Problem 1. (20 points) A particle moves through space according to the vector-valued function given by $\vec{r}(t) = (3t+1)\vec{i} + 2\vec{j} + (6t^2+1)\vec{k}$, where $-\infty < t < \infty$.

(a) Calculate the particle's velocity vector.

(b) Calculate the particle's acceleration.

(c) Calculate the curvature of the particle's trajectory.

(d) When is the curvature maximized and what is the maximum curvature?

(a)

$$\vec{v}(t) = \vec{r}\,'(t) = 3\vec{i} + 0\vec{j} + 12t\,\vec{k}$$

(b)

$$\vec{a}(t) = \vec{v}\,'(t) = 0\vec{i} + 0\vec{j} + 12\vec{k}$$

(c)

$$\mathcal{K}(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)|^3} = \frac{\left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 12t \\ 0 & 0 & 12 \end{vmatrix} \right\|}{(9+144t^2)^{3/2}}$$

$$= \frac{|\langle 3, 0, 12t \rangle \times \langle 0, 0, 12 \rangle|}{\left[\sqrt{9+(12t)^2}\,\right]^3}$$

$$= \frac{|0\vec{i} - \vec{j}\cdot 36 + 0\vec{k}|}{9^{3/2}\cdot(1+16t^2)^{3/2}} = \frac{36}{27\,(1+16t^2)^{3/2}} = \frac{4}{3\,(1+16t^2)^{3/2}}$$

(d)

$$\mathcal{K}'(t) = \frac{4}{3}\cdot\left(-\frac{3}{2}\right)\cdot\frac{32t}{(1+16t^2)^{5/2}}$$

$$\mathcal{K}'(0) = 0$$

		0	
$\mathcal{K}'(t)$	+ + + + +	0	− − − − − − −
$\mathcal{K}(t)$	$\nearrow$	$\mathcal{K}(0)$	$\searrow$

Therefore the curvature is maximized when $t=0$.

Maximum curvature

$$\mathcal{K}(0) = \frac{4}{3}$$

Problem 2. (10 points) Let $P = (0,0,0)$, $Q = (2,1,1)$ and $R = (-4,3,1)$.

(a) Find a normal vector to the plane which contains the triangle PQR.

(b) Find the area of the triangle PQR.

(c) Find the equation of the plane containing the triangle PQR.

$\vec{PQ} = \langle 2,1,1 \rangle \qquad \vec{PR} = \langle -4,3,1 \rangle$

(a) $\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = -2\vec{i} - 6\vec{j} + 10\vec{k}$

(b) Area triangle $PQR = \frac{1}{2}$ area parallelogram determined by $\vec{PQ}$ and $\vec{PR}$

$= \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2}\sqrt{4+36+100} = \sqrt{1+9+25} = \sqrt{35}$

(c) $-2x - 6y + 10z = 0.$

Problem 3. (5 points) Find the equation of the tangent line to the curve defined by the position vector $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ at the point $(2, 4, 8)$.

$\vec{r}\,'(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$ $\qquad$ $(t, t^2, t^3) = (2, 4, 8)$ when $t = 2$

therefore $\vec{r}\,'(2) = \vec{i} + 4\vec{j} + 12\vec{k}$

Parametric equations for the tangent line:

$$\begin{cases} x = 2 + t \\ y = 4 + 4t \\ z = 8 + 12t \end{cases}$$

Problem 4. (10 points) Find a parametric equation for the line L which passes through the point $(3, 1, -6)$ and is parallel to the intersection of the two planes $3x - 2y - 5z = 7$ and $4x + 3y + 5z = 2$.

vector parallel to the line of intersection of the two planes:

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -5 \\ 4 & 3 & 5 \end{vmatrix} = 5\vec{i} - \vec{j} \cdot 35 + \vec{k} \cdot 17$$

Parametric equation for L:

$$\begin{cases} x = 3 + 5t \\ y = 1 - 35t \\ z = -6 + 17t \end{cases}$$

Problem 5. Say $\vec{u}$ and $\vec{v}$ are vectors such that $|\vec{u}| > 0$ and $|\vec{v}| > 0$. Suppose also that $\vec{u} \times \vec{v} = \vec{v} \times \vec{u}$. What can you say about the angle between $\vec{u}$ and $\vec{v}$?

In general $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

Using the hypothesis we obtain $\vec{v} \times \vec{u} = -(\vec{v} \times \vec{u})$

Hence $\vec{v} \times \vec{u} = \vec{0}$ and $|\vec{v} \times \vec{u}| = |\vec{0}|$

$|\vec{v} \times \vec{u}| = |\vec{v}|\,|\vec{u}| \sin\theta$, $\theta \in [0, \pi]$ is the angle between $\vec{u}$ and $\vec{v}$

Since $|\vec{u}|\,|\vec{v}| \sin\theta = 0$, $|\vec{u}| > 0$, $|\vec{v}| > 0$ we have $\sin\theta = 0$ and hence $\theta \in \{0, \pi\}$.

Problem 6. (10 points) The position of a particle traveling in the three dimensional space is given by the equation $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$.

(a) Find the distance traveled by the particle from time $t = 0$ to the time $t = 10$.

(b) Where is the particle when it has traveled a distance of $\pi\sqrt{2}$?

(a)

$$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$L = \int_0^{10} \sqrt{2}\,dt = \sqrt{2} \cdot 10.$$

(b) $L(t_0)$ - traveled distance from $t = 0$ to the time t_0

$$L(t_0) = \int_0^{t_0} \sqrt{2}\,dt = \sqrt{2} \cdot t_0$$

$$L(t_0) = \pi\sqrt{2} \Rightarrow t_0 = \pi$$

$$\vec{r}(\pi) = \cos\pi\,\vec{i} + \sin\pi\,\vec{j} + \pi\vec{k} = -\vec{i} + \pi\vec{k}$$

this is where the particle is when it has traveled $\pi\sqrt{2}$.

