THE FEJÉR-RIESZ THEOREM

By a **trigonometric polynomial** is meant an expression in one of the equivalent forms $a_0 + \sum_1^n [a_j \cos(jt) + b_j \sin(jt)]$ or $\sum_{-n}^n c_j e^{ijt}$. When the values of a trigonometric polynomial are real for all real t, the coefficients a_j, b_j in the first form are necessarily real, and those in the second form satisfy $\bar{c}_j = c_{-j}$ for all indices j. L. Fejér [2] was the first to note the importance of the class of trigonometric polynomials that assume only nonnegative real values. His conjecture of the form of such a function was proved by F. Riesz.

Fejér-Riesz Theorem. *A trigonometric polynomial* $w(e^{it}) = \sum_{-n}^n c_j e^{ijt}$ *which assumes real and nonnegative values for all real* t *is expressible in the form*

$$w(e^{it}) = \left|p(e^{it})\right|^2$$

for some polynomial $p(z) = \sum_0^n a_j z^j$. *The polynomial can be chosen so that it has no roots in* $D = \{z : |z| < 1\}$, *and then it is unique except for a multiplicative constant of modulus one.*

The proof is based on the observation that $w(z) = \sum_{-n}^n c_j z^j$ satisfies $\overline{w(1/\bar{z})} = w(z)$ as a function of the complex variable z. If $c_{-n} \neq 0$, then $q(z) = z^n w(z)$ is a polynomial of degree $2n$ with $q(0) \neq 0$. The roots of $q(z)$ of modulus $\neq 1$ occur in pairs $\alpha, 1/\bar{\alpha}$ having equal multiplicity. Roots of unit modulus have even multiplicity. It follows that $w(z) = c\, \Pi_1^r\, [(z - \alpha_j)(z^{-1} - \bar{\alpha}_j)]$, where $\alpha_1, \ldots, \alpha_r$ have modulus ≥ 1 and c is a positive constant. The desired representation is obtained with $p(z) = \sqrt{c}\, \Pi_1^r\, (z - \alpha_j)$. See [4] for a variation of this method and an application in spectral theory.

A generalization of the Fejér-Riesz theorem plays an important role in the theory of orthogonal polynomials.

Szegő's Theorem. *Let* $w(e^{it})$ *be a nonnegative function which is integrable with respect to normalized Lebesgue measure* $d\sigma = dt/(2\pi)$ *on the unit circle* $\partial D = \{e^{it} : 0 \leq t < 2\pi\}$. *If*

$$\int_{\partial D} \log w(e^{it})\, d\sigma > -\infty,$$

then

$$w(e^{it}) = \left|h(e^{it})\right|^2 \qquad \sigma - a.e.,$$

where $h(e^{it})$ *is the boundary function of an outer function* $h(z)$ *on* D *which belongs to the Hardy class* H^2. *Such a function is unique up to a multiplicative constant of modulus one.*

The asymptotic properties of the polynomials which are orthogonal with respect to such a weight function $w(e^{it})$ are described in terms of the function $h(z)$ (Szegő [7], Ch. 12). The term **outer** here means that the set of functions of the form $h(z)k(z)$, where $k(z)$ is a polynomial, is dense in H^2. The log-integrability hypothesis is automatically satisfied when $w(e^{it})$ is a trigonometric polynomial, and then the outer function $h(z)$ is the polynomial $p(z)$ in the Fejér-Riesz theorem.

The Fejér-Riesz and Szegő theorems are prototypes for two kinds of hypotheses which assure the existence of similar representations of nonnegative functions. One type stipulates algebraic or analytical structure, the other that the given function is not too small. Nonnegativity on the the unit circle is often replaced by nonnegativity on the real line. For example, a theorem of

Ahiezer states that an entire function $w(z)$ of exponential type τ which is nonnegative on the real axis and satisfies

$$\int_{-\infty}^{\infty} \frac{\log^+ w(x)}{1+x^2}\, dx < \infty$$

can be written $w(x) = |f(x)|^2$ for x real where $f(z)$ is an entire function of exponential type $\tau/2$ which has no zeros in the open upper half-plane (see Boas [1]).

Related problems arise in linear prediction theory, but there the functions to be factored are operator valued. Such problems date to the 1940's and 1950's (see [3,6,8]). In this context, the term **spectral factorization** is used to describe the representation of nonnegative operator-valued functions. Operator extensions of the Fejér-Riesz theorem were proved in special cases by several authors, the final form being that given by M. Rosenblum:

Fejér-Riesz Theorem (Operator Version). *Let* $W(e^{it}) = \sum_{-n}^{n} C_j e^{ijt}$ *be a trigonometric polynomial whose coefficients are operators on a Hilbert space* $\mathcal{K}$ *and which assumes nonnegative selfadjoint values for all real* t*. Then*

$$W(e^{it}) = P(e^{it})^* P(e^{it})$$

for some outer polynomial $P(z) = \sum_{0}^{n} A_j z^j$ *whose coefficients are operators on* $\mathcal{K}$*.*

The term **outer** as used here is relative to the Hardy class $H^2_{\mathcal{K}}$ of functions with values in $\mathcal{K}$: the meaning is that the set of functions $P(z)k(z)$, where $k(z)$ is a polynomial with coefficients in $\mathcal{K}$, is dense in a subspace of $H^2_{\mathcal{K}}$ of the form $H^2_{\mathcal{M}}$ for some subspace $\mathcal{M}$ of $\mathcal{K}$.

Analogous theorems hold for operator-valued functions which satisfy hypotheses of Szegő type and for polynomials, rational functions, and entire functions of exponential type. However, the techniques used to prove the representation theorems in the scalar case are usually not applicable in the operator extensions. For example, the fundamental theorem of algebra, which is used in the proof of the Fejér-Riesz theorem, has no counterpart for operator-valued polynomials. A method due to D. Lowdenslager allows a unified approach to the operator extensions, and many results based on the method, including a proof of the operator Fejér-Riesz theorem, may be found in [5].

REFERENCES

1. R. P. Boas, *Entire Functions*, Academic Press, New York, 1954.
2. L. Fejér, *Über trigonometrische Polynome*, J. Reine Angew. Math. **146** (1916), 53–82.
3. H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables.* I,II, Acta Math. **99** (1958), 165–202; ibid. **106** (1961), 175–213.
4. F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar Publ. Co., New York, 1955.
5. M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford Mathematical Monographs, Oxford University Press, New York, 1985.
6. Ju. A. Rozanov, *Spectral theory of n-dimensional stationary stochastic processes with discrete time*, Uspehi Mat. Nauk **13** (1958), no. 2 (80), 93–142; Selected Translations in Mathematical Statistics and Probability **1** (1961), Amer. Math. Soc., Providence, RI, 253–306.
7. G. Szegő, *Orthogonal Polynomials*, 4-th ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1975.
8. N. Wiener and P. Masani, *The prediction theory of multivariate stochastic processes.* I,II, Acta Math. **98** (1957), 111–150; ibid. **99** (1958), 93–137.