Adaptive Control Design and Analysis

(Supplemental Notes)

Gang Tao

Department of Electrical and Computer Engineering
University of Virginia
Charlottesville, VA 22903

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Contents

Matrix Theory

Symmetric Matrices

For a matrix $M = M^T$, we have $M = \sum_{i=1}^{n} \lambda_i e_i e_i^T$ where λ_i and e_i are the eigenvalues and eigenvectors of M such that $e_i^T e_i = 1$ and $e_i^T e_j = 0$ with $i \neq j$. With $P = [e_1, e_2, \ldots, e_n]$ and $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, it follows that $M = P\Lambda P^T$, where $PP^T = P^T P = I$, and, in addition, for M nonsingular, that $M^{-1} = P\Lambda^{-1}P^T$.

For $M = M^T \geq 0$, we define $M^{1/2} = \sum_{i=1}^{n} \sqrt{\lambda_i} e_i e_i^T = P\Lambda^{1/2}P^T$ and express $M = M^{1/2}M^{1/2}$, where $(M^{1/2})^T = M^{1/2}$. It also follows that for M nonsingular, $(M^{1/2})^{-1} = \sum_{i=1}^{n} \frac{1}{\sqrt{\lambda_i}} e_i e_i^T = P\Lambda^{-1/2}P^T$. On the other hand, for $Q = P\Lambda^{1/2}$, we have $M = QQ^T$ (as compared with $M = M^{1/2}M^{1/2}$ for $(M^{1/2})^T = M^{1/2} = P\Lambda^{1/2}P^T$.

Radial Unboundedness Condition on $V(x)$ for Asymptotic Stability

One can draw the surface plots of $V(x) = c$, for different values of c, with

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2. \tag{1}$$

For this $V(x)$, such surface plots are closed curves for $c < 1$ but are open curves for $c > 1$, as $V(x)$ is not radially unbounded.

One can draw the phase-plane plot of $\dot{x} = f(x)$, by obtaining its numerical solutions for some typical initial conditions, for

$$\dot{x}_1 = -\frac{6x_1}{(1 + x_1^2)^2} + 2x_2,\ \dot{x}_2 = -\frac{2(x_1 + x_2)}{(1 + x_1^2)^2}. \tag{2}$$

For some initial conditions, the solution trajectories do not converge to the origin. (Are there any trajectories going to ∞?)

One can draw the vector field of $\dot{x}_1 = -\frac{6x_1}{(1+x_1^2)^2} + 2x_2$, $\dot{x}_2 = -\frac{2(x_1+x_2)}{(1+x_1^2)^2}$.

The solution $x = [x_1, x_2]^T$ of this system on the hyperbola $x_2 = \frac{2}{x_1 - \sqrt{2}}$ satisfies

$$g_1(x) = \frac{\dot{x}_2}{\dot{x}_1} = -\frac{1}{1 + 2\sqrt{2}x_1 + 2x_1^2}, \tag{3}$$

while the slope of this hyperbola is

$$g_2(x) = \frac{dx_2}{dx_1} = -\frac{1}{1 - 2\sqrt{2}x_1 + \frac{x_1^2}{2}}. \tag{4}$$

It follows that $0 > g_1(x) > g_2(x)$ for $x_1 > \sqrt{2}$ so that the trajectories to the right of the hyperbola branch in the first quadrant cannot cross the branch.

This example shows that for asymptotic stability, the radial unboundedness of $V(x)$ is a crucial condition [179], [351], [426].

Passivity of A Mass-Damper-Spring System

Consider a mass-damper-spring mechanical system with equation of motion

$$M\ddot{x} + D\dot{x} + Kx = F, \tag{5}$$

where M is the mass, D is the damping constant, K is the spring constant, x is the mass position, and F is the force acting on the mass.

The system energy is

$$V(x, \dot{x}) = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Kx^2 \tag{6}$$

whose time derivative is

$$\frac{d}{dt}V(x, \dot{x}) = F\dot{x} - D\dot{x}^2. \tag{7}$$

Over any time interval $[0, T]$, it follows that

$$V(x(T), \dot{x}(T)) = V(x(0), \dot{x}(0)) + \int_0^T F(t)\dot{x}(t)\,dt - \int_0^T D\dot{x}^2(t)\,dt. \tag{8}$$

Since $D \geq 0$, we have

$$-\int_0^T F(t)\dot{x}(t)\,dt \leq V(x(0), \dot{x}(0)), \tag{9}$$

which means that the energy extracted from the system is less than or equal to the initial system energy. From (7), the term $F\dot{x}$ clearly represents the system absorbed power from the input force F. For passivity analysis, the product of system input and output is defined as such power. In this sense we consider the velocity $v = \dot{x}$ as system output. Then, the system admittance (the reciprocal of impedence) is

$$G(s) = \frac{V(s)}{F(s)} = \frac{s}{Ms^2 + Ds + K}, \tag{10}$$

which is positively real (the mechanical-electric analogy pairs are “force vs. voltage” and “velocity vs. current”).

If we consider x as system output, the term Fx does not represents system power (i.e., $\int_0^T F(t)x(t)\,dt$ does not represents system energy) so that the passivity analysis is not applicable. In other words, the system transfer function for passivity analysis in terms of positive realness is defined in terms of system impedence (or admittance) relating curent (velocity) to voltage (force) (or voltage (force) to curent (velocity)).

Without the passivity property, the transfer function from the input force to the output position, $\frac{1}{Ms^2+Ds+K}$, cannot be positive real.

Positive Real Functions

A popular definition of positive real (PR) functions is that a function $h(s)$ of the complex variable $s = \sigma + j\omega$ is positive real if (i) $h(s)$ is real for real s, and (ii) $Re[h(s)] \geq 0$ for all s such that $\mathrm{Re}[s] > 0$.

One may “induce” a definition of strictly positive real (SPR) functions as: a function $h(s)$ of the complex variable $s = \sigma + j\omega$ is strictly positive real if (i) $h(s)$ is real for real s, and (ii) $Re[h(s)] > 0$ for all s such that $\mathrm{Re}[s] > 0$. This definition was once used in the early literature for SPR functions. It turns out that, like some other definitions of SPR functions, this definition does not capture the physical meaning of strictly positive realness, as indicated by the following example.

It is well-understood that a proper definition of strictly positive functions, which captures the physical meaning of strictly positive realness, is that $h(s)$ is strictly positive real if $h(s-\varepsilon)$ is positive real for some $\varepsilon > 0$. Based on this definition,

$$h(s) = \frac{s+1}{s^2+s+1} \tag{11}$$

is only positive real but not strictly positive real. From the expressions

$$\begin{aligned} h(s) &= \frac{\sigma + j\omega + 1}{(\sigma + j\omega)^2 + \sigma + j\omega + 1} \\ &= \frac{\sigma + 1 + j\omega}{\sigma^2 - \omega^2 + \sigma + 1 + j(\omega + 2\sigma\omega)} \\ &= \frac{(\sigma + 1 + j\omega)(\sigma^2 - \omega^2 + \sigma + 1 - j(\omega + 2\sigma\omega))}{(\sigma^2 - \omega^2 + \sigma + 1)^2 + (\omega + 2\sigma\omega)^2} \end{aligned} \tag{12}$$

$$\begin{aligned}\operatorname{Re}[h(s)] &= \frac{(\sigma+1)(\sigma^2-\omega^2+\sigma+1)+\omega(\omega+2\sigma\omega)}{(\sigma^2-\omega^2+\sigma+1)^2+(\omega+2\sigma\omega)^2}\\ &= \frac{\sigma^3+2\sigma^2+\sigma\omega^2+2\sigma+1}{(\sigma^2-\omega^2+\sigma+1)^2+(\omega+2\sigma\omega)^2},\end{aligned} \tag{13}$$

we see that $h(s)$ satisfies the above "induced" definition of SPR functions: (i) $h(s)$ is real for real s, and (ii) $Re[h(s)] > 0$ for all s such that $\operatorname{Re}[s] = \sigma > 0$. Hence, the conclusion is that this "induced" definition for SPR functions is not proper.

Note that this $h(s)$ is not SPR, because

$$\operatorname{Re}[h(j\omega)] = \frac{1}{(1-\omega^2)^2+\omega^2} \tag{14}$$

does not satisfy the necessary condition for SPRness: $\lim_{\omega^2\to\infty}\omega^2\operatorname{Re}[h(j\omega)] > 0$, or because for any chosen $\varepsilon > 0$,

$$\operatorname{Re}[h(j\omega-\varepsilon)] = \frac{-\varepsilon^3+2\varepsilon^2-\varepsilon\omega^2-2\varepsilon+1}{(\varepsilon^2-\omega^2-\varepsilon+1)^2+(\omega-2\varepsilon\omega)^2} < 0 \tag{15}$$

whenever $\omega^2 > (-\varepsilon^3+2\varepsilon^2-2\varepsilon+1)/\varepsilon$, that is, $h(s-\varepsilon)$ cannot be positive real for any $\varepsilon > 0$. ($P(\varepsilon) = -\varepsilon^3+2\varepsilon^2-2\varepsilon+1 = (\varepsilon-1)(-\varepsilon^2+\varepsilon-1) > 0$ for $\varepsilon \in (0,1)$ and $P(\varepsilon) < 0$ for $\varepsilon > 1$. For $h(s-\varepsilon)$ to be stable, $\varepsilon \in [0,0.5)$ is needed as $(s-\varepsilon)^2+s-\varepsilon+1 = s^2+(1-2\varepsilon)s+1-\varepsilon+\varepsilon^2$ and $1-\varepsilon+\varepsilon^2 > 0$ for any ε.)

