O(3) symmetric merons in an SU(3) Yang–Mills theory*

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ABSTRACT. We investigate irreducible, O(3) symmetric multiple-meron solutions to the classical SU(3) Yang–Mills equations in four-dimensional Euclidean space. The solutions have topological charge density equal to a sum of delta-functions with integer coefficients, and correspond to solutions of a system of two coupled singular elliptic equations. We prove the existence of two-meron solutions of the coupled system.

Infinite action solutions to the classical Yang–Mills equations in Euclidean space with topological charge density concentrated at points have been used by Glimm and Jaffe [1], and Callen et al. [2] in models for quark confinement. The first such meron solution was found by deAlfaro et al. [3] and was generalized to multiple merons on a line by Glimm and Jaffe [4]. Merons have only been studied in an SU(2) gauge theory, which can, of course, be embedded in the SU(3) theory believed to describe strong interactions. In this paper, we use the O(3) symmetric ansatz of Bais and Weldon [5] to investigate SU(3) merons which do not arise from an embedding of SU(2) in SU(3). We call these irreducible.

We find some new phenomena in SU(3). Finding irreducible meron solutions is reduced to solving a pair of coupled elliptic equations, analogous to the equation \( r^2 \Delta \psi = \psi^3 - \psi \) in the SU(2) case [4]. We prove that the equations have solutions corresponding to two-meron configurations. The charge density is a sum of unit delta-functions instead of the half delta-functions found in the SU(2) case. Although the word meron originated from fractional charges, we use it for integer charges as well because the solutions are similar to fractionally charged merons in other respects. They have regular, singular points and nonintegrable action.

Following Bais and Weldon [5], we define the following Hermitian, traceless matrices (\( a, b, l, m \) take values 1, 2, 3):

\[
(I_a)_{lm} = i \epsilon_{lam},
\]

\[
(Q_{ab})_{lm} = \delta_{al} \delta_{bm} + \delta_{am} \delta_{bl} - \frac{2}{3} \delta_{ab} \delta_{lm}.
\]

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Whenever ± appears in an equation, the equation is to be interpreted as two equations: one with ± replaced by +, and one with ± replaced by -. We denote by $\vec{K}^\pm$ the vector whose components are $K^\pm$. The following relations among the $M^\pm$ and $\vec{K}^\pm$ will be needed [5]:

\[
[K^-, M^+] = [K^+, M^-] = [M^+, M^-] = \vec{K}^+ \cdot \vec{K}^- - \vec{K}^- \cdot \vec{K}^+ = 0,
\]

\[
\vec{K}^+ \times \vec{K}^- = i\vec{k}(2M^+ - M^-), \quad \vec{K}^- \times \vec{K}^+ = i\vec{k}(2M^- - M^+),
\]

\[
\vec{K}^+ \times \vec{K}^- = 0,
\]

\[
\vec{\triangledown} \cdot \vec{K}^+ = \frac{1}{r} \vec{\triangledown} \times \vec{K}^+ = \frac{1}{r} \vec{\triangledown} \times \vec{K}^+,
\]

\[
\vec{\triangledown} \cdot \vec{K}^+ = \frac{1}{r} (-4M^+ + 2M^-), \quad \vec{\triangledown} \cdot \vec{K}^- = \frac{1}{r} (-4M^- + 2M^+),
\]

\[
\text{Tr} \, M^+ M^+ = 2/3, \quad \text{Tr} \, M^- M^- = 1/3,
\]

\[
\text{Tr} \, K^a_+ K^b_+ = \delta_{ab} - \delta_{ab} \bar{r} \bar{f}_b,
\]

\[
\text{Tr} \, K^a_- K^b_- = 0.
\]

