The four [10,5,4] binary codes

1 Preliminaries

There are four distinct $[10,5,4]$ binary codes. We shall prove this in a moderately elementary way, using the MacWilliams identities as the main tool. (For the outline of a computer proof, see [2].) Throughout the note, the codes are assumed to be linear codes over the binary field $\mathbb{F}_2$, the **alphabet** for the codes. If such a code is a k-dimensional subspace of the **ambient space** $\mathbb{F}_2^n$ of codes of length n, it is called an $[n,k]$ code. For a word c in $\mathbb{F}_2^n$, the **support** $\mathrm{supp}(c)$ of c is the set of coordinate positions at which c has nonzero entries, and the **weight** $wt(c)$ of c is $|\mathrm{supp}(c)|$. When the smallest nonzero weight among the words of an $[n,k]$ code (the codewords) is d, the code is labeled as an $[n,k,d]$ code. Informally, we shall refer to a word of weight w as "a w," and since the field is binary, we shall often identify $\mathrm{supp}(c)$ with c itself. The support $\mathrm{supp}(C)$ of a code C in $\mathbb{F}_2^n$ is the union of the supports of its words, and the size $|\mathrm{supp}(C)|$ is the **support length** $n(C)$ of C.

Here are the **MacWilliams identities**: let C be an $[n,k]$ code. Thet **dual code** $C^\perp$ of C is the set of words $a \in \mathbb{F}_2^n$ for which $a \cdot c = 0$ for all $c \in C$ (where $a \cdot c$ is the standard dot product, the sum of the products of the coordinates of a and c). The members of $C^\perp$ show the dependencies of the coordinate functions on C; that is, if $b \in C^\perp$, then the sum of the coordinates in $\mathrm{supp}(b)$ is 0 on C. Let A_i be the number of words in C of weight i, and let B_i be the number of words of $C^\perp$ of weight i (standard abbreviations; if the code needs indicating, one writes something like $A_i(C)$). Then for $0 \le m \le n$,

$$\sum_{i=m}^{n} \binom{i}{m} A_i = 2^{k-m} \sum_{j=0}^{m} \binom{n-j}{m-j} (-1)^j B_j. \tag{1}$$

There are many proofs of these equations. For example, with proper choice of functions, they are immediate from the Poisson summation formula for discrete Fourier transformations. There are several combinatorial proofs involving useful counting arguments. For some of these proofs and the generalization to other alphabets, see the texts by Huffman and Pless [5] and Hill [4].

At $m = 1$, the right side of (1) is $2^{k-1}(n - B_1)$; as B_1 is the number of coordinate positions at which all the words of C have 0 entries, $n - B_1 = n(C)$. Thus the equation reads

$$\sum_{i=1}^{n} iA_i = \sum_{c \in C} wt(c) = 2^{k-1} n(C). \tag{2}$$

This means the average codeword weight is $n(C)/2$, and for that reason equation (2) is sometimes called the **Average Weight Equation** (AWE). In particular, if a and b are different nonzero words in an $[n,k,d]$ code, then

$$wt(b) \le 2n(\langle a,b \rangle) - d - wt(a) \le 2n - d - wt(a). \tag{3}$$

If X is a set of coordinate positions, the **shortened** code C_0 of an $[n, k]$ code C at X is the the code of length $n - |X|$ obtained by deleting the entries in X from the codewords in C having 0s at the positions in X. The dimension of C_0 is at least $k - |X|$, more if there are dependencies among the coordinates in X. In particular, when $X = \text{supp}(b)$ for some $b \in C^{\perp}$, then $\dim(C_0) \geq k - wt(b) + 1$. So if one knew that there was no $[n - wt(b), k - wt(b) + 1, d]$ code, one could conclude that $B_{wt(b)} = 0$.

A final item of importance is the **Griesmer bound**: for an $[n, k, d]$ code,

$$n \geq d + \left\lceil \frac{d}{2} \right\rceil + \cdots + \left\lceil \frac{d}{2^{k-1}} \right\rceil .$$

For a proof, see [5, Theorem 2.7.4].