Parameter Projection Properties

The property (3.183) follows from the observation: if $\theta_j(t) = \theta_j^a$ and $g_j(t) < 0$, then $f_j(t) = -g_j(t) > 0$ and $\theta_j(t)-\theta_j^* = \theta_j^a-\theta_j^* \leq 0$, so that $(\theta_j(t)-\theta_j^*)f_j(t) \leq 0$; and if $\theta_j(t) = \theta_j^b$ and $g_j(t) > 0$, then $f_j(t) = -g_j(t) < 0$ and $\theta_j(t)-\theta_j^* = \theta_j^b-\theta_j^* \geq 0$, so that $(\theta_j(t)-\theta_j^*)f_j(t) \leq 0$.

Similarly, the property (3.214) follows from the observation: if $\theta_j(t)+g_j(t) > \theta_j^b$, then $f_j(t) = \theta_j^b-\theta_j(t)-g_j(t) < 0$ and $\theta_j(t)-\theta_j^*+g_j(t)+f_j(t) = \theta_j^b-\theta_i^* \geq 0$, so that $f_j(t)(\theta_j(t)-\theta_j^*+g_j(t)+f_j(t)) \leq 0$; and if if $\theta_j(t)+g_j(t) < \theta_j^a$, then $f_j(t) = \theta_j^b-\theta_j(t)-g_j(t) > 0$ and $\theta_j(t)-\theta_j^*+g_j(t)+f_j(t) = \theta_j^a-\theta_i^* \leq 0$, so that $f_j(t)(\theta_j(t)-\theta_j^*+g_j(t)+f_j(t)) \leq 0$.

Explicit Swapping Lemma

The swapping lemma (5.331) was an important lemma in the development of a stable model reference adaptive control system. It states that for a stable and proper rational function $h(s)$ with a minimal realization $h(s)=c(sI-A)^{-1}b+d$ and two vector signals $\theta(t)$ and $\omega(t)$, it follows that

$$\theta^T(t)h(s)[\omega](t)-h(s)[\theta^T\omega](t)=h_c(s)[h_b(s)[\omega^T]\dot{\theta}](t), \tag{16}$$

where $h_c(s)=c(sI-A)^{-1}$ and $h_b(s)=(sI-A)^{-1}b.$ Here we derive an alternative form of this lemma, explicitly in terms of the parameters of the funtion $h(s)$.

Denoting $P_m(s)=s^{n^*}+a_{n^*-1}s^{n^*-1}+\cdots+a_1s+a_0$, for vector signals $\theta(t)$ and $\omega(t)$, from (5.138) and with $a_{n^*}=1$, we have

$$\begin{aligned}&\theta^T(t)\frac{1}{P_m(s)}[\omega](t)-\frac{1}{P_m(s)}[\theta^T\omega](t)\\ &=\sum_{i=1}^{n^*}\left(\frac{\sum_{j=0}^{n^*-i}a_{n^*-j}s^{n^*-i-j}}{P_m(s)}\right)\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right](t).\end{aligned} \tag{17}$$

Introducing $F(s)=f_{n^*-1}s^{n^*-1}+\cdots+f_1s+f_0$, we express

$$\begin{aligned}&F(s)\left[\theta^T(t)\frac{1}{P_m(s)}[\omega]\right](t)\\ &=\left(f_{n^*-1}s^{n^*-2}+\cdots+f_1\right)\left[\dot{\theta}^T\frac{1}{P_m(s)}[\omega]+\theta^T\frac{s}{P_m(s)}[\omega]\right](t)+\theta^T(t)\frac{f_0}{P_m(s)}[\omega](t)\\ &=\left(f_{n^*-1}s^{n^*-2}+\cdots+f_1\right)\left[\dot{\theta}^T\frac{1}{P_m(s)}[\omega]\right](t)\\ &\quad+\left(f_{n^*-1}s^{n^*-3}+\cdots+f_2\right)\left[\dot{\theta}^T\frac{s}{P_m(s)}[\omega]+\theta^T\frac{s^2}{P_m(s)}[\omega]\right](t)\\ &\quad+\theta^T(t)\frac{f_1s+f_0}{P_m(s)}[\omega](t)=\cdots\\ &=\sum_{i=1}^{n^*-1}\left(\sum_{j=1}^{n^*-i}f_{n^*-j}s^{n^*-i-j}\right)\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right](t)+\theta^T(t)\frac{F(s)}{P_m(s)}[\omega](t)\end{aligned} \tag{18}$$

and use it to derive

$$\begin{aligned}&\frac{F(s)}{P_m(s)}[\theta^T\omega](t)-\theta^T(t)\frac{F(s)}{P_m(s)}[\omega](t)\\ &=\sum_{i=1}^{n^*-1}\left(\sum_{j=1}^{n^*-i}f_{n^*-j}s^{n^*-i-j}\right)\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right](t)\\ &\quad-F(s)\left[\sum_{i=1}^{n^*}\frac{\sum_{j=0}^{n^*-i}a_{n^*-j}s^{n^*-i-j}}{P_m(s)}\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right]\right](t)\\ &=\sum_{i=1}^{n^*-1}\left(\sum_{j=1}^{n^*-i}f_{n^*-j}s^{n^*-i-j}-\frac{F(s)\sum_{j=0}^{n^*-i}a_{n^*-j}s^{n^*-i-j}}{P_m(s)}\right)\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right](t)\end{aligned}$$

$$
\begin{aligned}
&- \frac{F(s)}{P_m(s)}\left[\dot{\theta}^T \frac{s^{n^*-1}}{P_m(s)}[\omega]\right](t) \\
&= \sum_{i=1}^{n^*-1} \frac{\alpha_i(s)}{P_m(s)}\left[\dot{\theta}^T \frac{s^{i-1}}{P_m(s)}[\omega]\right](t) - \frac{F(s)}{P_m(s)}\left[\dot{\theta}^T \frac{s^{n^*-1}}{P_m(s)}[\omega]\right](t). \qquad (19)
\end{aligned}
$$

Hence, we obtain the *explicit swapping lemma*:

$$
\frac{F(s)}{P_m(s)}[\theta^T\omega](t) - \theta^T(t)\frac{F(s)}{P_m(s)}[\omega](t) = \sum_{i=1}^{n^*} \frac{\alpha_i(s)}{P_m(s)}\left[\dot{\theta}^T \frac{s^{i-1}}{P_m(s)}[\omega]\right](t), \qquad (20)
$$

where

$$
\alpha_i(s) = P_m(s)\sum_{j=1}^{n^*-i} f_{n^*-j}s^{n^*-i-j} - F(s)\sum_{j=0}^{n^*-i} a_{n^*-j}s^{n^*-i-j},\ i = 1,2,\ldots,n^*-1, \quad (21)
$$

are polynominals of degrees n^*-1 or less, and $\alpha_{n^*}(s) \stackrel{\triangle}{=} -F(s)$.

