FACTORIZATION OF ANALYTIC FUNCTIONS AND OPERATOR INEQUALITIES

ROBERT B. LEECH[1]

An isometry S on Hilbert space such that $S^{*n} \to 0$ strongly as $n \to \infty$ will be called a **shift operator**. We fix a Hilbert space $\mathcal{H}$ and a shift operator S on $\mathcal{H}$. We write $\mathcal{C} = \ker S^*$ and we let P_n denote the projection mapping $\mathcal{H}$ onto $S^n\mathcal{C}$ for each nonnegative integer n. A contraction T on $\mathcal{H}$ will be called **S-analytic** if $TS = ST$.

Theorem. *Let A and B be S-analytic contractions on $\mathcal{H}$. A necessary and sufficient condition that $A = BC$ for some S-analytic contraction C is that $AA^* \leq BB^*$.*

A special case of this theorem [3, Proposition V.5.3] is used by Sz.-Nagy and Foiaş to show the existence of "nontrivial" factorizations. This special case is presented in another form by de Branges and Rovnyak [1, Theorem 3]. The proof of the theorem has been translated from the context of the de Branges-Rovnyak model and makes extensive use of the following lemma, which can be found in [2].

Lemma. *Let A and B be bounded operators with final space $\mathcal{H}$. A necessary and sufficient condition that $A = BC$ for some contraction C is that $AA^* \leq BB^*$.*

Proof of Theorem. Since $BB^* - AA^* \leq BB^*$, we have

$$BB^* - AA^* = BTT^*B^*$$

for some contraction T. Also,

$$\begin{aligned} BB^* - AA^* &\leq BB^* - AA^* + AP_0A^* \\ &= BB^* - ASS^*A^* \\ &= BP_0B^* + S(BB^* - AA^*)S^* \\ &= B(P_0 + STT^*S^*)B^* \\ &= B(P_0 + STS^*)(P_0 + ST^*S^*)B^* \end{aligned}$$

and, therefore,

$$BT = B(P_0 + STS^*)U$$

for some contraction U. We define a sequence of operators $\{V_n\}$ by $V_0 = T$, $V_{n+1} = (P_0 + SV_nS^*)U$. It can be verified inductively that for each n $BT = BV_n$, $V_n^*V_n \leq I$, and

$$V_n = \sum_{k=0}^{n-1} S^k P_0 (US^*)^k U + S^n T (S^*U)^n.$$

[1] This is a transcription of an unpublished paper. The manuscript was circulated to various people by J. W. Helton.

Since $P_N V_n = P_N V_{N+1}$ for $n \geq N+1$, the sequence $\{V_n\}$ converges weakly to an operator V, $|V| \leq 1$, $BV = BT$, and $V = (P_0 + SVS^*)U$. Therefore,

$$\begin{aligned} I - VV^* &= I - (P_0 + SVS^*)UU^*(P_0 + SV^*S^*) \\ &\geq I - (P_0 + SVS^*)(P_0 + SV^*S^*) \\ &= S(I - VV^*)S^*. \end{aligned}$$

Since

$$\begin{aligned} AP_0A^* &= (BB^* - ASS^*A^*) - (BB^* - AA^*) \\ &= B(P_0 + SVV^*S^* - VV^*)B^* \\ &= B(I - VV^* - S(I - VV^*)S^*)B^*, \end{aligned}$$

we have $AP_0 = BRP_0$, where

$$R = (I - VV^* - S(I - VV^*)S^*)^{1/2}Q$$

for some contraction $Q \in \mathcal{B}(\mathcal{C}, \mathcal{H})$. From the inequality

$$\begin{aligned} \sum_M^N S^n RR^* S^{*n} &\leq \sum_M^N S^n (I - VV^* - S(I - VV^*)S^*)S^{*n} \\ &\leq S^M (I - VV^*) S^{*M}, \end{aligned}$$

which holds for $0 < M < N$, it follows that the expression

$$\sum_0^\infty S^n R P_0 S^{*n}$$

converges strongly to a contraction C. Clearly $CS = SC$, and the relation

$$AP_0 = BRP_0 = BC$$

implies that $A = BC$. □

References

1. L. de Branges and J. Rovnyak, *Appendix on square summable power series, canonical models in quantum scattering theory*, Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965), Wiley, New York, 1966, pp. 295–392.
2. R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415.
3. B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North-Holland Publishing Co., Amsterdam, 1970.

Comments on the paper of R. B. Leech

Minor problems at the end of the proof are easily repaired. It is not clear that the series defining C converges strongly. Work instead with the adjoint $C^* = \sum_0^\infty S^n P_0 R^* S^{*n}$, which is given by a strongly convergent series and defines an S-analytic operator C. By construction $AP_0 = BRP_0$. Thus $AS^n P_0 S^{*n} = BS^n RP_0 S^{*n}$ for all n, yielding $A = BC$.

Supplementary References

1. D. Alpay and V. Bolotnikov, *On tangential interpolation in reproducing kernel Hilbert modules and applications*, Topics in interpolation theory (Leipzig, 1994), Oper. Theory Adv. Appl., vol. 95, pp. 37–68, Birkhäuser Verlag, Basel, 1997.
2. D. Alpay, V. Bolotnikov, and Ph. Loubaton, *On interpolation for Hardy functions in a certain class of domains under moment type constraints*, preprint.
3. D. Alpay, V. Bolotnikov, and Y. Peretz, *On the tangential interpolation problem for H_2 functions*, Trans. Amer. Math. Soc. **347** (1995), no. 2, 675–686.
4. D. Alpay, A. Dijksma, J. Rovnyak, and H. S. V. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, Oper. Theory Adv. Appl., vol. 96, Birkhäuser Verlag, Basel, 1997.
5. ———, *Realization and factorization in reproducing kernel Pontryagin spaces*, (1998), preprint.
6. J. A. Ball, *Interpolation problems and Toeplitz operators on multiply connected domains*, Integral Equations and Operator Theory **4** (1981), no. 2, 172–184.
7. L. de Branges, *Kreĭn spaces of analytic functions*, J. Funct. Anal., **81** (1988), 219–259.
8. M. A. Dritschel and J. Rovnyak, *Extension theorems for contraction operators on Kreĭn spaces*, Extension and interpolation of linear operators and matrix functions (Basel), Oper. Theory Adv. Appl., vol. 47, Birkhäuser, 1990, pp. 221–305.
9. J. W. Helton, with the assistance of J. A. Ball, C. R. Johnson, and J. N. Palmer, *Operator Theory, analytic functions, matrices, and electrical engineering*, CBMS Regional Conference Series in Mathematics, vol. 68, Amer. Math. Soc., Providence, RI, 1987.
10. M. Rosenblum, *A corona theorem for countably many functions*, Integral Equations and Operator Theory **3** (1980), no. 1, 125–137.
11. M. Rosenblum and J. Rovnyak, *Hardy classes and operator theory*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1985, Oxford Science Publications; Dover reprint, 1997.