Solutions practice test

Problem 1

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & -2 & 1 \\ 3 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & -4 & 3 \\ 0 & -3 & 2 \end{vmatrix} = 1 \begin{vmatrix} -4 & 3 \\ -3 & 2 \end{vmatrix} = 1\cdot(-8+9) = -1 \neq 0$$

$\Rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ are linearly independent

Problem 2 From properties of matrices $(AB)^T = B^T A^T$
$\det A^T = \det A$

Ψ fundamental matrix of $\vec{x}' = A\vec{x} \iff \begin{cases} \dfrac{d}{dt}\Psi(t) = A\,\Psi(t) \\ \det \Psi(t) \neq 0 \end{cases}$ (*)

$\det \Psi(t) \neq 0 \Rightarrow \det \Psi(t)^T \neq 0 \Rightarrow \det\{\Psi(t)^T\}^{-1} \neq 0$

$\Psi^T.\{\Psi^T\}^{-1} = I_n$ and differentiate w.r.to. t: $\dfrac{d}{dt}\Psi^T.\{\Psi^T\}^{-1} + \Psi^T \dfrac{d}{dt}\{\Psi^T\}^{-1} = 0$

This implies

$$\Psi^T \frac{d}{dt}\{\Psi^T\}^{-1} = -\left(\frac{d}{dt}\Psi^T\right).\{\Psi^T\}^{-1} \underset{\text{because (*)}}{=} -(A\Psi(t))^T\cdot\{\Psi^T\}^{-1}$$

$$= -\Psi^T A^T \{\Psi^T\}^{-1}$$

Multiplying to the left by $\{\Psi^T\}^{-1}$ (which exists as $\det \Psi(t)^T \neq 0$)

we obtain

$$\begin{cases} \dfrac{d}{dt}\{\Psi^T\}^{-1} = -A^T\{\Psi^T\}^{-1} \\ \det\{\Psi^T\}^{-1} \neq 0 \end{cases}$$

This shows $\{\Psi(t)^T\}^{-1}$ is a fundamental matrix associated to the system
$\vec{x}' = -A^T\vec{x}$

Problem 3

$$\vec{x}\,' = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & -1 & 4 \end{pmatrix} \vec{x}$$

eigenvalues

$$\det \begin{pmatrix} 3-\lambda & -1 & 1 \\ 1 & 1-\lambda & 1 \\ 4 & -1 & 4-\lambda \end{pmatrix} = (3-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 4 & 4-\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 1-\lambda \\ 4 & -1 \end{vmatrix}$$

$$= (3-\lambda)\left((\lambda-4)(\lambda-1)+1\right) + \cancel{4}-\lambda-\cancel{4} + -1-4+4\lambda$$

$$= (3-\lambda)(\lambda^2-5\lambda+5) + 3\lambda-5$$

$$= 3\lambda^2 - 15\lambda + 15 - \lambda^3 + 5\lambda^2 - 5\lambda + 3\lambda - 5$$

$$= -\lambda^3 + 8\lambda^2 - 17\lambda + 10$$

$$= -(\lambda^3 - 8\lambda^2 + 17\lambda - 10) = -(\lambda-1)\cdot(\lambda^2 - 7\lambda + 10) = -(\lambda-1)(\lambda-2)(\lambda-5)$$

$$1 - 8 + 17 - 10 = 0$$

$$\begin{array}{l|l} \lambda^3 - 8\lambda^2 + 17\lambda - 10 & \lambda - 1 \\ \hline \lambda^3 - \lambda^2 & \lambda^2 - 7\lambda + 10 \\ \hline \quad -7\lambda^2 + 17\lambda & \\ \quad -7\lambda^2 + 7\lambda & \\ \hline \qquad 10\lambda - 10 & \end{array}$$

Real distinct eigenvalues $\lambda_1 = 1$ $\quad \lambda_2 = 2$ $\quad \lambda_3 = 5$

eigenvector $\sim \lambda_1 = 1$

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 4 & -1 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2y_1 - y_2 + y_3 = 0 \\ y_1 + y_3 = 0 \\ 4y_1 - y_2 + 3y_3 = 0 \end{cases}$$

$$\begin{cases} y_1 - y_2 = 0 \\ y_1 + y_3 = 0 \\ y_1 - y_2 = 0 \end{cases} \qquad y_1 = y_2 = -y_3$$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

eigenvector ~ 3

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

general solution

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

$$x(0) = \begin{pmatrix} c_1 + c_2 \\ c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{aligned} c_1 &= -c_2 \\ c_2 &= 1 \qquad c_1 = -1 \end{aligned}$$