For a proper transfer function $h(s) = f_{n^*} + \frac{F(s)}{P_m(s)}$, it follows that

$$
h(s)[\theta^T\omega](t) - \theta^T(t)h(s)[\omega](t) = \sum_{i=1}^{n^*} \frac{\alpha_i(s)}{P_m(s)}\left[\dot{\theta}^T \frac{s^{i-1}}{P_m(s)}[\omega]\right](t). \qquad (22)
$$

Discrete-Time Swapping Lemma

Let a stable and proper rational function $h(z)$ have a minimal realization $h(z) = c(zI-A)^{-1}b+d$ and $\theta(t)$ and $\omega(t)$ be two vector signals, and denote $h_c(z) = c(zI-A)^{-1}$ and $h_b(z) = (zI-A)^{-1}b$. Then,

$$
\theta^T(t)h(z)[\omega](t) - h(z)[\theta^T\omega](t) = h_c(z)[(h_b(z)z)[\omega^T](z-1)[\theta]](t). \qquad (23)
$$

Proof: Using the discrete-time convolution: $y(t) = C\sum_{i=0}^{t-1}A^{t-i-1}Bu(i)$ for $y(t) = C(zI-A)^{-1}B[u](t)$, and its modification: $w(t) = C\sum_{i=0}^{t}A^{t-1}Bu(i)$ for $w(t) = C(zI-A)^{-1}Bz[u](t) = C(zI-A)^{-1}B[u](t+1)$, we express

$$
\begin{aligned}
&h_c(z)[(h_b(z)z)[\omega^T](z-1)[\theta]](t) \\
&= c\sum_{\tau=0}^{t-1}A^{t-\tau-1}\sum_{i=0}^{\tau}A^{\tau-i}b\,\omega^T(i)(\theta(\tau+1)-\theta(\tau)) \\
&= c\sum_{i=0}^{t-1}A^{t-i-1}b\,\omega^T(i)\theta(t) \\
&\quad + c\sum_{\tau=0}^{t-2}A^{t-\tau-1}\sum_{i=0}^{\tau}A^{\tau-i}b\,\omega^T(i)\theta(\tau+1) - c\sum_{\tau=0}^{t-1}A^{t-\tau-1}\sum_{i=0}^{\tau}A^{\tau-i}b\,\omega^T(i)\theta(\tau)
\end{aligned}
$$

$$
\begin{aligned}
&= \theta^T(t)h(z)[\omega](t) + c\sum_{\sigma=1}^{t-1} A^{t-\sigma}\sum_{i=0}^{\sigma-1} A^{\sigma-i-1} b\,\omega^T(i)\theta(\sigma) \\
&\quad - c\sum_{\tau=0}^{t-1} A^{t-\tau-1} b\,\omega^T(\tau)\theta(\tau) - c\sum_{\tau=1}^{t-1} A^{t-\tau-1}\sum_{i=0}^{\tau-1} A^{\tau-i} b\,\omega^T(i+1)\theta(\tau) \\
&= \theta^T(t)h(z)[\omega](t) - h(z)[\theta^T\omega](t), \qquad (24)
\end{aligned}
$$

where, by definition,

$$
c\sum_{i=0}^{t-1} A^{t-i-1} b\,\omega(i) = h(z)[\omega](t) \qquad (25)
$$

$$
c\sum_{\tau=0}^{t-1} A^{t-\tau-1} b\,\omega^T(\tau)\theta(\tau) = h(z)[\theta^T\omega](t). \qquad (26)
$$

This is the discrete-time version of the swapping lemma (5.331), whose explicit version can also be derived, similar to the continuous-time case above.

Convergence of Discrete-Time Gradient Algorithm

This section gives the proof of Theorem 3.2 which specifies the conditions for parameter convergence of the discrete-time gradient algorithm (3.83).

Consider the linear time-invariant system described by the difference equation

$$
P(z)[y](t) = Z(z)[u](t), \qquad (27)
$$

where $y(t) \in R$ and $u(t) \in R$ are the measured system input and output, and

$$
P(z) = z^n + p_{n-1}z^{n-1} + \cdots + p_1 z + p_0, \qquad (28)
$$

$$
Z(z) = z_m z^m + z_{m-1}z^{m-1} + \cdots + z_1 z + z_0. \qquad (29)
$$

Choose a stable polynomial $\Lambda(z) = z^n + \lambda_{n-1}z^{n-1} + \cdots + \lambda_1 z + \lambda_0$ whose zeros are in $|z| < 1$, and express (27) as

$$
y(t) = \theta^{*T}\phi(t), \qquad (30)
$$

where

$$
\begin{aligned}
\theta^* = [&z_0, z_1, \ldots, z_{m-1}, z_m, \\
&\lambda_0 - p_0, \lambda_1 - p_1, \ldots, \lambda_{n-2} - p_{n-2}, \lambda_{n-1} - p_{n-1}]^T \in R^{n+m+1}, \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
\phi(t) = \Big[&\frac{1}{\Lambda(z)}[u](t), \frac{z}{\Lambda(z)}[u](t), \ldots, \frac{z^{m-1}}{\Lambda(z)}[u](t), \frac{z^m}{\Lambda(z)}[u](t), \\
&\frac{1}{\Lambda(z)}[y](t), \frac{z}{\Lambda(z)}[y](t), \ldots, \frac{z^{n-2}}{\Lambda(z)}[y](t), \frac{z^{n-1}}{\Lambda(z)}[y](t)\Big]^T \in R^{n+m+1}. \qquad (32)
\end{aligned}
$$

Assume that the system input $u(t)$ has $n+m+1$ frequencies, that is,

$$u(t) = \alpha_0 + \sum_{i=1}^{N} (\alpha_i \sin \omega_i t + \alpha_{N+i} \cos \omega_i t), \tag{33}$$

where $\omega_j \neq \omega_k + 2\pi l$ for $j \neq k$ and an integer l, $\alpha_0 = 0$ and $N = \frac{n+m+1}{2}$ for an even $n+m+1$ (or $\alpha_0 \neq 0$ and $N = \frac{n+m}{2}$ for an odd $n+m+1$), and $\alpha_i \neq 0$, $i = 1, 2, \ldots, 2N$.

Property 1 *There exist an integer $\delta_1 > 0$ and a constant $\alpha_1 > 0$ such that*

$$\sum_{t=\sigma}^{\sigma+\delta_1} U(t, t+n+m) U^T(t, t+n+m) \geq \alpha_1 I, \; \forall \sigma \geq 0, \tag{34}$$

where $U(t, t+n+m) = [u(t), u(t+1), \ldots, u(t+n+m)]^T \in R^{n+m+1}$.

Proof:

Analysis of "Bursting Phenomenon" in Robust Adaptive Control

As illustrated in [134] that adaptive control systems with a fixed σ-modification may show some bursting phenomenon. Consider the adaptive system analyzed on page 232 (for Theorem 5.8) with $\sigma_1(t) = \sigma_0$ being a constant in (5.198):

$$\dot{\theta}(t) = -\text{sign}[k_p]\Gamma\omega(t)e(t) - \Gamma\sigma_1(t)\theta(t), \; t \geq 0$$

(that is, with a fixed σ-modification). For V given in (5.43): $V = e^2 + |k_p|\tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$, we have from (5.204) that

$$\begin{aligned}
\dot{V} &\leq -a_m e^2(t) + \frac{\bar{d}^2(t)}{a_m} - 2|k_p|\sigma_0 \tilde{\theta}^T(t)\theta(t) \\
&= -a_m e^2(t) + \frac{\bar{d}^2(t)}{a_m} - 2|k_p|\sigma_0 \tilde{\theta}^T(t)\tilde{\theta}(t) + 2|k_p|\sigma_0 \tilde{\theta}^T(t)\theta^* \\
&\leq -a_m e^2(t) + \frac{\bar{d}^2(t)}{a_m} - |k_p|\sigma_0 \tilde{\theta}^T(t)\tilde{\theta}(t) + |k_p|\sigma_0 \theta^{*T}\theta^*.
\end{aligned} \tag{35}$$

For $\Gamma = \gamma I$ and $a_m > \gamma\sigma_0$ (that is, with a small σ_0), we have

$$\dot{V} \leq -\gamma\sigma_0 V + \frac{\bar{d}^2(t)}{a_m} + |k_p|\sigma_0 \theta^{*T}\theta^* \tag{36}$$

which, for $|\bar{d}(t)| \leq d_0$, leads to

$$\lim_{t\to\infty} V(t) \leq \frac{d_0^2}{a_m \gamma \sigma_0} + \frac{|k_p|}{\gamma} \theta^{*T} \theta^*. \tag{37}$$

This implies that the upper bound for the tracking error $e(t) = y(t) - y_m(t)$, for $d(t) = d_0 = 0$ (in the absence of disturabnces $d(t)$), may be as large as $\frac{|k_p|}{\gamma}\theta^{*T}\theta^*$, independent of σ_0. We already knew that for $d(t) = 0$ and $\sigma_1(t) = \sigma_0 = 0$, the adaptive control system ensures that $\lim_{t\to\infty} e(t) = 0$. Now from the above analysis we know that for a small $\sigma_0 \neq 0$, the error bound on $|e(t)|$ can be up to $\frac{|k_p|}{\gamma}\theta^{*T}\theta^*$. On the other hand, from

$$\dot{V} \leq -a_m e^2(t) + \frac{\bar{d}^2(t)}{a_m} - 2|k_p|\sigma_0 \tilde{\theta}^T(t)\theta(t) \tag{38}$$

and the adaptive control system signal boundedness, similar to (5.204), we have

$$\int_{t_1}^{t_2} e^2(t)\, dt \leq \gamma_0 + k_0(t_2 - t_1)\bar{d}_0^2 + c_0(t_2 - t_1)\sigma_0 \tag{39}$$

for some constant $c_0 > 0$, where the σ_0 related term is due to using a fixed σ-modification $\sigma_1(t) = \sigma_0$, instead of the switching σ-modification which leads to (5.205) and in turn to (5.204). The inequality (39) implies that, when $d_0 = 0$ (in the absence of disturabnces), we have the mean error

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} e^2(t)\, dt \leq \frac{\gamma_0}{t_2 - t_1} + c_0 \sigma_0 \tag{40}$$

which is of the magnitude of $c_0\sigma_0$, but the absolute error $|e(t)|$, as from (37), could be as large as $\frac{|k_p|}{\gamma}\theta^{*T}\theta^*$, independent of σ_0. This is the so-called “bursting phenomenon” of robust adaptive control with a fixed σ-modification: the tracking error $e(t)$ may go to a large value independent of σ_0 for a small interval of time but in the mean sense the error $e(t)$ is of the order of σ_0. This analytically explains what was observed in the simulation results [134].