2 The $[10, 5, 4]$ codes

Now we specialize to $[10, 5, 4]$ codes (the Griesmer bound shows that the minimum weight of a $[10, 5]$ code is at most 4). First of all, there is no $[9, 5, 4]$ code: since the parameters $[8, 5, 4]$, $[7, 4, 4]$, $[6, 3, 4]$, and $[5, 2, 4]$ all violate the Griesmer bound, $B_1 = B_2 = B_3 = B_4 = 0$ for such a code. Thus the right sides of the MacWilliams identities (1) for $m = 0, \ldots, 4$ are all independent of the code. That yields five equations which can be solved for $A_4, \ldots, A_8$ in terms of A_9. In particular, $A_8 = -7 - 5A_9$, which is impossible because the A_i are nonnegative. This nonexistence of a $[9, 5, 4]$ code means that for a $[10, 5, 4]$ code, $B_1 = 0$. In dealing with the $[10, 5, 4]$ codes, we shall consider various possibilities for certain word weights. Let C be a $[10, 5, 4]$ code from here on.

2.1 Code 1

To begin with, suppose that $B_2 > 0$ and let $b \in C^{\perp}$ have weight 2 (b is a 2 in $C^{\perp}$). Let C_0 be the $[8, 4, 4]$ code obtained by shortening C at supp(b). This code C_0 is well-known, famous, and unique; it is the **first order Reed-Muller code** $R(1, 3)$ [5, Section 1.10]. Here "unique" means that any two such codes can be made the same by permuting the coordinates of one to produce the other. The uniqueness is quite easy: set up a **generator matrix** G_0 for C_0, a matrix of codewords whose row space is the code, in the form $G_0 = [I_4|M]$, M being a 4×4 matrix like the identity matrix I_4 (permuting some coordinates if needed). Then the rows of M must each contain three 1s, and two rows cannot be the same without giving a word of weight 2 in C_0 as the sum of the corresponding rows of G_0. With some further permuting, G_0 must be

$$G_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} .$$

The code C_0 is **self-dual**: $C_0^\perp = C_0$. In particular, any three coordinates are independent, because there is no 3 in $C^\perp$. So they can be chosen to serve in the role of the first three coordinates here, with appropriate permuting of the remaining positions. Thus the **group of the code**, the permutation group of the coordinate positions leaving the code as a whole invariant, is triply transitive.

Now we can set up a generator matrix for C by extending G_0 with two columns of 0s for the first four rows and selecting an additional word in C for the fifth row (always making appropriate permutations). Since the last two positions will be the support of b, this fifth word has two 1s there. We claim that it can be chosen to have weight 4, that is, that its projection onto the first eight positions be of weight 2. For if not, the code obtained from C by deleting the last coordinate (a **punctured** code) would be a $[9, 5, 4]$ code, which, as we saw, does not exist. The triple transitivity (double is enough) of the group of C_0 allows us to take the two 1s in the first eight positions for a weight 4 fifth row to be any two we want. Thus with appropriate permutation, a generator matrix for C must be

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

and of course this shows that such a code would be be unique. The code generated by this matrix is indeed a $[10, 5, 4]$ code, and the weight enumerators for C and $C^\perp$ are

$i =$	0	1	2	3	4	5	6	7	8	9	10
$A_i =$	1	0	0	0	18	0	8	0	5	0	0
$B_i =$	1	0	1	0	6	16	6	0	1	0	1

From here on, then, we take $B_2 = 0$.

2.2 Code 2

Suppose first that $A_{10} \neq 0$; then necessarily $A_{10} = 1$. If u is the corresponding all-1 word in C, then for $c \in C$, $u + c \in C$ is the "complement" of c obtained by switching 0s and 1s in c. Thus $A_6 = A_4$ and $A_7 = A_8 = A_9 = 0$. Solving the first three MacWilliams equations along with these additional relations gives $A_4 = A_6 = 15$ and $A_5 = 0$. The weight distributions are

$i =$	0	1	2	3	4	5	6	7	8	9	10
$A_i =$	1	0	0	0	15	0	15	0	0	0	1
$B_i =$	1	0	0	0	15	0	15	0	0	0	1

Thus C is **formally self-dual**: C and $C^\perp$ have the same distributions. By the famous **Assmus-Mattson theorem** [5, Theorem 8.4.2], the 4s (their supports, really) are the **blocks** of a 2-$(10, 4, 2)$ design: each pair of coordinate positions is in exactly two 4s (Chapter 8 of [5] has an introduction to design theory). We can see this fact directly, however: first, two different 4s cannot meet in three positions, otherwise their sum would have weight 2. Thus each pair of coordinate positions appears in at most two 4s. But there are $\binom{10}{2} = 45$ pairs, and the 4s show $15 \times \binom{4}{2} = 90$ pairs. So each pair must appear exactly twice. Each position is then in six 4s. To set up a generator matrix for C, start with two 4s meeting in just one position, x; only three 4s meet a 4 containing x in x and another position. The complement of the sum of the two selected 4s is a third 4 meeting each of the other two just in x. On permuting, we see this array for the three 4s:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Now start three more rows of 4s meeting the first 4 in pairs as shown:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & & & & & & \\ 1 & 0 & 1 & 0 & & & & & & \\ 1 & 0 & 0 & 1 & & & & & & \end{bmatrix}.$$