Solution of the IVP

$$\vec{x}(t) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

Problem 5 $\quad \vec{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \vec{x}$

eigenvalues

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((\lambda-1)^2 + 4\right) = (1-\lambda)(\lambda^2 - 2\lambda + 5)$$

$$\lambda_1 = 1, \quad \lambda_2 = 1+2i, \quad \lambda_3 = 1-2i$$

Fundamental set

$$\vec{y} e^{t}, \quad e^{t}(\vec{a}\cos 2t - \vec{b}\sin 2t), \quad e^{t}(\vec{a}\sin 2t + \vec{b}\cos 2t)$$

$\vec{y}$ eigenvector associated to $\lambda_1 = 1$ $\qquad \vec{a} + i\vec{b}$ eigenvector ~ to $\lambda_2 = 1+2i$

eigenvector ~ $\lambda_1 = 1$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad \begin{aligned} y_1 - y_3 &= 0 \\ 3y_1 + 2y_2 &= 0 \end{aligned} \qquad \begin{pmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{pmatrix}$$

eigenvector ~ 2

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} y_1 - y_2 + y_3 &= 0 \\ 4y_1 - y_2 + 2y_3 &= 0 \end{aligned} \quad \Big| \; II - I$$

$$3y_1 + y_3 = 0 \qquad y_1 = 1 \quad y_3 = -3 \qquad \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

$$y_2 = y_1 + y_3 \qquad y_2 = -2$$

eigenvector ~ 5

$$\begin{pmatrix} -2 & -1 & 1 \\ 1 & -4 & 1 \\ 4 & -1 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} -2y_1 - y_2 + y_3 &= 0 \\ y_1 - 4y_2 + y_3 &= 0 \\ 4y_1 - y_2 - y_3 &= 0 \end{aligned}$$

$$\left.\begin{aligned} y_2 &= -2y_1 + y_3 \\ &= 4y_1 - y_3 \end{aligned}\right| \Rightarrow -6y_1 + 2y_3 = 0 \Rightarrow 3y_1 = y_3$$

$$y_1 = 1 \quad y_3 = 3$$

$$y_2 = -2 + 3 = 1$$

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

General solution

$$\vec{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}e^{2t} + c_3\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}e^{5t} \qquad c_1, c_2, c_3 \in \mathbb{R}$$

Problem 4 $\quad \vec{x}\,' = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

eigenvalues

$$\det\begin{pmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{pmatrix} = (\lambda-4)(\lambda-1) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$$

eigenvector ~ 2

$$\begin{pmatrix} 1-2 & 1 \\ -2 & 4-2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenvector ~ $x_2 = 1+2i$

$$\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} y_1 &= 0 \\ y_2 - iy_3 &= 0 \end{aligned} \qquad \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

General solution

$$c_1\begin{pmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{pmatrix} e^t + c_2\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\sin(2t)\right\}e^t + c_3\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\sin(2t) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\cos(2t)\right\}e^t$$

Problem 7

$$(x,y) = x_1\bar{y}_1 + x_2\bar{y}_2$$

$$|(x,y)|^2 = (x_1\bar{y}_1 + x_2\bar{y}_2)(\bar{x}_1 y_1 + \bar{x}_2 y_2) = |x_1|^2|y_1|^2 + x_1\bar{x}_2\bar{y}_1 y_2 + \bar{x}_1 x_2 y_1\bar{y}_2 + |x_2|^2|y_2|^2$$

$$\leq |x_1|^2|y_1|^2 + \underbrace{2|x_1||x_2||y_1||y_2|}_{\leq} + |x_2|^2|y_2|^2$$

$$|x_1|^2|y_2|^2 + |x_2|^2|y_1|^2$$ (in general $2ab \leq a^2+b^2$ for any a, b real numbers; in our case $a = |x_1||y_2|$ and $b = |x_2||y_1|$)

therefore $|(x,y)|^2 \leq |x_1|^2|y_1|^2 + |x_1|^2|y_2|^2 + |x_2|^2|y_1|^2 + |x_2|^2|y_2|^2$

$$= |x_1|^2\underbrace{(|y_1|^2 + |y_2|^2)}_{\|y\|^2} + |x_2|^2\underbrace{(|y_1|^2 + |y_2|^2)}_{\|y\|^2}$$

$$= \|y\|^2(|x_1|^2 + |x_2|^2) = \|x\|^2\|y\|^2$$

the inequality follows by taking square root of both sides.