Robust MRAC with A Switching σ-Modification

To derive (5.224), for $V(\tilde{\theta}, \tilde{\rho}) = |\rho^*|\tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \gamma^{-1}\tilde{\rho}^2$, using

$$\varepsilon(t) = \rho^* \tilde{\theta}^T(t)\zeta(t) + \tilde{\rho}(t)\xi(t) + \mu\eta(t),\ \eta(t) = \Delta(s)[u](t) \tag{41}$$

we have

$$\dot{V} = -2\frac{\varepsilon^2(t)}{m^2(t)} + 2\frac{\varepsilon(t)\eta(t)}{m^2(t)} - 2\sigma_1(t)|\rho^*|\tilde{\theta}^T(t)\theta(t) - 2\sigma_2(t)\tilde{\rho}(t)\rho(t)$$

$$
\begin{aligned}
&= -\frac{\varepsilon^2(t)}{m^2(t)} - \left(\frac{\varepsilon(t)}{m(t)} - \mu\frac{\eta(t)}{m(t)}\right)^2 + \mu^2\frac{\eta^2(t)}{m^2(t)} \\
&\quad -2\sigma_1(t)|\rho^*|\tilde{\theta}^T(t)\theta(t) - 2\sigma_2(t)\tilde{\rho}(t)\rho(t) \\
&\leq -\frac{\varepsilon^2(t)}{m^2(t)} + \mu^2\frac{\eta^2(t)}{m^2(t)} - 2\sigma_1(t)|\rho^*|\tilde{\theta}^T(t)\theta(t) - 2\sigma_2(t)\tilde{\rho}(t)\rho(t) \qquad (42)
\end{aligned}
$$

which is (5.224). Since $\frac{|\eta(t)|}{m(t)} \leq b_0$ for some constant $b_0 > 0$, we have $\dot{V} < 0$ if

$$2\sigma_1(t)|\rho^*|\tilde{\theta}^T(t)\theta(t) + 2\sigma_2(t)\tilde{\rho}(t)\rho(t) > \mu^2 b_0^2. \qquad (43)$$

For $\tilde{\theta} = \theta - \theta^*$ and $\tilde{\rho} = \rho - \rho^*$, by definition, we have

$$\sigma_1(t)|\rho^*|\tilde{\theta}^T(t)\theta(t) \geq 0,\ \sigma_2(t)\tilde{\rho}(t)\rho(t) \geq 0,\ t \geq 0 \qquad (44)$$

$$\lim_{\|\theta\|_2\to\infty} \tilde{\theta}^T\theta = \infty,\ \lim_{|\rho|\to\infty} \tilde{\rho}\rho = \infty. \qquad (45)$$

This implies that there exist constants $\theta^0 > 0$ and $\rho^0 > 0$ such that $\|\theta(t)\|_2 \geq \theta^0$ or/and $|\rho(t)| \geq \rho^0$ implies that $\dot{V} < 0$. Hence, the boundedness of $\theta(t)$ and $\rho(t)$ is ensured. One choice of such (θ^0, ρ^0) is

$$
\begin{aligned}
\theta^0 &= \max\{2M_1, \sqrt{\frac{1}{4\sigma_{10}}(2\mu^2 b_0^2 + \|\theta^*\|_2^2 + \rho^{*2})} + \frac{1}{2}\|\theta^*\|\} \\
\rho^0 &= \max\{2M_2, \sqrt{\frac{1}{4\sigma_{20}}(2\mu^2 b_0^2 + \|\theta^*\|_2^2 + \rho^{*2})} + \frac{1}{2}|\rho^*|\}. \qquad (46)
\end{aligned}
$$

Discrete-time MRAC System Example

Consider the first-order plant

$$y(t+1) = a_p y(t) + b_p u(t) \qquad (47)$$

with two unknown parameters a_p and b_p and choose the reference model

$$y_m(t+1) = -a_m y_m(t) + b_m r(t) \qquad (48)$$

with $|a_m| < 1$ for stability. We use the adaptive controller structure

$$u(t) = k_1(t)y(t) + k_2(t)r(t), \qquad (49)$$

where $k_1(t)$ and $k_2(t)$ are estimates of the unknown parameters

$$k_1^* = \frac{-a_p - a_m}{b_p},\ k_2^* = \frac{b_m}{b_p}. \qquad (50)$$

In this case the tracking error equation becomes

$$e(t+1) = -a_m e(t) + b_p \tilde{k}_1(t) y(t) + b_p \tilde{k}_2(t) r(t), \tag{51}$$

where $\tilde{k}_1(t) = k_1(t) - k_1^*$ and $\tilde{k}_2(t) = k_2(t) - k_2^*$.

Defining $\rho^* = b_p$ and

$$\theta^* = [k_1^*, k_2^*]^T,\ \theta = [k_1, k_2]^T, \tag{52}$$

$$\omega(t) = [y(t), r(t)]^T \tag{53}$$

and introducing the filtered vector signal

$$\zeta(t) = \frac{1}{z+a_m}[\omega](t) = \left[\frac{1}{z+a_m}[y](t), \frac{1}{z+a_m}[r](t)\right]^T = [\zeta_1(t), \zeta_2(t)]^T, \tag{54}$$

where, as a notation, $\zeta_1(t) = \frac{1}{z+a_m}[y](t)$ denotes the output of the system with transfer function $\frac{1}{z+a_m}$ and input $y(t)$ (it satisfies the equation: $\zeta_1(t+1) = -a_m\zeta_1(t) + y(t)$, for generating $\zeta_1(t)$ from $y(t)$), we rewrite (51) as

$$\begin{aligned} e(t) &= \frac{\rho^*}{z+a_m}[\theta^T\omega - \theta^{*T}\omega](t) \\ &= \rho^*\left(\frac{1}{z+a_m}[\theta^T\omega](t) - \theta^{*T}\zeta(t)\right). \end{aligned} \tag{55}$$

The design task is to find adaptive laws to update the parameter estimates $\theta(t)$ and $\rho(t)$ (which is an estimate of ρ^*) such that the estimation error

$$\varepsilon(t) = e(t) - \rho(t)\left(\frac{1}{z+a_m}[\theta^T\omega](t) - \theta^T(t)\zeta(t)\right) \tag{56}$$

is small in some sense. With (55), this error can be expressed as

$$\varepsilon(t) = \rho^*\tilde{\theta}^T(t)\zeta(t) + \tilde{\rho}(t)\xi(t), \tag{57}$$

where

$$\xi(t) = \theta^T(t)\zeta(t) - \frac{1}{z+a_m}[\theta^T\omega](t) \tag{58}$$

$$\tilde{\theta}(t) = \theta(t) - \theta^*,\ \tilde{\rho}(t) = \rho(t) - \rho^*. \tag{59}$$

We choose the gradient adaptive laws for $\theta(t)$ and $\rho(t)$:

$$\theta(t+1) = \theta(t) - \frac{\mathrm{sign}[b_p]\Gamma\varepsilon(t)\zeta(t)}{m^2(t)},\ 0 < \Gamma = \Gamma^T < \frac{2}{b_p^0}I_2, \tag{60}$$

$$\rho(t+1) = \rho(t) - \frac{\gamma\varepsilon(t)\xi(t)}{m^2(t)},\ 0 < \gamma < 2, \tag{61}$$

where sign$[b_p]$ is the sign of b_p, $b_p^0 \geq |b_p|$ is a known upper bound on $|b_p|$, and

$$m(t) = \sqrt{1 + \zeta^T(t)\zeta(t) + \xi^2(t)}. \tag{62}$$

The stability analysis of this MRAC system is given in Section 6.3.1.

