explicitly mention invariant subspaces or not.

Example. If $T\,\mathbf{x} = \lambda\,\mathbf{x},\ \mathbf{x} \neq 0$, then

$$\mathcal{M} = \{\alpha\,\mathbf{x} \colon \alpha \in \mathbf{C}\}$$

is a 1-dimensional invariant subspace for T.

This simple observation and an induction argument shows that a linear transformation on a finite-dimensional space has invariant subspaces of all possible dimensions.

Theorem 3.1. *Every operator T on an n-dimensional space $\mathcal{V}$ has a totally ordered chain*

$$\{0\} = \mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \cdots \subsetneq \mathcal{M}_n = \mathcal{V}$$

of invariant subspaces.

Exercises. (1) *Find a linear transformation which has a unique totally ordered chain of invariant subspaces, one for each dimension.*

(2) *Find a linear transformation which has more than one totally ordered chain of invariant subspaces, one for each dimension.*

4. Models for operators

If $\mathcal{M} \subseteq \mathcal{V}$ is an invariant subspace for $T \colon \mathcal{V} \to \mathcal{V}$, then

$$T_{\mathcal{M}} = T|\mathcal{M}$$

is a new linear transformation:

$$T_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{M}.$$

Think of T as a parent, $T_{\mathcal{M}}$ a child. The parent has many children, one for each invariant subspace. How many children can one parent have?

Definition 4.1 (G. C. Rota [5])**.** *Let $\mathcal{M} \subseteq \mathcal{V}$ be an invariant subspace for $T \colon \mathcal{V} \to \mathcal{V}$. Let*

$$R \colon \mathcal{H} \to \mathcal{H}$$

be a second operator. We call $T_{\mathcal{M}} = T|\mathcal{M}$ a **model** *for R if there is a bounded operator $V \colon \mathcal{H} \to \mathcal{M}$ such that V is one-to-one and onto and*

$$R = V^{-1} T_{\mathcal{M}} V.$$

Rota's Question: *Is there a universal operator? Can a single operator T be used to model all operators?*

This is impossible in finite-dimensional spaces, because the eigenvalues and eigenvectors of $T_{\mathcal{M}}$ are eigenvalues and eigenvectors for T and therefore of very limited form.

Rota showed that in infinite-dimensional spaces it is possible to construct a universal operator.

5. Hilbert spaces

We add an axiom that is important in infinite-dimensional spaces.

Definition 5.1. *By a* **Hilbert space** *we mean an inner product space* $\mathcal{H}$ *which satisfies the* **completeness axiom:**

- *Every Cauchy sequence in* $\mathcal{H}$ *is convergent.*

A sequence $\{\,\mathbf{x}_n\}_1^\infty$ in $\mathfrak{H}$ is **Cauchy** if for every $\varepsilon > 0$ there is an integer N such that $\|\,\mathbf{x}_m - \mathbf{x}_n\| < \varepsilon$ whenever $m, n > N$. The completeness axiom requires that for every Cauchy sequence, there is a vector $\mathbf{x}$ in $\mathcal{H}$ such that $\|\,\mathbf{x} - \mathbf{x}_n\| \to 0$ as $n \to \infty$.

Example. $\mathbf{C}^n$ is a Hilbert space. For any Cauchy sequence of vectors

$$\mathbf{x}_1 = \begin{bmatrix} x_{11} \\ \vdots \\ x_{1n} \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} x_{21} \\ \vdots \\ x_{2n} \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} x_{31} \\ \vdots \\ x_{3n} \end{bmatrix}, \ \dots$$

the components are Cauchy sequences of complex numbers. Since every Cauchy sequence of complex numbers converges,

$$\begin{gathered} x_{11}, x_{21}, x_{31} \to \xi_1, \\ \cdots \\ x_{1n}, x_{2n}, x_{3n} \to \xi_n, \end{gathered}$$

for some numbers $\xi_1, \dots, \xi_n$, and hence

$$\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3, \dots \to \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}.$$

Example. Let ℓ^2 be the set of all sequences of complex numbers

$$\mathbf{x} = (x_0, x_1, x_2, \dots)$$

which satisfy

$$|x_0|^2 + |x_1|^2 + |x_2|^2 + \cdots < \infty.$$

We think of ℓ^2 as a space of infinite-dimensional vectors of finite length. Define

$$\begin{aligned} \alpha\,\mathbf{x} + \beta\,\mathbf{y} &= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \dots), \\ \langle\,\mathbf{x},\, \mathbf{y}\rangle &= x_0\bar{y}_0 + x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots \end{aligned}$$

for all

$$\begin{aligned} \mathbf{x} &= (x_0, x_1, x_2, \dots), \\ \mathbf{y} &= (y_0, y_1, y_2, \dots). \end{aligned}$$

We verify the axioms for a Hilbert space in much the same way as for $\mathbf{C}^n$, with additional attention to convergence of series.