Each of the last three rows has two more 1s. The two extra 1s for one of the rows cannot be among the 1s of row 2 or of row 3 without displaying two 4s meeting in three positions; and two 1s from different rows cannot be in the same position, since otherwise two positions are in three 4s. Up to permutation, then, there is only one way to complete the array:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Of course these rows are dependent; but they correctly span a $[10, 5, 4]$ code with the properties hypothesized. Again, the code is unique. Incidentally, it is not genuinely self-dual.

2.3 Code 3

In addition to the assumption that $B_2 = 0$, suppose now that $A_{10} = 0$. First assume that $A_9 > 0$. By (3), if a is a 9 and $b \neq a, 0$, then $wt(b) \leq 20 - 9 - 4 = 7$; in particular, $A_9 = 1$. Now in our binary case,

$$wt(c_1 + c_2) = wt(c_1) + wt(c_2) - 2\left|\text{supp}(c_1) \cap \text{supp}(c_2)\right|. \tag{4}$$

So if $wt(c_1)$ is odd, then $wt(c_2)$ and $wt(c_1+c_2)$ have opposite parity. Thus half the words of C have odd weight: $A_5 + A_7 + A_9 = 16$. On solving the first three MacWilliams equations along with the two additional relations just obtained, we find the distributions for C and $C^\perp$ to be

$$\begin{array}{rccccccccccc} i = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ A_i = & 1 & 0 & 0 & 0 & 13 & 9 & 2 & 6 & 0 & 1 & 0 \\ B_i = & 1 & 0 & 0 & 1 & 9 & 12 & 6 & 1 & 0 & 2 & 0 \end{array}$$

But those two 9s in $C^\perp$, in the presence of minimum weight 3 for $C^\perp$, contradict (3). Thus $A_9 = 0$.

If we continue to assume that there are words of odd weight in C, so that $A_5 + A_7 = 16$, we can solve as before and obtain the following distributions in terms of A_8:

$$\begin{array}{rcccccc} i = & 0 & 1 & 2 & 3 & 4 & 5 \\ A_i = & 1 & 0 & 0 & 0 & 15 - A_8 & 6 + 2A_8 \\ B_i = & 1 & 0 & 0 & (5 - A_8)/2 & 5 + A_8 & (27 + A_8)/2 \end{array}$$

$$\begin{array}{rccccc} i = & 6 & 7 & 8 & 9 & 10 \\ A_i = & 0 & 10 - 2A_8 & A_8 & 0 & 0 \\ B_i = & 10 - 2A_8 & (A_8 - 5)/2 & A_8 & (5 - A_8)/2 & 0 \end{array}$$

Then $B_3, B_7 \geq 0$ force $A_8 = 5$, giving the distributions

$$\begin{array}{rccccccccccc} i = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ A_i = & 1 & 0 & 0 & 0 & 10 & 16 & 0 & 0 & 5 & 0 & 0 \\ B_i = & 1 & 0 & 0 & 0 & 10 & 16 & 0 & 0 & 5 & 0 & 0 \end{array}$$

Again we have a formally self-dual code. If a and b are two different 8s in C sharing 0s, then $n(\langle a, b \rangle) = 9$ and (3) is contradicted. So the five 8s could be taken as the first rows of a generator matrix (their sum is 0). An additional row must show that B_2 is 0, and the only way to do this is by having one 1 and one 0 under the pair of 0s in each 8. Thus the easy unique generating matrix is

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

(up to coordinate permutations, as usual). To get $C^\perp$, switch the 0s and 1s in the last row; having words of odd weight, C is not genuinely self-dual. This matrix indeed gives a code with the desired weight distribution.