Parametrization of (9.102)

To define the parameters Θ_i^*, $i = 1, 2, 20, 3$, in (9.105) for the nominal version of the multivariable MRAC controller (9.102), we divide (9.105) from the right by $P_0^{-1}(D)$ and use the left factorization of $G_0(D) = P_l^{-1}(D)Z_l(D)$ such that $G_0(D) = P_l^{-1}(D)Z_l(D) = Z_0(D)P_0^{-1}(D)$ (see the proof of Lemma 9.3), to obtain

$$\begin{aligned} &\Theta_1^{*T}A(D) + (\Theta_2^{*T}A(D) + \Theta_{20}^*\Lambda(D))P_l^{-1}(D)Z_l(D) \\ &= \Lambda(D)(I - \Theta_3^*\xi_m(D)P_l^{-1}(D)Z_l(D)). \end{aligned} \tag{63}$$

Expressing $\Lambda(D)\Theta_3^*\xi_m(D)P_l^{-1}(D)$ as

$$\Lambda(D)\Theta_3^*\xi_m(D)P_l^{-1}(D) = Q_l(D) + R_l(D)P_l^{-1}(D) \tag{64}$$

for some $M \times M$ polynomial matrices $Q_l(D)$ and $R_l(D)$ such that $\partial_{ci}[R_l(D)] < \partial_{ci}[P_l(D)] \leq \nu$. Then, we define Θ_1^*, Θ_2^* and Θ_{20}^* from

$$\Theta_2^{*T}A(D) + \Theta_{20}^*\Lambda(D) = -R_l(D), \tag{65}$$

$$\Theta_1^{*T}A(D) = \Lambda(D)I_M - Q_l(D)Z_l(D) \tag{66}$$

(recall that $\partial[\Lambda(D)] = \nu - 1$ and $\partial[A(D)] = \nu - 2$ for the controller structure (9.102)).

From (9.102) (that is, (63)), we have the plant–model matching equation

$$I_M - \Theta_1^{*T}F(D) - (\Theta_2^{*T}F(D) + \Theta_{20}^*)G_0(D) = \Theta_3^*W_m^{-1}(D)G_0(D) \tag{67}$$

where $W_m(D) = \xi_m^{-1}(D)$. From this equation with $\lim_{D\to\infty}\Theta_3^*W_m^{-1}(D)G_0(D) = I_M$ as from the definition of $\xi_m(D)$, we have

$$\lim_{D\to\infty}\Theta_1^{*T}F(D) = 0, \tag{68}$$

which implies that $\partial[\Theta_1^{*T}A(D)] \leq \nu - 2$, that is, (66) is solvable.

Plant Signal Identities for MIMO Cases

It has been shown that there exist Θ_1^*, Θ_2^* and Θ_3^* to satisfy the matching equation

$$\Theta_1^{*T} A(D) P_0(D) + \Theta_2^{*T} A(D) Z_0(D) = \Lambda(D)(P_0(D) - \Theta_3^* \xi_m(D) Z_0(D)), \tag{69}$$

which is nominally (mathematically) equivalent to

$$I_M - \Theta_1^{*T} F(D) - \Theta_2^{*T} F(D) G_0(D) = \Theta_3^* W_m^{-1}(D) G_0(D). \tag{70}$$

To obtain the plant parametrized signal identity (similar to that in (5.30) for the SISO case with $M = 1$), we consider (9.84) and (9.85):

$$\Theta_2^{*T} A(D) = Q_l(D) P_l(D) - \Lambda(D) K_p^{-1} \xi_m(D) \tag{71}$$

$$\Theta_1^{*T} A(D) = \Lambda(D) I_M - Q_l(D) Z_l(D). \tag{72}$$

and obtain the signal identities:

$$\Theta_2^{*T} \frac{A(D)}{\Lambda(D)}[y](t) = \frac{1}{\Lambda(D)} Q_l(D) P_l(D)[y](t) - K_p^{-1} \xi_m(D)[y](t) \tag{73}$$

$$\Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) = u(t) - \frac{1}{\Lambda(D)} Q_l(D) Z_l(D)[u](t). \tag{74}$$

Using the open-loop plant signal identity: $P_l(D)[y](t) = Z_l(D)[u](t)$, we finally have the plant parametrized signal identity:

$$u(t) = \Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^{*T} \frac{A(D)}{\Lambda(D)}[y](t) + K_p^{-1} \xi_m(D)[y](t). \tag{75}$$

This identity holds for any input signal $u(t)$, in a feedback structure.

Similarly, for the controller structure (9.102) with nominal parameters:

$$u(t) = \Theta_1^{*T} \omega_1(t) + \Theta_2^{*T} \omega_2(t) + \Theta_{20}^* y(t) + \Theta_3^* r(t), \tag{76}$$

in which $\partial[\Lambda(D)] = \nu - 1$, we have the polynomial matching equation

$$\begin{aligned} &\Theta_1^{*T} A(D) P_0(D) + (\Theta_2^{*T} A(D) + \Theta_{20}^* \Lambda(D)) Z_0(D) \\ &= \Lambda(D)(P_0(D) - \Theta_3^* \xi_m(D) Z_0(D)) \end{aligned} \tag{77}$$

and the transfer matrix matching equation

$$I_M - \Theta_1^{*T} F(D) - (\Theta_2^{*T} F(D) + \Theta_{20}^*) G_0(D) = \Theta_3^* W_m^{-1}(D) G_0(D). \tag{78}$$

The nominal parameters Θ_1^*, Θ_2^*, Θ_{20}^* and Θ_3^* are defined from

$$(\Theta_2^{*T} A(D) + \Theta_{20}^* \Lambda(D)) = -R_l(D) = Q_l(D)P_l(D) - \Lambda(D)K_p^{-1}\xi_m(D) \quad (79)$$

$$\Theta_1^{*T} A(D) = \Lambda(D)I_M - Q_l(D)Z_l(D), \quad (80)$$

that is, dividing $\Lambda(D)K_p^{-1}\xi_m(D)$ on the right by $P_l(D)$ to get $R_l(D)$ and $Q_l(D)$. From these equations, we obtain the signal identities:

$$(\Theta_2^{*T} \frac{A(D)}{\Lambda(D)} + \Theta_{20}^*)[y](t) = \frac{1}{\Lambda(D)} Q_l(D)P_l(D)[y](t) - K_p^{-1}\xi_m(D)[y](t) \quad (81)$$

$$\Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) = u(t) - \frac{1}{\Lambda(D)} Q_l(D)Z_l(D)[u](t). \quad (82)$$

Using the open-loop plant signal identity: $P_l(D)[y](t) = Z_l(D)[u](t)$, we have the plant signal identity in a feedback form:

$$u(t) = \Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^{*T} \frac{A(D)}{\Lambda(D)}[y](t) + \Theta_{20}^* y(t) + K_p^{-1}\xi_m(D)[y](t) \quad (83)$$

which also holds for any input signal $u(t)$, similar to that in (75).

Both plant parametrized signal identities (75) and (83) are useful for adaptive control: either verify the nominal controller structure

$$u(t) = \Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^{*T} \frac{A(D)}{\Lambda(D)}[y](t) + \Theta_{20}^* y(t) + \Theta_3^* r(t) \quad (84)$$

for the matching equation (69), or

$$u(t) = \Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^{*T} \frac{A(D)}{\Lambda(D)}[y](t) + \Theta_{20}^* y(t) + \Theta_3^* r(t) \quad (85)$$

for the matching equation (77), where $\Theta_3^* r(t) = K_p^{-1}\xi_m(D)[y_m](t)$ for the reference output $y_m(t) = \xi_m^{-1}(D)[r](t)$, leading to $\xi_m(D)[y - y_m](t) = 0$ exponentially, or they motivate the adaptive controller structure

$$u(t) = \Theta_1^T \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^T \frac{A(D)}{\Lambda(D)}[y](t) + \Theta_{20} y(t) + \Theta_3 r(t) \quad (86)$$

for the matching equation (69), or

$$u(t) = \Theta_1^T \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^T \frac{A(D)}{\Lambda(D)}[y](t) + \Theta_{20} y(t) + \Theta_3 r(t) \quad (87)$$

for the matching equation (77), leading to the desired tracking error equation:

$$e(t) = y(t) - y_m(t) = \xi_m^{-1}(D)K_p[\tilde{\Theta}^T \omega](t) \quad (88)$$

for $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ with

$$\Theta(t) = [\Theta_1^T(t), \Theta_2^T(t), \Theta_3(t)]^T, \ \Theta^* = [\Theta_1^{*T}, \Theta_2^{*T}, \Theta_3^*]^T \tag{89}$$

$$\omega(t) = [\omega_1^T(t), \omega_2^T(t), r^T(t)]^T \tag{90}$$

for the matching equation (69), or

$$\Theta(t) = [\Theta_1^T(t), \Theta_2^T(t), \Theta_{20}(t), \Theta_3(t)]^T, \ \Theta^* = [\Theta_1^{*T}, \Theta_2^{*T}, \Theta_{20}^*, \Theta_3^*]^T \tag{91}$$

$$\omega(t) = [\omega_1^T(t), \omega_2^T(t), y^T(t), r^T(t)]^T \tag{92}$$

for the matching equation (69). Both tracking error equations can be used to derive some desired estimation errors for designing stable adaptive laws to update the controller parameters $\Theta(t)$.