Since a Hilbert space is an inner product space, previous definitions apply. Let us briefly review them. An **operator** is a mapping

$$T\colon \mathcal{H} \to \mathcal{H}$$

such that $T(\alpha\,\mathbf{x} + \beta\,\mathbf{y}) = \alpha T\,\mathbf{x} + \beta T\,\mathbf{y}$. An operator $T\colon \mathcal{H} \to \mathcal{H}$ is

- **isometric** if $\|T\,\mathbf{x}\| = \|\,\mathbf{x}\|$ for every x in $\mathcal{V}$,
- **contractive** if $\|T\,\mathbf{x}\| \leq \|\,\mathbf{x}\|$ for every x in $\mathcal{V}$,
- **bounded** if $\|T\,\mathbf{x}\| \leq M\|\,\mathbf{x}\|$ for every x in $\mathcal{V}$ and some constant M.

Some differences arise in infinite-dimensional spaces. Not all operators are bounded.

Theorem 5.2. *The class of all bounded operators on a Hilbert space $\mathcal{H}$ coincides with the set of continuous operators, that is, operators T on $\mathcal{H}$ such that*

$$\mathbf{x}_n \to \mathbf{x} \quad \Longrightarrow \quad \mathbf{x}_n \to T\,\mathbf{x}.$$

Another difference is that not all subspaces of a Hilbert space satisfy the completeness axiom.

Theorem 5.3. *The class of subspaces of a Hilbert space $\mathcal{H}$ which satisfy the completeness axiom coincides with the class of closed subspaces, that is, subspaces $\mathcal{M}$ of $\mathcal{H}$ such that*

$$\mathbf{x}_n \in \mathcal{M} \;\; \textit{and} \;\; \mathbf{x}_n \to \mathbf{x} \quad \Longrightarrow \quad \mathbf{x} \in \mathcal{M}.$$

Here I shall speak only of continuous operators and closed subspaces.

Example. Nothing is lost from the finite-dimensional theory. In $\mathbf{C}^n$,

$$U_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad S_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

define isometric and contraction operators, respectively.

Example. On ℓ^2, the operator

$$U\colon (x_0, x_1, x_2, \dots) \to (0, x_0, x_1, \dots)$$

is isometric, and

$$S\colon (x_0, x_1, x_2, \dots) \to (x_1, x_0, x_2, \dots)$$

is a contraction.

6. THE MOTHER OF ALL OPERATORS

Recall that operators on a finite-dimensional Hilbert space have many invariant subspaces. In fact, any operator T on a finite-dimensional space has a totally ordered chain of invariant subspaces,

$$\{0\} = \mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \cdots \subsetneq \mathcal{M}_n = \mathcal{V},$$

and all of these subspaces are closed by finite dimensionality.

Open Problem. *Given any bounded operator T on an infinite-dimensional Hilbert space $\mathcal{H}$, does there exist a closed subspace $\mathcal{M}$ other than the zero subspace and full space such that*

$$T\mathcal{M} \subseteq \mathcal{M}?$$

Rota was thinking about this problem when he posed his question of the existence of a universal operator. Let us re-pose the question about a universal operator in the context of Hilbert space operators.

Definition 6.1. *Let T be a continuous operator on a Hilbert space $\mathcal{K}$, and let $T_{\mathcal{M}} = T|\mathcal{M}$ be its restriction to a closed invariant subspace $\mathcal{M}$. Let*

$$R\colon \mathcal{H} \to \mathcal{H}$$

be a second operator on a Hilbert space $\mathcal{H}$. We call $T_{\mathcal{M}}$ a **model** *for R if there is a bounded operator $V : \mathcal{H} \to \mathcal{M}$ such that such that V is one-to-one and onto and*

$$R = V^{-1} T_{\mathcal{M}} V.$$

If T is a bounded operator on a Hilbert space $\mathcal{H}$, and if $\rho \neq 0$ is any scalar, then the operators T and ρT have exactly the same invariant subspaces. Thus from the point of view of the invariant subspace problem, nothing is lost by replacing T by a nonzero constant multiple.

The "mother of all operators" is a generalization of the shift operator S on ℓ^2. Let $\mathcal{H}$ be a Hilbert space, and define $\mathcal{K}$ to be the Hilbert space of all sequences

$$\mathbf{x} = (\, \mathbf{x}_0, \, \mathbf{x}_1, \, \mathbf{x}_2, \dots)$$

where $\mathbf{x}_0, \, \mathbf{x}_1, \, \mathbf{x}_2, \dots$ are vectors in $\mathcal{H}$ such that

$$\|\, \mathbf{x}_0\|^2 + \|\, \mathbf{x}_1\|^2 + \|\, \mathbf{x}_2\|^2 + \cdots < \infty.$$

