Problem 1. (10 points) Find the volume of the region bounded above by the sphere $x^2+y^2+z^2=2$ and below by the paraboloid $z=x^2+y^2$.

Curve of intersection

$$\left.\begin{array}{l} z = x^2+y^2 \\ x^2+y^2+z^2 = 2 \end{array}\right. \Rightarrow z^2+z-2=0 \Rightarrow \left.\begin{array}{l} z=1 \\ z=-2 \\ z \geq 0 \end{array}\right\} \Rightarrow z=1$$

Intersection $\longrightarrow$ circle of center $(0,0,1)$ and radius 1

therefore $\text{Volume} = \iint_D \left(\int_{x^2+y^2}^{\sqrt{2-x^2+y^2}} dz \right) dA$ where $D = \{(x,y) \mid x^2+y^2 \leq 1\}$

Passing to polar coordinates:

$$\text{Volume} = \int_0^{2\pi} \int_0^1 \left(\int_{r^2}^{\sqrt{2-r^2}} dz \right) r\,dr\,d\theta = 2\pi \int_0^1 \left(\sqrt{2-r^2} - r^2 \right) r\,dr$$

$$= 2\pi \left[\int_0^1 \sqrt{2-r^2}\, r\,dr - \frac{1}{4} \right] = 2\pi \left[\int_{\sqrt{2}}^1 u\,(-u)\,du - \frac{1}{3} \right] = 2\pi \left[\frac{u^3}{3} \Big|_1^{\sqrt{2}} - \frac{1}{4} \right]$$

$$= 2\pi \left[\frac{2\sqrt{2}}{3} - \frac{7}{12} \right]$$

$2-r^2 = u^2 \Rightarrow -2r\,dr = -2u\,du$

Problem 2. (10 points) Evaluate

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$$

Polar coordinates

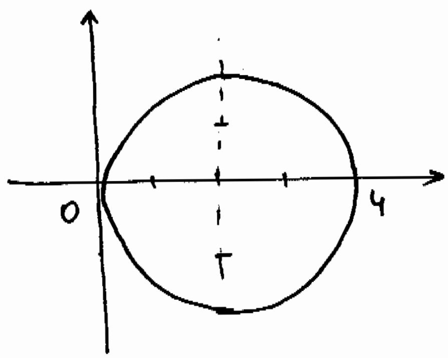
$\theta \in [0, \pi/2]$ $\quad r \in [0, 3]$ $\quad \sqrt{x^2+y^2} = r$

Jacobian: r

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^3 r^2\, dr\, d\theta = \frac{\pi}{2} \cdot \frac{r^3}{3} \Big|_0^3 = \frac{9\pi}{2}$$

Problem 3. (10 points) Evaluate

$$\iint_D (x^2+y^2)\,dA, \quad \text{where} \quad D = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 4x\}.$$



$$x^2+y^2 \le 4x \iff (x-2)^2+y^2 \le 4$$

Polar coordinates

$$x = r\cos\theta \qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$y = r\sin\theta$$

$$x^2+y^2 = r^2$$

Jacobian r

For a fixed θ $\quad r \in [0, 4\cos\theta]$ (indeed $x^2+y^2 = r^2 \le 4r\cos\theta = 4x \Rightarrow r \le 4\cos\theta$).

$$\int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta} r^2 \cdot r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{4^4}{4}\cdot\cos^4\theta\,d\theta = 64\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \Rightarrow \cos^4\theta = \frac{(1+\cos 2\theta)^2}{4} = \frac{1+2\cos 2\theta+\cos^2 2\theta}{4} =$$

$$= \frac{1+2\cos 2\theta}{4} + \frac{1+\cos 4\theta}{8}$$

Therefore

$$64\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta = 64\int_{-\pi/2}^{\pi/2}\frac{1+2\cos 2\theta}{4}\,d\theta + 64\int_{-\pi/2}^{\pi/2}\frac{1+\cos 4\theta}{8}\,d\theta$$

$$= \frac{64}{4}\cdot\pi + \underbrace{\frac{64}{2}\cdot\frac{\sin 2\theta}{2}\Big|_{-\pi/2}^{\pi/2}}_{0} + \frac{64}{8}\cdot\pi + \underbrace{\frac{64}{8}\cdot\frac{\sin 4\theta}{4}\Big|_{-\pi/2}^{\pi/2}}_{0}$$

$$= 16\pi + 8\pi = 24\pi$$

Problem 4. (10 points) Find $\int_C xyz\,ds$ where the curve C is described by the position vector

$$\vec{r}(t) = \left\langle t, \tfrac{2}{3}\sqrt{2t^3}, \tfrac{t^2}{2} \right\rangle, \quad \text{for} \quad t \in [0,1].$$