Adaptive Robot Control for Time-Varying Parameters

Consider the manipulator dynamic equation (9.586):

$$D(q,t)\ddot{q} + \frac{\partial D(q,t)}{\partial t}\dot{q} + C(q,\dot{q},t)\dot{q} + \phi(q,t) = u. \tag{93}$$

With (9.590): $v = \dot{q}_d - \Lambda_0(q - q_d)$, $s = \dot{q} - v$, $e = q - q_d$, we have (9.592):

$$\begin{aligned} & D(q,t)\dot{s} + C(q,\dot{q},t)s \\ &= u - D(q,t)\dot{v} - C(q,\dot{q},t)v - \phi(q,t) - \frac{\partial D(q,t)}{\partial t}\dot{q} \\ &\stackrel{\triangle}{=} u - Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\theta^*(t) - \frac{\partial D(q,t)}{\partial t}\dot{q}, \end{aligned} \tag{94}$$

for some known function matrix $Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t) \in R^{n \times n_\theta}$ and unknown parameter vector $\theta^*(t) \in R^{n_\theta}$ which may be time-varying.

In this study, we consider the case when

$$\theta^*(t) = \theta_0^* + \delta\theta^*(t) \tag{95}$$

where θ_0^* is a constant vector and $\delta\theta^*(t)$ is the variation of $\theta^*(t)$ with respect to θ_0^* (note that both θ_0^* and $\delta\theta^*(t)$ are unknown). We will develop and analyze some alternative adaptive control schemes to that presented in (9.607)–(9.608).

Adaptive Control Scheme I

As an alternative scheme to (9.607)–(9.608), we use the control law

$$u(t) = Y(q, q_d, \dot{q}, \dot{q}_d, \ddot{q}_d, t)\theta_0(t) - m(t)\psi(t) - m_1(t)\psi_1(t) - K_D s(t), \quad (96)$$

$$m(t) = k_0 \|\dot{q}(t) + v(t)\| f(q),\ k_0 > 0,\ \psi(t) = m(t)s(t), \quad (97)$$

$$m_1(t) = k_1 \|Y(q, q_d, \dot{q}, \dot{q}_d, \ddot{q}_d, t)\|,\ k_1 > 0,\ \psi_1(t) = m_1(t)s(t), \quad (98)$$

and the update law for the estimate $\theta_0(t)$ of θ_0^*:

$$\dot{\theta}_0(t) = -\Gamma\left(Y^T(q, q_d, \dot{q}, \dot{q}_d, \ddot{q}_d, t)s + \sigma(t)\theta_0(t)\right),,\ \Gamma = \Gamma^T > 0 \quad (99)$$

where $\sigma(t)$ is a switching signal similar to that in (9.610), using a design parameter $\sigma_0 > 0$ and the knowledge of the upper bound M_0 on $\|\theta_0^*\|$:

$$\sigma(t) = \begin{cases} 0 & \text{if } \|\theta_0(t)\| < M_0, \\ \sigma_0(\frac{\|\theta_0(t)\|}{M_0} - 1) & \text{if } M_0 \le \|\theta_0(t)\| < 2M_0, \\ \sigma_0 & \text{if } \|\theta_0(t)\| \ge 2M_0. \end{cases} \quad (100)$$

This adaptive control scheme has the properties: all signals in the closed-loop system are bounded, and the tracking error $e(t) = q(t) - q_d(t)$ satisfies

$$\int_{t_1}^{t_2} \|e(t)\|^2\, dt \le \alpha_0 \left(\frac{\gamma^2}{k_0^2} + \frac{\gamma_1^2}{k_1^2}\right)(t_2 - t_1) + \beta_0 \quad (101)$$

for some constants $\alpha_0 > 0$, $\beta_0 > 0$ and any $t_2 > t_1 \ge 0$, where $\gamma_1 > 0$ is the upper bound on $\sup_{t\ge 0}\|\delta\theta^*(t)\|$. Moreover, $e(t) \in L^2$ and $\lim_{t\to\infty} e(t) = 0$ in the absence of parameter time variations, that is, when $\delta\theta^*(t) = 0$ and $\frac{\partial D(q,t)}{\partial t} = 0$.

The proof of these properties is based on the positive definite function

$$V_0(s, \tilde{\theta}_0) = \frac{1}{2}(s^T D s + \tilde{\theta}_0^T \Gamma^{-1}\tilde{\theta}_0),\ \tilde{\theta}_0(t) = \theta_0(t) - \theta_0^*,\ D = D(q(t), t), \quad (102)$$

which has the following time derivation:

$$\begin{aligned} \dot{V}_0 &= -s^T(t)K_D s(t) - m^2(t)s^T(t)s(t) - m_1^2(t)s^T(t)s(t) - \sigma(t)\tilde{\theta}_0^T(t)\theta_0(t) \\ &\quad -\frac{1}{2}s^T(t)\frac{\partial D(q,t)}{\partial t}(\dot{q}(t) + v(t)) - s^T(t)Y(q, q_d, \dot{q}, \dot{q}_d, \ddot{q}_d, t)\delta\theta^*(t) \\ &\le -s^T(t)K_D s(t) - \left(m(t)\|s(t)\| - \frac{\gamma}{4k_0}\right)^2 + \frac{\gamma^2}{16k_0^2} \\ &\quad -\left(m_1(t)\|s(t)\| - \frac{\gamma_1}{2k_1}\right)^2 + \frac{\gamma_1^2}{4k_1^2} - \sigma(t)\tilde{\theta}_0^T(t)\theta_0(t). \end{aligned} \quad (103)$$

With this adaptive control scheme, as indicated by (101), the tracking performance can be influenced by the design parameters k_0 and k_1 in the feedback control law (96)–(98) (one may increase k_0 and k_1 to reduce the tracking error $e(t)$).

Adaptive Control Scheme II

A different adaptive control scheme can be developed, employing a switching control law which uses adaptive estimates of parameter variation uncertainty bounds, to improve system tracking performance.

To derive such a scheme, we denote the parameter variation uncertainties as

$$g(q,\dot{q},q_d,\dot{q}_d,t) = \frac{1}{2}\frac{\partial D(q,t)}{\partial t}(\dot{q}+v) = [g_1,g_2,\ldots,g_n]^T \tag{104}$$

$$h(q,\dot{q},q_d,\dot{q}_d,\ddot{q}_d,t) = Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\delta\theta^*(t) = [h_1,g_2,\ldots,h_n]^T \tag{105}$$

and make use of the bounding relationship

$$|g_i(q,\dot{q},q_d,\dot{q}_d,t)| \le a_i^*\alpha_i(q,\dot{q},q_d,\dot{q}_d,t),\ i=1,2,\ldots,n \tag{106}$$

$$|h_i(q,\dot{q},q_d,\dot{q}_d,\ddot{q}_d,t)| \le b_i^*\beta_i(q,\dot{q},q_d,\dot{q}_d,\ddot{q}_d,t),\ i=1,2,\ldots,n \tag{107}$$

for some unknown constants a_i^* and b_i^*, and known functions $\alpha_i(q,\dot{q},q_d,\dot{q}_d,t)$ and $\beta_i(q,\dot{q},q_d,\dot{q}_d,\ddot{q}_d,t)$, $i=1,2,\ldots,n$.

If the parameters a_i^* and b_i^* were known, one could use the control law

$$u(t) = Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\theta_0(t) - \phi^*(t) - \phi_1^*(t) - K_D s(t), \tag{108}$$

$$\phi^*(t) = [\mathrm{sgn}[s_1(t)]a_1^*\alpha_1, \mathrm{sgn}[s_2(t)]a_2^*\alpha_2,\ldots,\mathrm{sgn}[s_n(t)]a_n^*\alpha_n]^T, \tag{109}$$

$$\phi_1^*(t) = [\mathrm{sgn}[s_1(t)]b_1^*\beta_1, \mathrm{sgn}[s_2(t)]b_2^*\beta_2,\ldots,\mathrm{sgn}[s_n(t)]b_n^*\beta_n]^T, \tag{110}$$

where $\theta_0(t)$ is updated from (99), and the sgn function is defined as

$$\mathrm{sgn}[w] = \begin{cases} 1 & \text{if } w>0, \\ 0 & \text{if } w=0, \\ -1 & \text{if } w<0. \end{cases} \tag{111}$$

Form V_0 defined in (102), this control law leads to

$$\begin{aligned}
\dot{V}_0 &= -s^T(t)K_D s(t) - s^T(t)\phi^*(t) - s^T(t)\phi_1^*(t) - \sigma(t)\tilde{\theta}_0^T(t)\theta_0(t) \\
&\quad -\frac{1}{2}s^T(t)\frac{\partial D(q,t)}{\partial t}(\dot{q}(t)+v(t)) - s^T(t)Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\delta\theta^*(t) \\
&= -s^T(t)K_D s(t) - \sum_{i=1}^{n}|s_i(t)|a_i^*\alpha_i - \sum_{i=1}^{n}|s_i(t)|b_i^*\beta_i - \sigma(t)\tilde{\theta}_0^T(t)\theta_0(t) \\
&\quad - \sum_{i=1}^{n}s_i(t)g_i - \sum_{i=1}^{n}s_i(t)h_i \\
&\le -s^T(t)K_D s(t).
\end{aligned} \tag{112}$$

The last equality follows from the facts that $\sum_{i=1}^{n}|s_i(t)|a_i^*\alpha_i-\sum_{i=1}^{n}s_i(t)g_i\geq 0$, $\sum_{i=1}^{n}|s_i(t)|b_i^*\beta_i-\sum_{i=1}^{n}s_i(t)h_i\geq 0$ and $\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t)\geq 0$. From (112), one may conclude that all signals in the closed-loop system are bounded, and the tracking error $e(t)=q(t)-q_d(t)$ converges to zero as t goes to ∞.

