**Problem 1.** (10 points) Find the limit if exists, or show that the limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}.$$

$$\lim_{(x,0)\to(0,0)} f(x,0) = \lim_{(x,0)\to(0,0)} 0 = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} f(x,y) = \lim_{x\to 0} \frac{2x^2}{3x^2} = \frac{2}{3} \neq 0$$

Therefore the limit DNE

**Problem 2.** (10 points) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = x^3 - 6xy + 8y^3.$$

(a) Find the critical points of $f$.
(b) Is $(0, 0)$ a local extreme or a saddle point?

(a) $f_x(x,y) = 3x^2 - 6y$
$f_y(x,y) = -6x + 24y^2$

$\Rightarrow \begin{cases} 2y = x^2 \Rightarrow 4y^2 = x^4 \\ x - 4y^2 = 0 \Rightarrow x - x^4 = 0 \Rightarrow x(1-x^3) = 0 \Rightarrow \begin{cases} x = 0 \\ x = 1 \end{cases} \end{cases}$

$y = \frac{x^2}{2}$

therefore the critical points are $(0,0)$ and $(1, \frac{1}{2})$

(b) $f_{xx}(x,y) = 6x \qquad f_{xx}(0,0) = 0$
$f_{yy}(x,y) = 48y \qquad f_{yy}(0,0) = 0$

$f_{xy}(x,y) = f_{yx}(x,y) = -6$

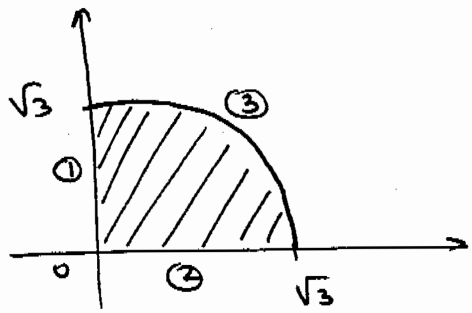
$f$ infinitely many times differentiable on $\mathbb{R}^2 \Rightarrow$ the second derivative test applies

$D(0,0) = 0 - (-6)^2 = -36 < 0 \Rightarrow (0,0)$ is a saddle point

**Problem 3.** (15 points) Find the absolute maximum and minimum values of the function $f$ on the set $D = \{(x,y) : x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 3\}$, where

$$f(x,y) = xy^2.$$

Domain D:




Critical points inside D:

$$\left.\begin{array}{l} f_x(x,y) = y^2 \\ f_y(x,y) = 2xy \end{array}\right\} \Rightarrow \begin{array}{l} y = 0 \\ x \in [0,\sqrt{3}] \end{array} \begin{array}{l} \rightarrow \text{not inside D} \\ \text{(on the bounda} \end{array}$$

Therefore the are no critical points inside D

Analysis on the boundary

① $(0,y)$ $y \in [0,\sqrt{3}]$ $f(0,y) = 0$

② $(x,0)$ $x \in [0,\sqrt{3}]$ $f(x,0) = 0$

③ $x^2 + y^2 = 3 \Rightarrow y^2 = 3 - x^2 \Rightarrow f(x,y) = x \cdot (3 - x^2) = 3x - x^3$

$(3x - x^3)' = 3 - 3x^2 \Rightarrow$ $x^2 = 1 \Rightarrow \left.\begin{array}{l} x = \pm 1 \\ x \geq 0 \end{array}\right\} \Rightarrow x = 1$

$y^2 = 3 - 1$

$\left.\begin{array}{l} y = \pm\sqrt{2} \\ y \geq 0 \end{array}\right\} \Rightarrow$

$\Rightarrow y = \sqrt{2}$

Therefore CP at $(1,\sqrt{2})$

Candidates for absolute max/min:

Edge ① $\rightarrow f(0,y) = 0$

Edge ② $\rightarrow f(x,0) = 0$

$\}\longrightarrow$ absolute minimum value $= 0$.

$(1,\sqrt{2}) \longrightarrow f(1,\sqrt{2}) = 2 \longrightarrow$ absolute maximum value $= 2$

**Problem 4.** (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$.

(a) State the $\varepsilon$ and $\delta$ definition of the fact that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(b) Show, using the $\varepsilon$ and $\delta$ definition of the limit, that

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin(xy) = 0.$$

(a) For any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x,y) - 0| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

(b) Let $\varepsilon > 0$ and $\delta = \sqrt{\varepsilon} > 0$

then, if $0 < \sqrt{x^2+y^2} < \delta$

$$|f(x,y) - 0| = |(x^2+y^2)\sin(xy)| \underset{|\sin(xy)| \le 1}{\le} x^2+y^2 < \delta^2 = \varepsilon$$

this shows $\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin(xy) = 0$

**Problem 5.** (15 points) Determine the set of points at which the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, where

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$a(x,y) = x^3$ continuous on $\mathbb{R}^2$

$b(x,y) = x^2+y^2$ continuous on $\mathbb{R}^2$ $\quad b(x,y) = 0 \iff x^2+y^2 = 0 \iff (x,y) = (0,$

therefore

$\dfrac{a(x,y)}{b(x,y)}$ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$. That is $f(x,y) = \dfrac{x^3}{x^2+y^2}$ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$

