SOLUTIONS PRACTICE PROBLEMS IV

Problem 1 Find the limit below if it exists. Justify your answer:

$$\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$$

Solution in a neighborhood of the point $(4,3)$ x, $y+1$ are positive numbers and we can rewrite the denominator as

$$x-y-1 = x-(y+1) = (\sqrt{x}-\sqrt{y+1})(\sqrt{x}+\sqrt{y+1})$$

therefore $\displaystyle\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\cancel{\sqrt{x}-\sqrt{y+1}}}{(\cancel{\sqrt{x}-\sqrt{y+1}})(\sqrt{x}+\sqrt{y+1})} =$

$$= \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{1}{\sqrt{x}+\sqrt{y+1}} = \frac{1}{\sqrt{4}+\sqrt{4}} = \frac{1}{4}$$

Problem 2 Find the limit below if it exists. Justify your answer

$$\lim_{(x,y,z)\to(0,0,0)} \cos\left(\frac{xy-yz+xz}{x^2+y^2+z^2}\right)$$

Solution

Approach $(0,0,0)$ along the x-axis

$$\lim_{(x,0,0)\to(0,0,0)} \cos\left(\frac{0}{x^2}\right) = 1$$

Approach $(0,0,0)$ along the first bisector in the (x,y) plane

$$\lim_{(x,0,x)\to(0,0,0)} \cos\left(\frac{x^2}{2x^2}\right) = \cos\frac{1}{2} \neq 1$$

Therefore the limit DNE

Problem 3

Find the limit below if it exists. Justify your answer

$$\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+x+y^2}$$

Solution

Approach $(0,0)$ along the x-axis: $\lim_{(x,0)\to(0,0)} \frac{2x}{x^2+x} = \lim_{x\to 0} \frac{2}{x+1} = 2$

Approach $(0,0)$ along the y-axis $\lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0 \neq 2$

therefore the limit DNE

Problem 4 Determine whether the function $f(x,y) = \begin{cases} xy\,\frac{x^2-y^2}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

is continuous at $(0,0)$. Find $f_x(0,0)$ and $f_y(0,0)$ if they exist.

Solution

We have $|f(x,y)| = |xy|\,\frac{|x^2-y^2|}{x^2+y^2}$

clearly $x^2-y^2 \le x^2+y^2$ and $y^2-x^2 \le x^2+y^2$. This implies $|x^2-y^2| \le x^2+y^2$

and $\frac{|x^2-y^2|}{x^2+y^2} \le 1$

therefore $0 \le |f(x,y)| \le |xy|$ and by the Squeeze Theorem

$$0 \le \lim_{(x,y)\to(0,0)} |f(x,y)| \le \lim_{(x,y)\to(0,0)} |xy| = 0.$$

this gives $\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$ Now $-|f(x,y)| \le f(x,y) \le |f(x,y)|$

and by the Squeeze theorem $f(x,y) \to 0$ as $(x,y)\to(0,0)$

Therefore $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ and f is continuous at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \frac{0-0}{h} = 0 .$$

Problem 5 Find the limit

$$\lim_{(x,y,z)\to(1,-1,0)} \frac{(\ln x)y - xz + 3}{2x - 2y + z}$$

Solution

$f(x,y,z) = (\ln x)y - xz + 3$ is a continuous function when $x>0$, $y \in \mathbb{R}$, $z \in \mathbb{R}$

$g(x,y,z) = 2x - 2y + z$ is a continuous function on $\mathbb{R}^3$

$$\lim_{(x,y,z)\to(1,-1,0)} f(x,y,z) = f(1,-1,0) = (\ln 1)(-1) - (1\cdot 0) + 3 = 3$$

$$\lim_{(x,y,z)\to(1,-1,0)} g(x,y,z) = g(1,-1,0) = 2 - 2(-1) + 0 = 4 \neq 0$$

therefore the quotient $\frac{f(x,y,z)}{g(x,y,z)}$ is a continuous function at $(1,-1,0)$

and

$$\lim_{(x,y,z)\to(1,-1,0)} \frac{(\ln x)y - xz + 3}{2x - 2y + z} = \frac{f(1,-1,0)}{g(1,-1,0)} = \frac{3}{4}$$

Problem 6 Find the limit

$$\lim_{(x,y)\to(3,4)} \frac{2x + 3y - 1}{\sqrt{x^2+y^2} - 5}$$

Solution

$$\lim_{\substack{x\to 3 \\ x<3}} \frac{2x + 12 - 1}{\sqrt{x^2+16} - 5} = -\infty$$

(the top approaches 17
the bottom approaches 0 with negative values)

$$\lim_{\substack{x\to 3\\ x>3}} \frac{2x+12-1}{\sqrt{x^2+16}-5} = +\infty$$ (the top approaches 17, the bottom approaches 0 with positive values)

Therefore $\lim_{(x,y)\to(3,4)} \dfrac{2x+3y-1}{\sqrt{x^2+y^2}-5}$ DNE

<u>Problem 7</u> Consider $f(x,y,z) = x - \sqrt{y^2+z^2}$. Find f_x, f_y, f_z

<u>Solution</u>

$f_x(x,y,z) = 1 \quad f_y(x,y,z) = -\frac{1}{2}(y^2+z^2)^{-1/2}\cdot 2y = \dfrac{-y}{\sqrt{y^2+z^2}}$

$$f_z(x,y,z) = -\frac{1}{2}(y^2+z^2)^{-1/2}\cdot 2z = \frac{-z}{\sqrt{y^2+z^2}}$$

<u>Problem 8</u> Determine the set of points at which the function is continuous

$$f(x,y) = \begin{cases} \dfrac{x^2y\cos y}{2x^2+y^2} & (x,y)\neq(0,0) \\ 1 & (x,y)=(0,0) \end{cases}$$

<u>Solution</u> Clearly f is continuous on $\mathbb{R}^2\setminus\{(0,0)\}$ (that is at all points in the plane different from the origin) as both top and bottom are continuous functions and $2x^2+y^2=0$ if and only if $x=0$ and $y=0$.

Checking continuity at $(0,0)$

$$0 \le |f(x,y)| \le \frac{|x^2y||\cos y|}{|2x^2+y^2|} \le \frac{|y|\,|x^2|}{2x^2+y^2} \le |y| \xrightarrow[\text{as }(x,y)\to(0,0)]{} 0$$

however $x^2 \le 2x^2+y^2$ and therefore $\dfrac{x^2}{2x^2+y^2} \le 1$

By the squeeze theorem

$\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$ and proceeding as described in the solution of problem 4 we conclude

$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \neq 1 = f(0,0)$. Therefore f is not continuous at $(0,0)$

<u>Conclusion</u> f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$