When the parameters a_i^* and b_i^* are unknown, one can use the control law

$$u(t)=Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\theta_0(t)-\phi(t)-\phi_1(t)-K_Ds(t), \tag{113}$$

$$\phi(t)=[\mathrm{sgn}[s_1(t)]a_1(t)\alpha_1,\mathrm{sgn}[s_2(t)]a_2(t)\alpha_2,\ldots,\mathrm{sgn}[s_n(t)]a_n(t)\alpha_n]^T, \tag{114}$$

$$\phi_1(t)=[\mathrm{sgn}[s_1(t)]b_1(t)\beta_1,\mathrm{sgn}[s_2(t)]b_2(t)\beta_2,\ldots,\mathrm{sgn}[s_n(t)]b_n(t)\beta_n]^T, \tag{115}$$

where $\theta_0(t)$ is updated from (99), and the parameters $a_i(t)$ and $b_i(t)$ are estimates of a_i^* and b_i^* and updated from the adaptive laws:

$$\dot{a}_i(t)=\kappa_{ai}|s_i(t)|\alpha_i(q,\dot{q},q_d,\dot{q}_d,t),\ \kappa_{ai}>0,\ i=1,2,\ldots,n, \tag{116}$$

$$\dot{b}_i(t)=\kappa_{bi}|s_i(t)|\beta_i(q,\dot{q},q_d,\dot{q}_d,\ddot{q}_d,t),\ \kappa_{bi}>0,\ i=1,2,\ldots,n. \tag{117}$$

Consider the positive definite function

$$V(s,\tilde{\theta}_0,\tilde{a}_i,\tilde{b}_i)=\frac{1}{2}(s^TDs+\tilde{\theta}_0^T\Gamma^{-1}\tilde{\theta}_0+\sum_{i=1}^{n}\kappa_{ai}^{-1}\tilde{a}_i^2+\sum_{i=1}^{n}\kappa_{bi}^{-1}\tilde{b}_i^2), \tag{118}$$

where $\tilde{\theta}_0(t)=\theta_0(t)-\theta_0^*$, $D=D(q(t),t)$, $\tilde{a}_i(t)=a_i(t)-a_i^*$, $\tilde{b}_i(t)=b_i(t)-b_i^*$, $i=1,2,\ldots,n$. Using (94), (113), (99), (116) and (117), we have

$$\begin{aligned}
\dot{V}&=-s^T(t)K_Ds(t)-s^T(t)\phi(t)-s^T(t)\phi_1(t)-\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t)\\
&\quad-\frac{1}{2}s^T(t)\frac{\partial D(q,t)}{\partial t}(\dot{q}(t)+v(t))-s^T(t)Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\delta\theta^*(t)\\
&\quad+\sum_{i=1}\tilde{a}_i\kappa_{ai}^{-1}\dot{a}_i+\sum_{i=1}\tilde{b}_i\kappa_{bi}^{-1}\dot{b}_i\\
&=-s^T(t)K_Ds(t)-\sum_{i=1}^{n}|s_i(t)|a_i(t)\alpha_i-\sum_{i=1}^{n}|s_i(t)|b_i(t)\beta_i-\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t)\\
&\quad-\sum_{i=1}^{n}s_i(t)g_i-\sum_{i=1}^{n}s_i(t)h_i+\sum_{i=1}\tilde{a}_i\kappa_{ai}^{-1}\dot{a}_i+\sum_{i=1}\tilde{b}_i\kappa_{bi}^{-1}\dot{b}_i\\
&\leq-s^T(t)K_Ds(t)-\sum_{i=1}^{n}|s_i(t)|a_i(t)\alpha_i-\sum_{i=1}^{n}|s_i(t)|b_i(t)\beta_i-\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t)\\
&\quad+\sum_{i=1}^{n}|s_i(t)|a_i^*\alpha_i+\sum_{i=1}^{n}|s_i(t)|b_i^*\beta_i+\sum_{i=1}\tilde{a}_i\kappa_{ai}^{-1}\dot{a}_i+\sum_{i=1}\tilde{b}_i\kappa_{bi}^{-1}\dot{b}_i\\
&=-s^T(t)K_Ds(t)-\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t)\\
&\leq-s^T(t)K_Ds(t).
\end{aligned} \tag{119}$$

This result also implies that all signals in the closed-loop system are bounded, and the tracking error $e(t) = q(t) - q_d(t)$ converges to zero as t goes to ∞.

However, since the adaptive control scheme (113) uses the sgn functions in $\phi(t)$ and $\phi_1(t)$ and such switching signals are discontinuous when $s_i(t)$ passes through zero, it may lead to chattering of system response.

Remark 1 The adaptive control scheme (113) may have certain advantage for performance even if when the parameters a_i^* and b_i^* are known. This is because the parameters a_i^* and b_i^* are only the upper bounds for the parameter variation uncertainties g_i and h_i in (104) and (105), and some smaller (and unknown) bounds may exist and can be estimated by the adaptive laws (116) and (117). The use of smaller bounds is desirable because it leads to smaller control signals. In this case, the adaptive laws (116) and (117) can be modified by setting

$$\dot{a}_i(t) = 0,\ t \geq \tau \ \text{ if } a_i(\tau) = a_i^*, \tag{120}$$

$$\dot{b}_i(t) = 0,\ t \geq \tau \ \text{ if } b_i(\tau) = b_i^* \tag{121}$$

With this modification, we also have $\dot{V} \leq -s^T(t)K_D s(t)$, as desired. $\square$

Adaptive Control Scheme III

As mentioned above the use of the discontinuous sgn functions in $\phi(t)$ and $\phi_1(t)$ in the adaptive control scheme (113) may cause chattering of system response. To overcome possible chatterings, we can modify the control law (113) as

$$u(t) = Y(q, q_d, \dot{q}, \dot{q}_d, \ddot{q}_d, t)\theta_0(t) - \hat{\phi}(t) - \hat{\phi}_1(t) - K_D s(t), \tag{122}$$

$$\hat{\phi}(t) = [\text{sat}[s_1(t); \varepsilon_1]a_1(t)\alpha_1, \ldots, \text{sat}[s_n(t); \varepsilon_n]a_n(t)\alpha_n]^T, \tag{123}$$

$$\hat{\phi}_1(t) = [\text{sat}[s_1(t); \eta_1]b_1(t)\beta_1, \ldots, \text{sat}[s_n(t); \eta_n]b_n(t)\beta_n]^T, \tag{124}$$

where the sat function is defined as

$$\text{sat}[s_i; x_i] = \begin{cases} 1 & \text{if } s_i > x_i, \\ \frac{s_i}{x_i} & \text{if } |s_i| \leq x_i, \\ -1 & \text{if } s_i < -x_i \end{cases} \tag{125}$$

for some chosen $x_i > 0$ (for $x_i = \varepsilon_i$ or $x_i = \eta_i$ in (123) and (124)), $i = 1, 2, \ldots, n$, with the associated indicator funtions

$$\chi[s_i; x_i] = \begin{cases} 1 & \text{if } |s_i| > x_i, \\ 0 & \text{if } |s_i| \leq x_i. \end{cases} \tag{126}$$

Such functions have the property: $\chi[s_i;x_i](1-\chi[s_i;x_i])=0$ (that is, $\chi[s_i;x_i]=0$ whenever $1-\chi[s_i;x_i]=1$, and $\chi[s_i;x_i]=1$ whenever $1-\chi[s_i;x_i]=0$).