Problem 7. (10 points) The planes $4x - 5y + z = 2$ and $4x - 5y + z = 3$ are parallel. Find the distance between them.

$\Pi_1 \quad 4x - 5y + z = 2 \qquad P_0(-1,-1,1)$ is a point on Π_1.

L - line through $P_0 \perp \Pi_2$

$$\begin{cases} x = -1 + 4t \\ y = -1 - 5t \\ z = 1 + t \end{cases}$$

$P_1 = L \cap \Pi_2 \qquad 4(-1+4t) - 5(-1-5t) + (1+t) = 3$

$$-4 + 16t + 5 + 25t + 1 + t = 3$$

$$2 + 42t = 3 \Rightarrow t = \frac{1}{42}$$

$$P_1\left(-1 + \frac{4}{42},\ -1 - \frac{5}{42},\ 1 + \frac{1}{42}\right)$$

$$\text{dist}(P_0, P_1) = \sqrt{\left(\frac{4}{42}\right)^2 + \left(\frac{5}{42}\right)^2 + \left(\frac{1}{42}\right)^2} = \frac{1}{42}\sqrt{42} = \frac{1}{\sqrt{42}}$$

Problem 8. (10 points) Let $\vec{r}(t) = (\cos^3 t)\vec{i} + (\sin^3 t)\vec{j} + \cos(2t)\vec{k}$, where $0 \le t \le \frac{\pi}{2}$.

(a) Find the unit tangent vector to $\vec{r}(t)$ at $t = \frac{\pi}{4}$

(b) Find the length of the curve.

(a) $\vec{r}\,'(t) = 3\cos^2 t\,(-\sin t)\vec{i} + 3\sin^2 t\cos t\,\vec{j} - 2\sin(2t)\vec{k}$

$|\vec{r}\,'(t)| = \sqrt{9\cos^4 t\cdot\sin^2 t + 9\sin^4 t\cos^2 t + 4\sin^2(2t)}$

$= \sqrt{9\sin^2 t\cos^2 t\,(\underbrace{\cos^2 t+\sin^2 t}_{1}) + 16\sin^2 t\cos^2 t} =$

$= \sqrt{25\sin^2 t\cos^2 t} = 5\,|\sin t\cos t| = 5\sin t\cos t$

since on $[0,\frac{\pi}{2}]$ both

sin and cos are ≥ 0

$|\vec{r}\,'(\frac{\pi}{4})| = 5\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{5}{2}$

$\vec{r}\,'(\pi/4) = -3\cdot\frac{1}{2\sqrt{2}}\vec{i} + 3\cdot\frac{1}{2\sqrt{2}}\vec{j} - 2\vec{k}$

$\vec{T}\,'(\frac{\pi}{4}) = -\frac{3}{5\sqrt{2}}\vec{i} + \frac{3}{5\sqrt{2}}\vec{j} - \frac{4}{5}\vec{k}$

(b) $L = \int_0^{\pi/2} |\vec{r}\,'(t)|\,dt = \int_0^{\pi/2} 5\sin t\cos t\,dt = \frac{5}{2}\int_0^{\pi/2} \sin 2t\,dt$

$= \frac{5}{2}\cdot\left.\frac{(-\cos 2t)}{2}\right|_0^{\pi/2} = \frac{5}{4} + \frac{5}{4} = \frac{5}{2}$

Problem 9. (10 points) Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

speed $(t) = |\vec{v}(t)|$ constant $\Rightarrow$ speed$^2(t) = |\vec{v}(t)|^2$ constant

therefore $\frac{d}{dt}|\vec{v}(t)|^2 = 0$

However $\frac{d}{dt}|\vec{v}(t)|^2 = \frac{d}{dt}\,\vec{v}(t)\cdot\vec{v}(t) = 2\vec{v}(t)\cdot\vec{v}\,'(t)$

$$= 2\vec{v}(t)\cdot\vec{a}(t)$$

Hence $2\vec{v}(t)\cdot\vec{a}(t) = 0$ i.e. $\vec{v}(t) \perp \vec{a}(t)$.

Problem 10. (10 points) Find an equation for the plane parallel to the plane $2x - y + 2z = -4$ if the point $(3, 2, -1)$ is equidistant from the two planes.

$\vec{n} = \langle 2, -1, 2 \rangle$

L line through $(3,2,-1)$ perpendicular on both planes

$$\begin{cases} x = 3 + 2t \\ y = 2 - t \\ z = -1 + 2t \end{cases}$$

Intersection of L with the first plane:

$2(3+2t) - (2-t) + 2(-1+2t) = -4$

$4t + t + 4t + 6 - 2 - 2 = -4 \quad 9t = -6 \Rightarrow t = -\frac{2}{3}$

Intersection of L with the second plane: $t = \frac{2}{3}$ since $(3,2,-1)$ (corresponding to $t=0$) is equidistant from the two planes.

$P\left(3 + \frac{4}{3},\ 2 - \frac{2}{3},\ -1 + \frac{4}{3}\right) = \left(\frac{13}{3}, \frac{4}{3}, \frac{1}{3}\right)$

Equation of the plane

$2\left(x - \frac{13}{3}\right) - \left(y - \frac{4}{3}\right) + 2\left(z - \frac{1}{3}\right) = 0$