The $O(3)$ symmetric ansatz for the gauge field [5] is

\[
A_o = iM^+ a_o^+ + iM^- a_o^-,
\]

\[
\vec{A} = iM^+ \alpha_0^+ - \vec{r} \times \vec{\Phi}^+ [(1 + \phi^+_1)/r] + i\vec{R}^+ (\phi^+_0/r) + iM^- \alpha_0^- - \vec{r} \times \vec{\Phi}^- [(1 + \phi^-_1)/r] + i\vec{R}^- (\phi^-_0/r),
\]

where $\alpha_0^\pm$, $\alpha_0^\pm$, $\phi_0^\pm$, and $\phi_1^\pm$ depend only on $r = |\vec{r}|$ and $t$. The ansatz is consistent with gauge transformations $\exp(ia^+ M^+)$, $\exp(ia^- M^-)$ where $\alpha^\pm$ depends on $r$ and $t$ only. This is a natural generalization of Witten's ansatz in $SU(2)$ [6]. We can regard $\vec{\Phi}^\pm = \begin{pmatrix} \phi_0^\pm \\ \phi_1^\pm \end{pmatrix}$ as a two component Higgs field.
interacting with a U(1) gauge field \( a^± = a_0^± dt + a_r^± dr \) in the half plane \( \mathbb{R}^2_+ = \{(r, t)|r > 0\} \) with metric \( r^2(dr^2 + dt^2) \). The covariant derivatives are defined as \( D^± \phi^± = d\phi^± + a^± \epsilon^± \) where 
\[
\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

The SU(3) field strengths \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \) break into an electric part \( E_i = F_{0i} \) and a magnetic part \( B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} \) [5]:

\[
\begin{align*}
\vec{E} &= iM^* \phi^0_\alpha - i \frac{\dot{r}}{r} \times \vec{K}^* D_0^0 \phi^0_\alpha + i \frac{K^+}{r} D_0^0 \phi^0_\alpha + \\
&\quad + iM^* \phi^r_0 - i \frac{\dot{r}}{r} \times \vec{K}^* D_0^0 \phi^r_0 + i \frac{K^-}{r} D_0^0 \phi^r_0, \\
\vec{B} &= iM^* \frac{\dot{r}}{r^3} \left( 2|\phi^0|^2 - |\phi^r|^2 - 1 \right) + i \frac{\dot{r}}{r} \times \vec{K}^* D_r^0 \phi^0_\alpha + \frac{iK^+}{r} D_r^0 \phi^0_\alpha - \\
&\quad - iM^* \frac{\dot{r}}{r^3} \left( 2|\phi^0|^2 - |\phi^r|^2 - 1 \right) + i \frac{\dot{r}}{r} \times \vec{K}^* D_r^0 \phi^r_\alpha + \frac{iK^-}{r} D_r^0 \phi^r_\alpha.
\end{align*}
\]

Here \( f^± = da^± \) are the field strengths defined in \( \mathbb{R}^2_+ \), i.e. \( f^0_\alpha = \partial_\alpha a^0_\gamma - \partial_\gamma a^0_\alpha \). The SU(3) topological charge density is

\[
\Sigma(r, t) = -\frac{1}{3(2\pi)^2} \epsilon_{\mu\nu\rho\sigma} Tr F_{\mu\nu} F_{\rho\sigma} = \frac{1}{4\pi^2 r^2} \left( f^0_\alpha + k^0_\alpha + f^r_\alpha + k^r_\alpha \right).
\]

Here, \( k^± = dI^± \) in \( \mathbb{R}^2_+ \), with \( I^± = \phi^±_0 D^±_r \phi^±_0 - \phi^r_0 D^±_0 \phi^±_0 \), i.e. \( k^0_\alpha = \partial_\alpha I^+_0 - \partial_\gamma I^0_\gamma \). The charge density is the 4-divergence of the current

\[
\begin{align*}
J_\mu &= -\frac{1}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} Tr (A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma), \\
\vec{J} &= -\frac{1}{4\pi^2 R^2} \left( a^0_0 + I^0_0 + a^r_0 + I^r_0 \right) \hat{r}, \\
J_0 &= \frac{1}{4\pi^2 R^2} \left( a^0_0 + I^0_0 + a^r_0 + I^r_0 \right).
\end{align*}
\]