Define

$$\alpha\, \mathbf{x} + \beta\, \mathbf{y} = (\alpha\, \mathbf{x}_1 + \beta\, \mathbf{y}_1, \alpha\, \mathbf{x}_2 + \beta\, \mathbf{y}_2, \alpha\, \mathbf{x}_3 + \beta\, \mathbf{y}_3, \dots),$$
$$\langle\, \mathbf{x}, \, \mathbf{y} \rangle = \langle\, \mathbf{x}_0, \, \mathbf{y}_0 \rangle + \langle\, \mathbf{x}_1, \, \mathbf{y}_1 \rangle + \langle\, \mathbf{x}, \, \mathbf{y} \rangle + \cdots .$$

Then $\mathcal{K}$ is a Hilbert space, and the operator

$$S\colon (x_0, x_1, x_2, \dots) \to (x_1, x_0, x_2, \dots)$$

is a contraction.

Rota Model. *The operator S is universal in the sense that, for every bounded operator T on $\mathcal{H}$ there is a nonzero constant ρ and a closed invariant subspace $\mathcal{M}$ of S such that $S_{\mathcal{M}} = S|\mathcal{M}$ is a model for ρT.*

Idea of the proof. Let T be any bounded operator on $\mathcal{H}$, and consider any nonzero number ρ. Set $T_\rho = \rho T$.

The main step is to show that if ρ is sufficiently small and $\mathbf{x}_0$ is any vector in $\mathcal{H}$, then the sequence

$$\mathbf{x} = (\,\mathbf{x}_0, T_\rho\,\mathbf{x}_0, T_\rho^2\,\mathbf{x}_0, \dots\,),$$

belongs to $\mathcal{K}$, that is,

$$\|\,\mathbf{x}_0\|^2 + \|\,\mathbf{x}_1\|^2 + \|\,\mathbf{x}_2\|^2 + \cdots < \infty,$$

and the set of all such sequences is a closed subspace.

Then we can define a mapping

$$V \colon \mathcal{H} \to \mathcal{M}$$

by

$$V \colon\ \mathbf{x}_0 \to (\,\mathbf{x}_0, T_\rho\,\mathbf{x}_0, T_\rho^2\,\mathbf{x}_0, \dots\,)$$

for every $\mathbf{x}_0$ in $\mathcal{H}$.

All of the properties are now easily checked. $\square$

Actually, more is true. It is possible to choose V itself to be an isomorphism, that is, in addition

$$\langle V\,\mathbf{x}, V\,\mathbf{y}\rangle = \langle\,\mathbf{x},\,\mathbf{y}\rangle$$

for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{H}$.

Example. In ℓ^2, let $\mathcal{M}$ be the span of the vectors

$$\begin{aligned}\mathbf{b}_1 &= (1, \alpha_1, \alpha_1^2, \dots\,)\,,\\ &\cdots\\ \mathbf{b}_n &= (1, \alpha_n, \alpha_n^2, \dots\,)\,,\end{aligned}$$

where $\alpha_1, \dots, \alpha_n$ are fixed complex numbers of absolute value less than one. Then $\mathcal{M}$ is invariant under S, and the eigenvalues of $S|\mathcal{M}$ are precisely the numbers $\alpha_1, \dots, \alpha_n$. Thus already in ℓ^2, S is capable of serving as a model for operators with very general eigenvalues.

7. EPILOGUE

The end of one story is often the beginning of another. The study of the shift operator is the beginning of many stories.

The shift operator is very closely connected with complex analysis. The Hardy space H^2 is defined as the set of power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

such that

$$|a_0|^2 + |a_1|^2 + |a_2|^2 + \cdots < \infty.$$

The inner product of two series $f(z) = \sum_0^\infty a_n z^n$ and $g(z) = \sum_0^\infty b_n z^n$ is defined by

$$\langle f, g \rangle = a_0 \bar{b}_0 + a_1 \bar{b}_1 + a_2 \bar{b}_2 + \cdots \ .$$

Thus H^2 is just another way of looking at ℓ^2, and so H^2 is a Hilbert space. It turns out that there is a very rich interplay between the study of the shift operator and analytic function theory. The Beurling-Lax theorem uses analytic function theory to describe all of the invariant subspaces of the shift operator in both the scalar and vector versions [1, 3].

The approach to the invariant subspace problem through the use of models has failed, but the efforts in this direction and related problems for shift operators have opened many new and unforseen directions. If I may get personal, I will mention my book with Marvin Rosenblum [4]. In particular, the book includes a statement and proof of the strong version of the universal model, but it also shows connections with quite different areas.

The idea of a model itself has survived in operator theory and indeed has many forms. It is beyond the scope of a discussion here. I tried to identify a number of these models in commentary that I recently wrote for the publication of a collection of papers by Gian-Carlo Rota [2].

As to the invariant subspace problem, it is open and a prize that somebody some day will claim.

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