2.4 Code 4

Finally, we assume that $B_2 = 0$ and $A_9 = A_{10} = 0$, and that C has no words of odd weight. The MacWilliams identities give the weight distributions

$i =$	0	1	2	3	4	5	6	7	8	9	10
$A_i =$	1	0	0	0	16	0	12	0	3	0	0
$B_i =$	1	0	0	2	7	12	7	2	0	0	1

If b_1 and b_2 are the two 3s in $C^\perp$, they cannot share a 1; for if they did, they would share just one 1 (to avoid a sum of weight 2; $C^\perp$ is a $[10, 5, 3]$ code). Then shortening C on the union of their supports would produce a $[5, 2, 4]$ code not conforming to the Griesmer bound. In addition, two different 8s in C cannot share 0s, just as before. So the three 8s can be taken as the first three rows of a generator matrix, which after permutation looks like

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The first three columns are at the support of b_1 and the second three support b_2. Any codeword c must have an even number of 1s in each of these supports to give $b_1 \cdot c = b_2 \cdot c = 0$. The sum of the three 8s is f, a 4 supported on the last four columns. Keeping f in view, one sees that two additional rows for a generator matrix can be taken to have 0s in the first three positions and two 1s in the second three, those not being in the same two spots. Adjusting by f, one sees two 1s in the last four positions for each of these two rows, the rows sharing one 1 there. The upshot is the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and the code exists and is unique.

3 Alternative presentations

Up to coordinate permutations, every $[10, 5, 4]$ code has a generator matrix of the form $[I_5|A]$. The computer search in [2] looked for the possibilities for A. A generator matrix for the dual code is then $[A^\intercal|I_5]$. However, in this section we shall look for some other forms of generator matrices.

3.1 Code 1

Both code 1 and code 3 have $A_8 = 5$; the five 8s were used to set up a generator matrix for code 3. The same could be done with code 1, since as for code 3, two 8s cannot have a common 0. But for code 1, we have to see $B_2 = 1$. That means a word not in the span of the 8s should have both 0 and 1 under all pairs of the 0s in the 8s except for one. This suggests the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Indeed, this generates a $[10, 5, 4]$ code with the weight distribution of code 1.

3.2 Code 2

This code was studied by means of a 2-(10,4,2) design. In general, a design can be described by its **incidence matrix** M: this is a matrix with rows indexed by the b blocks and columns indexed by the v points. In row i and column j there is a 1 if point j is in block i and a 0 if not. Matrix M looks like a generator matrix for a binary code, and this resemblance has been elaborated on by many people over the years; see the book by Assmus and Key [1]. One speaks of the code with generator matrix M as the code of the design. The alphabet does not need to be $\mathbb{F}_2$, but certain alphabets expose more details than others. There are three 2-$(10, 4, 2)$ designs, given explicitly in Table 1.25 of the *Handbook of Combinatorial Designs* [3]. The ranks of their incidence matrices over $\mathbb{F}_2$ (the **2-ranks**) are 6, 7, and 5. Thus only the third design corresponds to a $[10, 5]$ binary code, and indeed, the code is code 2.

Code 2 also has a generator matrix which is a **double circulant** of the form $[I_5|A]$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

3.3 Code 3

We can create a $[10, 4, 4]$ code by **replicating** a $[5, 4, 2]$ code–taking a generator matrix G_0 for the shorter code and using the matrix $G = [G_0|G_0]$. For the $[5, 4, 2]$ code, take the **even subcode** of $\mathbb{F}_2^5$, the set of words of even weight; this is a linear code by (4) read modulo 2. To get a $[10, 5, 4]$ code, we need one more word, and the word of weight 5 supported on the first half of G works:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Code 3 is the only one with words of odd weight. In fact, another approach to classifying the codes would be to start with those having words of even weight only. The presence of 5s in code 3 also suggests a double circulant:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and that does generate code 3. Incidentally, the other guess for a double circulant,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix},$$

produces code 2.

3.4 Code 4

One minor variation for producing this code comes from the fact that the shortening on the support of a 3 in $C^\perp$ is a $[7, 3, 4]$ code. This is a **simplex code** [5, Section 1.8]. Up to permutations, a generator matrix H for it can be taken in just one way:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

The group of this code is doubly transitive. There is visibly a 3 in its dual, supported in positions 4 to 6. Permute the columns of H to bring that 3 to the left:

$$H' = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

If we start with the matrix of three 8s for code 4 as before, and expand H' by three columns of 0s at the left end, we can stack the two matrices to get code 4, having arranged that both of the matrices now have a 3 orthogonal to their rows supported in positions 4 to 6. (The ending block of 1s in the 8s allows us to arrange the last four columns of H' any way we wish.) The result is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and that works.

References

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