After integrating over angular variables, the charge density in \( \mathbb{R}^2_+ \) is

\[
q(r, t) = \frac{1}{\pi} \left( k^0_\alpha + f^0_\alpha + k^r_\alpha + f^r_\alpha \right).
\]

The Yang–Mills equations \( \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0 \) become the following in our ansatz:

\[
\star d\psi^± = 4I^± - 2f^±, \\
\star d\psi^± = 4\psi^± - 2\psi^±.
\]

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\begin{align}
\ast D'_+ \ast D'_+ \phi^+_i - \phi^+_i (2 |\phi^+_i|^2 - |\phi^+_i|^2 - 1) &= 0, \\
\ast D'_- \ast D'_- \phi^-_i - \phi^-_i (2 |\phi^-_i|^2 - |\phi^-_i|^2 - 1) &= 0,
\end{align}

where \((\ast d^\pm f^\pm)_0 = r^2 \partial_t f^\pm + 2rf^\pm_0, (\ast d^\mp f^\mp)_0 = r^2 \partial_\theta f^\mp_0,\) and \(\ast D^\pm \ast D^\pm = r^2 D^\pm_\theta D^\pm_\theta, (\alpha = 0, r).\)

We look for solutions to (9) and (10) with

\begin{align}
Q(\vec{r}, t) &= \sum_{i=1}^n \alpha_i \delta(\vec{r}) \delta(t - t_i), \\
q(r, t) &= \sum_{i=1}^n 2\alpha_i \delta(r) \delta(t - t_i).
\end{align}

\(t_i\) are the positions of the merons on the \(t\)-axis, \((-\infty = t_0 < t_1 < \cdots < t_n < t_{n+1} = \infty)\) and \(\alpha_i\) are their charges (allowing \(\alpha_0 = 0\)). Then (8) and \(k^\pm = dl^\pm\) imply that (9) and (11) are equivalent to the following equations:

\begin{align}
I^+ &= \frac{1}{6} \ast d^\ast (2f^+ + f^-), & I^- &= \frac{1}{6} \ast d^\ast (2f^- + f^+), \\
\left[\frac{d}{2}\ast d^\ast + \frac{1}{\pi}\right] \left( f^+ + f^- \right) &= q(r, t) = \sum_{i=1}^n 2\alpha_i \delta(r) \delta(t - t_i).
\end{align}

One solution of (12) and (13) is

\begin{align}
f^\pm_\theta(r, t) &= \sum_{i=1}^n 2\pi \phi^\pm_\theta(t - t_i), & I^\pm &= 0, & \beta^+_i + \beta^-_i &= \alpha_i,
\end{align}

because for this \(f^\pm, \ast d^\pm f^\pm = 0\) due to the \(r^2\) in the metric. In what follows, we assume \(f^\pm_\theta\) is given by (14), and reduce the remaining Yang–Mills equations (10) to two coupled elliptic equations for two real-valued functions of \(r\) and \(t\). We will find that in gauges consistent with (4), \(\beta^+_i\) must be integers in order for \(A_n\) to be regular away from the merons.

Following Glimm and Jaffe [4], we find an \(a^\pm\) such that \(f^\pm = da^\pm\) where \(f^\pm\) is as in (14):

\begin{align}
a^\pm &= d\theta^\pm, & \Theta^\pm(r, t) &= \sum_{i=1}^n \beta^+_i \arg (-t + t_i + i\tau).
\end{align}