$$\vec{r}\,'(t) = \left\langle 1,\ \frac{2}{3}\,6t^2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2t^3}},\ t \right\rangle = \langle \underbrace{1}_{x'(t)},\ \underbrace{\sqrt{2t}}_{y'(t)},\ \underbrace{t}_{z'(t)} \rangle$$

$$\int_C xyz\,ds = \int_0^1 x(t)y(t)z(t)\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

$$= \int_0^1 \frac{t^3}{2} \cdot \frac{2}{3}\sqrt{2t^3} \cdot \underbrace{\sqrt{1+t^2+2t}}_{(t+1)^2}\,dt = \frac{\sqrt{2}}{3}\int_0^1 t^{4+\frac{1}{2}}(t+1)\,dt$$

$$= \frac{\sqrt{2}}{3}\int_0^1 t^{5+\frac{1}{2}}\,dt + \frac{\sqrt{2}}{3}\int_0^1 t^{4+\frac{1}{2}}\,dt$$

$$= \frac{\sqrt{2}}{3}\left[\frac{1}{6+\frac{1}{2}} + \frac{1}{5+\frac{1}{2}}\right] = \frac{2\sqrt{2}}{3}\left[\frac{1}{13}+\frac{1}{11}\right]$$

Problem 5. (10 points) Find the mass of a ball B given by $x^2 + y^2 + z^2 \leq 9$ if the density at any point is proportional to its distance from the origin.

$$\rho(x,y,z) = k\sqrt{x^2+y^2+z^2} \qquad k>0$$

Then

$$m = \iiint_E \rho(x,y,z)\,dV = k\iiint_E \sqrt{x^2+y^2+z^2}\,dV$$

Pass to spherical coordinates

$$x = \rho\sin\phi\cos\theta \qquad \sqrt{x^2+y^2+z^2} = \rho$$
$$y = \rho\sin\phi\sin\theta \qquad \text{Jacobian } \rho^2\sin\phi$$
$$z = \rho\cos\phi$$

$$k\int_0^{2\pi}\int_0^{\pi}\int_0^{3} \rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = k2\pi\cdot\int_0^{\pi}\frac{3^4}{4}\cdot\sin\phi\,d\phi$$
$$= k\frac{81\pi}{2}\cdot(-\cos\phi)\Big|_0^{\pi} = k\frac{81\pi}{2}[1+1]$$
$$= 81\pi k$$

Problem 6. (10 points) Evaluate the integral

$$\iint_R \frac{x+5y}{2x-y}\,dA,$$

where R is the parallelogram enclosed by the lines $x+5y=0$, $x+5y=5$, $2x-y=1$, and $2x-y=6$.

Let $\left.\begin{array}{l} u = x+5y \\ v = 2x-y \end{array}\right/ \Rightarrow \frac{u+5v}{11} = x \qquad \frac{2u-v}{11} = y$

The Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{11} & \frac{5}{11} \\ \frac{2}{11} & -\frac{1}{11} \end{vmatrix} = \left(-\frac{1}{11^2}\right) - \frac{10}{11^2} = \frac{-11}{11^2} = -\frac{1}{11}$$

Then $$\iint_R \frac{x+5y}{2x-y}\,dA = \iint_S \frac{u}{v} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA \quad \text{where } S = [0,5]\times[1,6]$$

$$= \int_0^5\int_1^6 \frac{u}{v}\cdot\left|\frac{1}{11}\right| dv\,du = \frac{1}{11}\int_0^5 u \ln 6\,du = \frac{25}{22}\ln 6$$

Problem 7. (10 points) Find the area of the part of the of the plane $2x + 5y + z = 10$ that lies inside the cylinder $y^2 + z^2 = 9$.

$$x = \frac{10 - 5y - z}{2}$$

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1 + \left(\frac{5}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \, r \, dr \, d\theta = 2\pi \cdot \int_0^3 \sqrt{\frac{4 + 25 + 1}{4}} \, r \, dr$$

$$= \sqrt{30} \, \pi \, \frac{9}{2} = \frac{9\pi\sqrt{30}}{2}$$

Problem 8. (10 points) Find the work done by the force field

$$\vec{F}(x,y,z) = z\vec{i} + x\vec{j} + y\vec{k},$$

in moving a particle from the point $(3,0,0)$ to the point $(0,\frac{\pi}{2},3)$ along:

(a) the segment joining these points.

(b) the helix described by $\vec{r}(t) = (3\cos t)\vec{i} + t\vec{j} + (3\sin t)\vec{k}$.