Checking continuity at the origin:

$$0 \leq \left|\frac{x^3}{x^2+y^2}\right| \leq |x| \cdot \frac{x^2}{x^2+y^2} \leq |x|$$

since $\dfrac{x^2}{x^2+y^2} \leq 1$

By the Squeeze theorem $\lim\limits_{(x,y)\to(0,0)} |f(x,y)| = 0$

Now $-|f(x,y)| \leq f(x,y) \leq |f(x,y)|$

$\to 0$ as $(x,y) \to (0,0)$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

i.e. $f$ is continuous at $(0,0)$

<u>Conclusion</u> $f$ is continuous on $\mathbb{R}^2$

**Problem 6.** (10 points) Let $f(x,y) = \sin\left(\frac{x}{y}\right) + \cos\left(\frac{y}{x^2}\right)$. Find $f_x(x,y)$ and $f_y(x,y)$.

$$f_x(x,y) = \frac{1}{y} \sin\left(\frac{x}{y}\right) - \sin\left(\frac{y}{x^2}\right) \cdot (-2)\frac{y}{x^3} = \frac{1}{y} \sin\left(\frac{x}{y}\right) + \frac{2y}{x^3} \sin\left(\frac{y}{x^2}\right)$$

$$f_y(x,y) = \cos\left(\frac{x}{y}\right) \cdot \left(-\frac{x}{y^2}\right) - \sin\left(\frac{y}{x^2}\right) \cdot \frac{1}{x^2}$$

**Problem 7.** (10 points) Find the directions in which the directional derivative of the function $f(x, y) = x^2 + \sin(xy)$ at the point $(1, 0)$ has the value 1.

$\nabla f(x,y) = \langle 2x + y\cos(xy) \ , x\cos(xy) \rangle$

$\nabla f(1,0) = \langle 2 , 1 \rangle$

$D_{\vec{u}} f(1,0) = \langle 2,1 \rangle \cdot \langle u_1, u_2 \rangle = 1$ where $\vec{u} = \langle u_1, u_2 \rangle$ is a unit vector

since $f$ is differentiable

$2u_1 + u_2 = 1 \Rightarrow u_2 = 1 - 2u_1$

then $u_1^2 + u_2^2 = u_1^2 + (1-2u_1)^2 = u_1^2 + 1 + 4u_1^2 - 4u_1 = 1 \Rightarrow$

$\Rightarrow 5u_1^2 - 4u_1 = 0 \Rightarrow u_1(5u_1 - 4) = 0 \Rightarrow \begin{cases} u_1 = 0 \\ u_2 = \frac{4}{5} \end{cases} \Rightarrow \begin{cases} u_2 = 1 \\ u_2 = -\frac{3}{5} \end{cases}$

therefore the two directions are:

$\langle 0, 1 \rangle$ , $\langle 4/5, -3/5 \rangle$

**Problem 8.** (10 points) Show that the tangent plane at any point on the cone $x^2 + y^2 = z^2$ passes through the origin.

let $(a,b,c) \in$ cone i.e $a^2+b^2=c^2$ (*)

the cone is a level surface of $F(x,y,z) = x^2+y^2-z^2$

The equation of the tangent plane to the cone at $(a,b,c)$:

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$$

$$\begin{cases} F_x(a,b,c) = 2a \\ F_y(a,b,c) = 2b \\ F_z(a,b,c) = -2c \end{cases}$$

therefore the eq of the tangent plane is:

$$2a(x-a) + 2b(x-b) - 2c(z-c) = 0$$

Check whether $(0,0,0)$ belongs to the tangent plane:

$$2a(0-a) + 2b(0-b) - 2c(0-c) \overset{?}{=} 0$$

$$-2[a^2+b^2-c^2] \overset{?}{=} 0 \quad \text{— yes because of (*)}$$

therefore $(0,0,0) \in$ tangent plane at the cone at $(a,b,c)$ for any $(a,b,c)$ on the cone.

**Problem 9.** (10 points) Show that the function $f(x,y) = \ln(x^2 + xy + y^2)$ satisfies

$$xf_x(x,y) + yf_y(x,y) = 2.$$

$$f_x = \frac{2x+y}{x^2+xy+y^2} \qquad f_y = \frac{2y+x}{x^2+xy+y^2}$$

then

$$x f_x(x,y) + y f_y(x,y) = \frac{2x^2+xy+xy+2y^2}{x^2+xy+y^2}$$

$$= \frac{2\cancel{(x^2+xy+y^2)}}{\cancel{x^2+xy+y^2}}$$

$$= 2$$

**Problem 10. (Bonus)** (10 points) Find $f_x(1,0)$ if $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ is given by

$$f(x,y) = x(x^2+y^2)^{-3/2} e^{\sin(x^2 y)}.$$

A solution different than anything presented on the exam:

$$f_x(1,0) = \lim_{h \to 0} \frac{f(1+h,0) - f(1,0)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)(1+h)^{-3} - 1}{h} = \lim_{h \to 0} \frac{\frac{1}{(1+h)^2} - 1}{h} =$$

$$= \lim_{h \to 0} \frac{1-(1+h)^2}{(1+h)^2 \cdot h} = \lim_{h \to 0} \frac{\cancel{1} - \cancel{1} - 2h - h^2}{(1+h)^2 \cdot h} = \lim_{h \to 0} \frac{-2-h}{(1+h)^2} = -2$$