\(\phi^\pm\) can be rotated into a single direction by defining

\begin{align}
\tilde{\phi}^\pm = \Theta^\pm \phi^\pm, & & \Theta^\pm = \begin{pmatrix}
\cos \theta^\pm & \sin \theta^\pm \\
-\sin \theta^\pm & \cos \theta^\pm
\end{pmatrix}.
\end{align}
\( \Theta^\pm \) rotates the covariant derivative into the ordinary derivative because \( d\Theta^\pm = a^\pm e \Theta^\pm \) and \( \Theta^\pm e(\Theta^\pm)^{-1} = e, \) so that \( \Theta^\pm D(\Theta^\pm)^{-1} = \Theta^\pm (d + a^\pm e) (\Theta^\pm)^{-1} = d. \) Moreover, \( \Theta^\pm \epsilon(\Theta^\pm)^{-1} = *, \) and \( |\phi^\pm|^2 = |\phi^\pm|^2 \) so that (10) becomes

\[
\star d \star \psi^\pm - \psi^\pm (2|\psi^\pm|^2 - |\phi^\pm|^2 - 1) = 0,
\]

\[
\star d \star \psi^\mp - \psi^\mp (2|\psi^\mp|^2 - |\phi^\mp|^2 - 1) = 0,
\]

where \( \star d \star = r^2 \Delta. \) The circulation density of \( \vec{\psi}^\pm \) is

\[
e_{ij} \star \vec{j} \star \psi^\pm = e_{ij} \phi_i \mathbf{D}^\pm \phi^j = f^\pm = 0.
\]

This implies that \( \vec{\psi}^\pm \) maintains a constant direction in connected sets where \( \vec{\psi}^\pm \neq 0. \) If \( \vec{\psi}^\pm \) solves (17), it can, at worst, change direction by \( \pi \) across curves where \( \vec{\psi}^\pm = 0. \) This means that (17) is equivalent to the following system:

\[
r^2 \Delta \psi^+ - \psi^+ (2(\psi^+)^2 - (\psi^\pm)^2 - 1) = 0,
\]

\[
r^2 \Delta \psi^- - \psi^- (2(\psi^-)^2 - (\psi^\pm)^2 - 1) = 0,
\]

where \( \psi^\pm \) is the component of \( \vec{\psi}^\pm \) along a fixed direction which is one of the two possible directions of \( \vec{\psi}^\pm \).

The action density for the \( SU(3) \) field is given in [5]:

\[
L(\vec{r}, t) = -\frac{1}{4} \text{Tr} F_{\mu \nu} F^{\mu \nu} = r^2 (D^\mu \phi^\mu)^2 + r^2 (D^\mu \phi_i)^2 + \frac{1}{2} (f^\phi)^2 + \frac{1}{2} (f^\phi)^2 + \frac{1}{2} f^\phi f^\phi + \frac{1}{2} f^\phi f^\phi + (\phi^\pm)^2 + (\phi^\pm)^2 + (\phi^\pm)^2 + (\phi^\pm)^2.
\]

For the case \( f^\phi = 0 \) at \( r > 0, \) we integrate \( L(\vec{r}, t) \) over angular variables and express the result in terms of \( \psi^+ \) and \( \psi^- \):

\[
I(r, t) = 8 \pi \left( \frac{1}{2} |\Delta \psi^+|^2 + \frac{1}{2} |\Delta \psi^-|^2 + (r^2) V(\psi^+, \psi^-) \right),
\]

\[
V(\psi^+, \psi^-) = \frac{1}{2} (1 - (\psi^+)^2 - (\psi^-)^2 - (\psi^+)^2 (\psi^-)^2 - (\psi^+)^2 (\psi^-)^2) \geq 0.
\]

The potential achieves its minimum \( V = 0 \) at the four points \( |\psi^+| = |\psi^-| = 1. \) Along the line \( \psi^+ = \chi \cos \omega, \psi^- = \chi \sin \omega, \) \( V \) is the double-well potential

\[
V(\omega, \chi) = \frac{1}{2} - \frac{1}{a^2} + \frac{a^2}{2} \left( \frac{4}{a^2} - \chi^2 \right),
\]

where \( a^2 = 5 + 3 \cos 4\omega \) (Figure 1).
Fig. 1. The potential $V(\psi^+, \psi^-)$. Dots indicate absolute minima. Curve indicates minima along rays through the origin.