(a) $\vec r(t) = (1-t)\langle 3,0,0\rangle + t\langle 0,\frac{\pi}{2},3\rangle \qquad t\in[0,1]$

$= \langle 3-3t, \frac{\pi}{2}t, 3t\rangle \qquad \vec r\,'(t) = \langle -3, \frac{\pi}{2}, 3\rangle$

$$W_{\text{Segment}} = \int \vec F\cdot d\vec r = \int_0^1 \vec F(\vec r(t))\cdot \vec r\,'(t)\,dt = \int_0^1 \langle 3t, 3-3t, \tfrac{\pi}{2}t\rangle\cdot\langle -3,\tfrac{\pi}{2},3\rangle\, dt$$

$$= \int_0^1 -9t + \frac{3\pi}{2} - \cancel{\frac{3\pi}{2}t} + \cancel{\frac{3\pi}{2}t}\,dt = \frac{3\pi}{2}-\frac{9}{2}$$

(b) $\vec r(t) = (3\cos t)\vec i + t\vec j + (3\sin t)\vec k$ at $(0,\frac{\pi}{2},3)$ when $t=\frac{\pi}{2}$

at $(3,0,0)$ when $t=0$.

$\vec r\,'(t) = \langle -3\sin t, 1, 3\cos t\rangle$

$$W_{\text{helix}} = \int \vec F\cdot d\vec r = \int_0^{\pi/2} \vec F(\vec r(t))\cdot \vec r\,'(t)\,dt = \int_0^{\pi/2} \langle 3\sin t, 3\cos t, t\rangle\cdot\langle -3\sin t, 1, 3\cos t\rangle\, dt$$

$$= \int_0^{\pi/2} -9\sin^2 t + 3\cos t + 3t\cos t\,dt = \int_0^{\pi/2} -9\,\frac{1-\cos 2t}{2} + 3\cos t + 3t\cos t\,dt$$

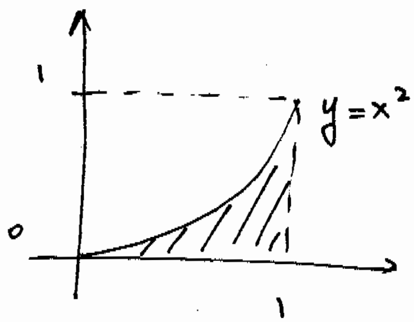
$$= -\frac{9\pi}{2\cdot 2} + \underbrace{\frac{9}{2}\,\frac{\sin 2t}{2}\Big|_0^{\pi/2}}_{0} + 3\sin t\Big|_0^{\pi/2} + \underbrace{3\int_0^{\pi/2} t\cos t\,dt}_{\text{integration by parts}} =$$

$$= -\frac{9\pi}{2\cdot 2} + 3 + 3t\sin t\Big|_0^{\pi/2} - 3\int_0^{\pi/2}\sin t\,dt = -\frac{9\pi}{2\cdot 2} + 3 + \frac{3\pi}{2} + 3\cos t\Big|_0^{\pi/2}$$

$$= -\frac{9\pi}{4} + \frac{6\pi}{4} = -\frac{3\pi}{4}$$

Problem 9. (10 points) Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy.$$



By Fubini's theorem

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy =$$

$$= \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx = \int_0^1 x^2 \sqrt{x^3+1}\, dx =$$

$x^3+1 = u \Rightarrow 3x^2 dx = du$

$$= \frac{2}{3}\frac{1}{3} (x^3+1)^{3/2} \Big|_0^1 = \frac{2}{3^2}\left[2^{3/2} - 1\right] = \frac{2}{9}\left[2\sqrt{2} - 1\right]$$

Problem 10. (10 points) Use spherical coordinates to evaluate the integral

$$\iiint_E z\, dV$$

where E is the solid lying between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$ in the first octant.

$x = \rho \sin\phi \cos\theta$ $\quad \rho \in [1,3]$

$y = \rho \sin\phi \sin\theta$ $\quad \theta \in [0, \frac{\pi}{2}]$

$z = \rho \cos\phi$ $\quad \phi \in [0, \frac{\pi}{2}]$ $\Rightarrow$

Jacobian $\rho^2 \sin\phi$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^3 \rho^3 \sin\phi \cos\phi \, d\rho\, d\phi\, d\theta = \frac{\pi}{2}\int_0^{\pi/2}\left[\frac{3^4}{4} - \frac{1}{4}\right]\cdot \frac{\sin 2\phi}{2}\, d\phi$$

$$= \frac{\pi}{4}\,\frac{80}{4}\cdot \underbrace{\frac{(-\cos 2\phi)}{2}\Big|_0^{\pi/2}}_{1} = 5\pi$$