The adaptive laws are also modified as

$$\dot{a}_i(t)=\chi[s_i;\varepsilon_i]\kappa_{ai}|s_i(t)|\alpha_i(q,\dot{q},q_d,\dot{q}_d,t),\ \kappa_{ai}>0,\ i=1,2,\ldots,n, \tag{127}$$

$$\dot{b}_i(t)=\chi[s_i;\eta_i]\kappa_{bi}|s_i(t)|\beta_i(q,\dot{q},q_d,\dot{q}_d,\ddot{q}_d,t),\ \kappa_{bi}>0,\ i=1,2,\ldots,n. \tag{128}$$

With this modification, we have

$$\begin{aligned}
\dot{V} &= -s^T(t)K_Ds(t)-s^T(t)\hat{\phi}(t)-s^T(t)\hat{\phi}_1(t)-\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t)\\
&\quad -\frac{1}{2}s^T(t)\frac{\partial D(q,t)}{\partial t}(\dot{q}(t)+v(t))-s^T(t)Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\delta\theta^*(t)\\
&\quad +\sum_{i=1}\tilde{a}_i\kappa_{ai}^{-1}\dot{a}_i+\sum_{i=1}\tilde{b}_i\kappa_{bi}^{-1}\dot{b}_i\\
&= -s^T(t)K_Ds(t)-\sum_{i=1}^n s_i(t)\mathrm{sat}[s_i;\varepsilon_i]a_i(t)\alpha_i-\sum_{i=1}^n s_i(t)\mathrm{sat}[s_i;\eta_i]b_i(t)\beta_i\\
&\quad -\sum_{i=1}^n s_i(t)g_i-\sum_{i=1}^n s_i(t)h_i+\sum_{i=1}\tilde{a}_i\kappa_{ai}^{-1}\dot{a}_i+\sum_{i=1}\tilde{b}_i\kappa_{bi}^{-1}\dot{b}_i-\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t)\\
&\le -s^T(t)K_Ds(t)-\sum_{i=1}^n s_i(t)\mathrm{sat}[s_i;\varepsilon_i]a_i(t)\alpha_i-\sum_{i=1}^n s_i(t)\mathrm{sat}[s_i;\eta_i]b_i(t)\beta_i\\
&\quad +\sum_{i=1}^n |s_i(t)|a_i^*\alpha_i+\sum_{i=1}^n |s_i(t)|b_i^*\beta_i+\sum_{i=1}\tilde{a}_i\kappa_{ai}^{-1}\dot{a}_i+\sum_{i=1}\tilde{b}_i\kappa_{bi}^{-1}\dot{b}_i-\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t)\\
&= -s^T(t)K_Ds(t)-\sum_{i=1}^n(1-\chi[s_i;\varepsilon_i])s_i(t)\mathrm{sat}[s_i;\varepsilon_i]a_i(t)\alpha_i\\
&\quad -\sum_{i=1}^n(1-\chi[s_i;\eta_i])s_i(t)\mathrm{sat}[s_i;\eta_i]b_i(t)\beta_i+\sum_{i=1}^n(1-\chi[s_i;\varepsilon_i])|s_i(t)|a_i^*\alpha_i\\
&\quad +\sum_{i=1}^n(1-\chi[s_i;\eta_i])|s_i(t)|b_i^*\beta_i-\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t).
\end{aligned} \tag{129}$$

This modified scheme would ensure the closed-loop signal boundedness but not the asymptotic convergence of the tracking error $e(t)=q(t)-q_d(t)$ to zero (only a bounded tracking error $e(t)=q(t)-q_d(t)$ of the order ε_i and η_i).

Derivation of (10.152)

In this case, the expression (5.30) also holds

$$u(t)=\phi_1^{*T}\frac{a(s)}{\Lambda(s)}[u](t)+\phi_2^{*T}\frac{a(s)}{\Lambda(s)}[y](t)+\phi_{20}^*y(t)+\phi_3^*P_m(s)[y](t) \tag{130}$$

(with θ_i^* replaced by ϕ_i^* as the new notation and the exponetially decaying term $\varepsilon_1(t)$ ignored). Recall (10.39):

$$u(t) = u_d(t) + (\theta - \theta^*)^T \omega(t) + d_N(t) \tag{131}$$

and (10.15) (with $a_s(t) = 0$ for simplicity):

$$u_d(t) = -\theta^T(t)\omega(t) \tag{132}$$

from which we have

$$u(t) = -\theta^{*T}\omega(t) + d_N(t). \tag{133}$$

Using (131) for $u(t)$ in the left side of (130) and (133) for $u(t)$ in the right side of (130), we obtain

$$\begin{aligned} &u_d(t) + (\theta - \theta^*)^T \omega(t) + d_N(t) \\ &= \phi_1^{*T} \frac{a(s)}{\Lambda(s)}[-\theta^{*T}\omega + d_N](t) + \phi_2^{*T} \frac{a(s)}{\Lambda(s)}[y](t) + \phi_{20}^* y(t) + \phi_3^* P_m(s)[y](t). \end{aligned} \tag{134}$$

Subtracting (134) from (10.151) and recalling (10.148), we have (10.152).

Backstepping Design with Nonsmooth Inverse Signal

For an adaptive inverse control scheme using a backstepping feedback design for the output-feedback nonlinear systems (see Section 10.6.3), the nonlinearity inverse signal $\omega(t)$ in (10.280), defined in (10.21) or (10.35) for the case of an adaptive dead-zone inverse $\widehat{DI}(\cdot)$ or an adaptive backlash inverse $\widehat{BI}(\cdot)$, is not smooth (it is not even continuous; but it is well-defined). A backstepping-based feedback control design needs certain smoothness condition. Will the nonsmoothness of $\omega(t)$ cause any problem? The answer is “no”. It turns out that the use of filters in the state observer helps to avoid the potential difficulty with nonsmoothness of such signals in a backstepping design.

Starting from (10.277), (10.283) and (10.284), we see that the nonsmooth signal $\omega(t)$ passes the filters $\frac{p_{ij}(s)}{\Lambda(s)}$, where $\frac{p_{ij}(s)}{\Lambda(s)}$ are the components of $(sI - A_0)^{-1}$ for $A_0 = A - kc$ with $k = [k_1, k_2, \ldots, k_n]^T$ and (A, c) in (10.272). This means that only the signals $\omega_{ij}(t)$ in (10.285) are associated with the nonsmooth signal $\omega(t)$. In fact, it is the signal $\hat{x}_2(t)$ defined in (10.128) that plays a key role in the backstepping design procedure, that is, only those of the signals $\omega_{ij}(t)$, for $i = 0, 1, \ldots, m$ and $j = 2$, are crucial in the backstepping design.

From the canonical form of A and c, we can obtain

$$\begin{aligned} &p_{02}(s) = s + k_1,\ p_{12}(s) = s(s + k_1),\ p_{22}(s) = s^2(s + k_1),\ \ldots, \\ &p_{n-2,2}(s) = s^{n-2}(s + k_1),\ p_{n-1,2}(s) = -k_2 s^{n-2} - k_3 s^{n-3} - \cdots - k_{n-1}s - k_n. \end{aligned} \tag{135}$$

This calculation can be verified by using the symbolic algebra operations in Matlab. For example, for $n = 5$, we can use

```
a=sym('[s+k1, -1, 0, 0, 0; k2, s, -1, 0, 0;
k3, 0, s, -1, 0; k4, 0, 0, s, -1; k5, 0, 0, 0, s]')
b=inv(a)
c=symmul(det(a),b)
```

to get the numerator matrix of $(sI - A_0)^{-1}$ as

```
[s^4,                          s^3,                s^2,
                                   s,                         1]
[-s^3*k2-k3*s^2-k4*s-k5, s^3*(s+k1),          s^2*(s+k1),
                               (s+k1)*s,                  s+k1]
[-s*(k3*s^2+k4*s+k5),-k3*s^2-k4*s-k5,s^2*(k2+s^2+s*k1),
                      s*(k2+s^2+s*k1),          k2+s^2+s*k1]
[-s^2*(k4*s+k5),          -s*(k4*s+k5),      -k4*s-k5,
              s*(k2*s+k3+s^3+s^2*k1), k2*s+k3+s^3+s^2*k1]
[-k5*s^3,                  -k5*s^2,            -k5*s,
     -k5,                          s^4+k1*s^3+k2*s^2+k3*s+k4]
```

In the backstepping design procedure, the derivatives $\omega_{i2}^{(j)}(t)$, $i = 0,1,\ldots,m$, $j = 0,1,\ldots,\rho-1$, are needed, where $\rho = n-m$. For the case of $\rho = n-m = 1$, no derivative of $\omega_{i2}(t)$ is used. For $\rho = n-m \geq 2$, the highest degree of $s^{\rho-1}p_{i2}(s)$, $i = 0,1,\ldots,m-1$, is $\rho-1+m = n-1$, and the degree of $s^{\rho-1}p_{m2}(s)$ is n, so that $\frac{s^{\rho-1}p_{i2}(s)}{\Lambda(s)}$, $i = 0,1,\ldots,m-1$, are strictly proper, and $\frac{s^{\rho-1}p_{m2}(s)}{\Lambda(s)}$ is proper. Therefore, the signals $\omega_{i2}^{(j)}(t)$, $i = 0,1,\ldots,m$, $j = 0,1,\ldots,\rho-1$, are well-defined, and so all other signals related the inverse signal $\omega(t)$ in the backstepping design procedure, that is, the nonsmoothness $\omega(t)$ does not cause any problem for the backstepping-based adaptive inverse control scheme.