Fig. 2. A possible set of strings connecting merons. The central two merons have $\psi^+$ and $\psi^-$ changing sign simultaneously. Roman numerals indicate the quadrant of $\arg(\psi^+ + i\psi^-)$.

In order that the $SU(3)$ field strengths be regular at $r = 0$ between the merons, we must require that $|\psi^+| = |\psi^-| = 1$ at $r = 0$ so that the term in $V$ in the action density (21) is not singular. We can assume that the set $\{t_i|1 \leq i \leq n\}$ includes every point where $\psi^+$ or $\psi^-$ changes sign. The condition that $A_{\mu}$ in (4) be regular at $r = 0$ is

$$\lim_{r \to 0} 1 + \phi^+_i = \lim_{r \to 0} \phi^-_i = 0. \tag{23}$$

For each segment $(t_i, t_{i+1})$, it will be possible to find a gauge such that (23) is satisfied, as long as $|\psi^+| = |\psi^-| = 1$ at $r = 0$. However, the charges $\beta^\pm_i$ can be changed by any amount by gauge transformations unless we restrict our attention to gauges (4) which are regular on both $(t_{i-1}, t_i)$ and $(t_i, t_{i+1})$. By (23), $\vec{\psi}^\pm$ does not change direction across $t_i$ in such gauges. According to (16), $\vec{\psi}^\pm$ changes direction by $\beta^\pm_i\pi$ for any $\theta^\pm$ consistent with (14). Therefore, when $\psi^\pm$ is constant across $t_i$, $\beta^\pm_i$ must be an even integer, and when $\psi^\pm$ changes sign across $t_i$, $\beta^\pm_i$ must be an odd integer. Although $\beta^\pm_i$ is still not gauge invariant, $\beta^\pm_i \mod 2$ is invariant.

In summary, finding meron solutions with $SU(3)$ topological charge density $Q(\vec{r}, t) = \Sigma_{i=1}^n (\beta^+_i + \beta^-_i)\delta(\vec{r})\delta(t - t_i)$ is reduced to solving (19) with boundary conditions

$$\psi^\pm = (-1)^{1+\sum_{i=1}^k \beta^\pm_i} \text{at } r = 0, \hspace{1cm} t_k < t < t_{k+1}. \tag{24}$$
The charge is quantized in integers, instead of half-integers as in \( SU(2) \), and is defined only modulo 2, instead of modulo 1 as in \( SU(2) \). The integer charge is not an artifact of a normalization convention. \( SU(2) \) monopoles embedded in an \( SU(3) \) theory have half-integer charges. For a solution to (19) and (24), we introduce strings as the lines for which \( \psi^* \) or \( \psi \) is zero. We expect that the strings will end at the monopoles, and divide the plane into disjoint regions where \( \omega = \arg(\psi^* + i\psi) \) is within one of the four quadrants of the unit circle (Figure 2).

If \( \beta_i^+ = \beta_i^- \equiv 0 \mod 2 \) for some \( i \), the boundary conditions (24) are as if there were no monopole at \( t_i \). The action density (21) would also be affected (if solutions to (19) and (24) exist). Such \( t_i \) are artifacts of a singular gauge transformation, not monopoles. The three other possibilities for \( \beta_i^\pm \mod 2 \) affect the boundary conditions in such a way that the action density is changed on sets of positive measure in \( \mathbb{R}^4 \) (if solutions to (19) and (24) exist). For these cases, \( t_i \) is a genuine monopole, not a gauge artifact. This includes the case \( \beta_i^+ = -\beta_i^- \equiv 1 \mod 2 \) which has total charge \( \beta_i^+ + \beta_i^- = 0 \) at \( t_i \).

We now treat the existence of solutions to (19) and (24). Note that (19) are the variational equations of (21). For the case \( \beta_i^+ = \beta_i^- \equiv 0 \mod 2 \) for all \( i \), \( \psi^+ = \psi^- \) is a solution, provided

\[
r^2 \Delta \psi^* = (\psi^*)^3 - \psi^*.
\]

Note (25) is the equation for monopoles in \( SU(2) \) [4]. A proof of existence of solutions to (25) and (24) was given by Jonsson et al. [7]. Their proof depended on a knowledge of the behavior of \( \psi^+ \) \( \psi^- \) in a neighborhood of each monopole. This knowledge was provided by a closed-form two-meron solution (one monopole at \( t = 0 \), one at \( t = \infty \)). No solution to (19) and (24) with \( \beta_i^+ \neq \beta_i^- \mod 2 \) is known in closed form. However, in what follows, we prove that such a solution exists in the two-meron case. This leads us to expect that an existence proof for (19) and (24) in the general case could be given along the lines of the proof of Jonsson et al. for (25). An alternative proof, using super- and sub-solution methods, has been given by Baker and Zipatti [8] and such methods may also be applicable here.

If \( \psi^+ \) and \( \psi^- \) depend only on \( \theta = \arg(-t + i\epsilon) \), then (19) becomes

\[
\sin^2 \theta (\psi^*)'' - \psi^*(2(\psi^*)^2 - (\psi^-)^2 - 1) = 0,
\]

\[
\sin^2 \theta (\psi^-)' - \psi^*(2(\psi^-)^2 - (\psi^-)^2 - 1) = 0,
\]

where \( \cdot \) denotes differentiation with respect to \( \theta \). The two-meron boundary conditions that we will study are

\[
\lim_{\theta \to 0} \psi^* = \lim_{\theta \to 0} \psi^* = \lim_{\theta \to \pi} \psi^* = \lim_{\theta \to \pi} (-\psi^*) = 1,
\]

(27)

corresponding to charges \( \beta_i^+ = 1, \beta_i^- \equiv 0 \mod 2 \). We note that \( \psi^+ = \psi^- = \cos \theta \) is a solution of (26) with boundary conditions corresponding to \( \beta_i^+ = \beta_i^- \equiv 1 \mod 2 \).

**THEOREM.** There exist two bounded, continuous functions \( \psi^+ \) and \( \psi^- \) defined in \( (0, \pi) \) satisfying (26) and (27). \( \psi^\pm \) possesses two bounded, continuous derivatives and \( \lim_{\theta \to 0} (\psi^\pm)' = \lim_{\theta \to \pi} (\psi^\pm)' = 0. \)

**Proof.** It is sufficient to work in the interval \((0, \pi/2]\) with boundary conditions \(\psi^*= (\psi')'= 0\) at \(\pi/2\) because setting \(\psi^*(\pi/2 + \theta) = -\psi^*(\pi/2 - \theta)\) and \(\psi^*(\pi/2 + \theta) = \psi^*(\pi/2 - \theta)\) will extend a solution and its first two derivatives to continuous functions on \((0, \pi]\).

We define four functions, \(u_1 < u_2\) and \(v_1 < v_2\) which satisfy the boundary conditions at 0 and at \(\pi/2\):

\[
\begin{align*}
u_1(\theta) &= \cos \theta, & u_2(\theta) &= 1 - (2\theta/\pi)^3, \\
 v_1(\theta) &= 1 - \frac{1}{2} \sin^2 \theta, & v_2(\theta) &= 1.
\end{align*}
\]

(28)

It will turn out that the solution satisfies \(u_1 < \psi^* < u_2\) and \(v_1 < \psi < v_2\). The following differential inequalities hold for \(\theta \in (0, \pi/2)\):

\[
\begin{align*}
sin^2 \theta u''_1 - u_1(2u_1^2 + u_1^2 - 1) &> 0 \\
 sin^2 \theta v''_1 - v_1(2v_1^2 - u_1^2 - 1) &> 0
\end{align*}
\]

(29)

\[
\begin{align*}
sin^2 \theta u''_2 - u_2(2u_2^2 - v_1^2 - 1) &< 0 \\
 sin^2 \theta v''_1 - v_2(2v_2^2 - u_2^2 - 1) &< 0.
\end{align*}
\]

The third inequality is proved by dividing by \(\theta^3\) and replacing \((\sin^2 \theta)/\theta^2\) by the smaller function \(1 - (4\theta/\pi)(1 - 2/\pi)\). The result is a sixth degree polynomial in \(\theta\) which is everywhere less than \(-1/4\). The other three inequalities are simple to prove.

We use \(u_i\) and \(v_i\) to define some regions in \([0, \pi/2] \times \mathbb{R}^2\) (Figure 3): \(D = ((\theta, \psi^*, \psi')0 < \theta < \pi/2, u_i(\theta) < \psi^* < u_2(\theta), v_i(\theta) < \psi < v_2(\theta)\); \(S = \partial D - [\pi/2] \times \mathbb{R}^2\). Let \(D_0\) be a neighborhood of \(D\) and \(D_\delta = D_0 \cap [\delta, \pi/2] \times \mathbb{R}^2\) for \(\delta > 0\). The equations (26) satisfy a Lipschitz condition in \(D_\delta\) if \(\delta > 0\). Therefore, the initial value problem (26) with initial conditions \(\psi^* = (\psi')' = 0, (\psi^*)' = b, \psi = c\) at \(\pi/2\) has a unique twice continuously differentiable solution as far as the boundary of \(D_\delta\). This solution is a curve \(\gamma_{bc}\) in \(D_0\), where \(\gamma_{bc}(\theta) = (\theta, \psi(\theta), \psi'(\theta))\). \(\gamma_{bc}\) depends continuously on \((b, c)\).

For \((b, c) \in \Gamma = (-6/\pi, -1) \times (1/2, 1)\), we have \(\gamma_{bc}(\pi/2) \in \partial D\) and \(\gamma_{bc}(\pi/2) \in D\). For each \((b, c) \in \Gamma\) there is associated a unique point \(T(b, c)\) where \(\gamma_{bc}\) first intersects \(S\). In the case \(\gamma_{bc}(\theta) \in D\) for all \(\theta \in (0, \pi/2)\), we define \(T(b, c) = (0, 1, 1) = \gamma_{bc}(0)\). As long as \(\gamma_{bc}(\theta) \in D\), we can divide (26) by \(\sin^2 \theta\) to obtain a uniform bound on \((\psi')''\) of order unity. Since \(u_i(0) = v_i(0) = 0\), such a bound implies that \(\lim_{\theta \to 0} (\psi')' = 0\). Therefore \(\gamma_{bc}\) is a solution to our problem if \(T(b, c) = (0, 1, 1)\).

We prove that the map \(T: \Gamma \to S\) is continuous. Since \(\gamma_{bc}\) depends continuously on \((b, c)\), the only way \(T\) could fail to be continuous is if \(\gamma_{bc}\) were tangent to \(S\) at \(T(b, c)\). For example, suppose \(\psi^* = u_1\) and \((\psi')' = u_1'\) at \(\theta_0 > 0\). Since \(\psi' \geq v_1\) and \(u_1' > 0\), (29) implies that \(u_1' \geq (\sin^2 \theta_0)^3 \psi^*(2(\psi')^2 - (\psi')^2 - 1) = (\psi')''\) (strict inequality if \(\theta_0 < \pi/2\)). Therefore, \(\psi' < u_1\)
for \( \theta \neq \theta_0 \) in some neighborhood of \( \theta_0 \). This contradicts the definition of \( T(b, c) \) as the first point where \( \gamma_{bc} \) intersects \( S \). The case where \( \gamma_{bc} \) is tangent to one of the other three faces of \( S \) are handled in the same fashion. Since \( T(b, c) \) cannot be a point of tangency, \( T \) must be continuous.

In order to conclude that there is a \( (b, c) \in \Gamma \) such that \( T(b, c) = (0, 1, 1) \), we must show that there is a curve \( C \subseteq \Gamma \) such that \( T(C) \) surrounds the point \((0, 1, 1)\) in \( S \). We outline a proof here. The face of \( S \) consisting of points \((\theta, \psi^*, \psi^-)\) with \( \psi^* = u_1 \) will be denoted by \( S_{u_1} \). The face consisting of points \((\theta, \psi^*, \psi^-)\) with \( \psi^- = u_1 \) will be denoted by \( S_{c_0} \) (Figure 3). If \((b_0, c_0) \in \partial\Gamma\), then \( \gamma_{bc} \) is tangent to \( \partial D \) at \( \pi/2 \). The remarks in the above paragraph on solutions tangent to \( \partial D \) apply, and we conclude that \( \gamma_{bc} \) is outside \( D \) for \( \theta \) near \( \pi/2 \). Since the differential equations satisfy a Lipschitz condition, the curve \( \gamma_{bc} \) and its derivative converge uniformly to \( \gamma_{bc} \) and its derivative as \( (b, c) \) approaches \( (b_0, c_0) \). Therefore, there is a \( \delta \) such that for \( (b, c) \in B_{\delta}((b_0, c_0)) \cap \Gamma \) the point \( T(b, c) \) has \( \theta \) coordinate close to \( \pi/2 \). If \( b_0 = -6/\pi \) there is a \( \delta' < \delta \) such that for \( (b, c) \in B_{\delta'}((b_0, c_0)) \cap \Gamma \) the curve \( \gamma_{bc} \) does not reach \( S \) until it exits by \( S_{u_2} \) or at a point no more than one-third of the way from \( S_{u_2} \) to \( S_{u_1} \) or \( S_{v_1} \) or \( S_{v_1} \). Similarly, if \( b_0 = -1 \), \( \gamma_{bc} \) does not reach \( S \) until it exits by \( S_{u_1} \) or at a point no more than one-third of the way from \( S_{u_1} \) to \( S_{u_2} \). Here, we have used uniform convergence of the derivative. If \( c_0 = 1/2 \), \( \gamma_{bc} \) exits by \( S_{v_1} \) or close to it, and if \( c_0 = 1 \), \( \gamma_{bc} \) exits by \( S_{u_1} \) or close to it (for \( (b, c) \) in an appropriate neighborhood of \( (b_0, c_0) \)). These neighborhoods of the points of \( \partial \Gamma \) form an open cover which admits a finite subcover by the compactness of \( \partial \Gamma \). Thus there is a \( \delta^* \) such that every point of \( \Gamma \) within \( \delta^* \) of \( \partial \Gamma \) is covered. Let \( C \) be a rectangle in \( \Gamma \) closer to \( \partial \Gamma \) than \( \delta^* \). By construction, the four sides of \( C \) are mapped by \( T \) into four regions which surround the point \((0, 1, 1)\) in \( S \). Since \( T \) is continuous, \( T(C) \) must surround the point \((0, 1, 1)\). If \( T(\Gamma) \) did not include \((0, 1, 1)\), then \( T(C) \) would have to jump as \( C \) is contracted to a point. Since \( T \) is continuous, we conclude that there is a \( (b, c) \) such that \( T(b, c) = (0, 1, 1) \), and the theorem is proved.

Fig. 3. The surface \( S \), its faces \( S_{u_1} \) and \( S_{v_1} \) and the curves that define them.

Fig. 4. A numerical solution of the two-meron equation.
In the above proof, we found that the solution to (26) and (27) lies in a domain which does not include the function \( \psi^+ = 1 \). Thus, both \( \psi^+ \) and \( \psi^- \) exhibit singular behavior at the origin of \( \mathbb{R}^2 \), even though \( \psi^- \) obeys constant boundary conditions. This can also be seen in Figure 4, which displays a numerical solution of (26) and (27).

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REFERENCES


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