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A note on future branching time

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I

In future branching time any point can have different possible futures but all points have a unique past. The sentence 'always *A*' will be true at a point in future branching time just in case '*A*' holds at every time in the past, present, and all possible futures of it. A point reached by first going to some past time and then into the future may be neither past, nor present, nor future to the point from which we started. This will be the case if we move into the past on one branch and then into the future on a different branch. In assessing the truth value of 'always *A*' in future branching time (when *A* contains no modal operators) we look only at points directly accessible without this kind of "dog-legging".

In moving to a future time we bypass certain branches. These may contain possibilities which are lost as we move past them. Conversely, in moving to a past time new branches become part of our future and possibilities can be gained. Monotonic gain and loss of possibility, with respect to movement into the past and future, is characteristic of future branching time.

If '□' is interpreted as 'always' we can ask for the modal logic which axiomatizes the sentences about 'always' which are valid in future branching time. The modal logic **K** can be axiomatized as follows:

I am indebted to David Lewis. Without his valuable help this could not have been written. In particular, axiom M was his suggestion. Professor G. E. Hughes has independently reached the same result in his paper [1], published in this issue of *Theoria*. His proof is different from the one offered here.

RULES

Necessitation: If A is a theorem, then $\Box A$ is a theorem.

Modus Ponens: If A, $A \supset B$ are theorems, then B is a theorem.

Substitution: If B is the result of uniformly substituting for propositional variables in A and A is a theorem, then B is a theorem.

AXIOMS

Propositional calculus tautologies.

$\Diamond A \equiv -\Box - A$.

$\Box(A \supset B) \supset (\Box A \supset \Box B)$.

We also consider the sentences

T $\Box A \supset A$.
B $A \supset \Box\Diamond A$.
4² $\Box\Box A \supset \Box\Box\Box A$.
S $\Diamond\Diamond A \mathbin{\&} \Diamond\Diamond B \supset \Diamond(\Diamond A \mathbin{\&} B)$.
M $\Diamond A \mathbin{\&} -\Diamond B \supset -\Diamond(-\Diamond A \mathbin{\&} \Diamond B)$.

Rescher and Urquhart conjectured that the logic of 'always' for future branching time is Thomas' **KBT4²** (see [2], pp. 129, 258 and [5], p. 59). In [4] Segerberg disproved this conjecture by noting that the sentence **S**, although valid in future branching time, is not a theorem of **KBT4²**. In this note it is proved that a sentence is valid in frames which represent one-way future branching time iff the sentence is a theorem of the modal logic **KTBM**.

II

A *frame* is an ordered pair $\langle I, R\rangle$ in which I is a non-empty set and R a 2-place relation on I. The frame $\langle I, R\rangle$ is reflexive, symmetric, or monotonic just in case R is a reflexive, symmetric, or monotonic relation. R is monotonic if it satisfies the condition

$$\forall\, i, j, k, h\ (iRj \mathbin{\&} jRk \mathbin{\&} iRh \supset (iRk \vee jRh)).$$

V is an *interpretation based upon the frame* $\langle I, R\rangle$ iff V is a function from sentences of modal logic to subsets of I such that for all sentences A in its domain

(1) $V(-A) = I - V(A)$,
(2) $V(A \mathbin{\&} B) = V(A) \cap V(B)$, etc.
(3) $V(\Box A) = \{i \in I\colon\ \forall j \in I(iRj \supset j \in V(A))\}$,
(4) $V(\Diamond A) = \{i \in I\colon\ \exists j \in I(iRj \mathbin{\&} j \in V(A))\}$.

A sentence is *valid* in the frame $\langle I, R\rangle$ iff for each interpretation V based upon $\langle I, R\rangle$, $V(A) = I$.

A frame $\langle I, R\rangle$ *represents future branching* time iff there exists a subrelation P of R such that (a) $R = P \cup \breve{P}$, (b) P is reflexive, (c) P is transitive, (d) $\forall i, j, k \in I(iPj \mathbin{\&} iPk \supset jRk)$, and (e) $P \cap \breve{P} = id$, where 'id' denotes the identity relation. Note that, given (a)—(e), condition (d) is equivalent to requiring P to be piecewise connected. 'iPj' is read 'j is past or present to i' and '$i\breve{P}j$' is read 'j is future or present to i'.

(We could have said that a frame $\langle I, R\rangle$ represents future branching time just in case there is a subrelation P of R which satisfies (a') $R = P \cup \breve{P} \cup id$, (b') P is asymmetric, (c), and (d). In this case 'iPj' is read 'j is past to i'. Since there exists a subrelation of R satisfying (a)—(e) iff there exists a subrelation of R satisfying (a'), (b'), (c), and (d), these formulations are equivalent.)

LEMMA 1. *If the frame* $\langle I, R\rangle$ *represents future branching time then it is reflexive, symmetric, and monotonic.*

LEMMA 2. **KTBM** *is sound and complete for reflexive, symmetric, and monotonic frames.*

Proof. Soundness. It is well known that the axioms **B** and **T** are valid in symmetric and reflexive frames, resp. We show **M** is valid in monotonic frames. Suppose **M** fails in the monotonic frame $\langle I, R\rangle$. Then for some $i \in I$, $i \in V(\Diamond A)$, $i \in V(-\Diamond B)$, and $i \in V(\Diamond(-\Diamond A \mathbin{\&} \Diamond B))$. So for some $j, k, h \in I$, iRh and $h \in V(A)$, iRj and $j \in V(-\Diamond A \mathbin{\&} \Diamond B)$, and jRk and $k \in V(B)$. Since $iRjRk$ and iRh, by monotonicity we have either iRk, which contradicts $i \in V(-\Diamond B)$, or jRh, which contradicts $j \in V(-\Diamond A)$.

Completeness: The *canonical frame* for **KTBM** is the frame $\langle I, R\rangle$ in which I is the set of all maximal **KTBM**-consistent sets and, for any $i, j \in I$, iRj iff $\forall A\ (A \in j \supset \Diamond A \in i)$. R is called the *canonical accessibility relation* for KTBM. The *canonical interpretation* V for KTBM is defined as follows: for any propositional variable P, $V(P)=\{i\in I\colon P\in i\}$. It is known that $\langle I, R\rangle$ is a frame, V is an interpretation based upon $\langle I, R\rangle$, and $V(A)=I$ only if A is a theorem of KTBM. Since the logic contains T and B, the canonical frame is reflexive and symmetric. We show it is monotonic. Suppose it is not. Then for some $i, j, k, h\in I$, iRj & jRk & iRh and $-(iRk)$ & $-(jRh)$. By $-(iRk)$, $A\in k$ and $-\Diamond A\in i$, for some A. By $-(jRh)$, $B\in h$ and $-\Diamond B\in j$, for some B. Since iRh, $\Diamond B\in i$ and so $-\Diamond A$ & $\Diamond B\in i$. By M $-\Diamond(\Diamond A$ & $-\Diamond B)\in i$. Since jRk and $A\in k$, $\Diamond A\in j$. So $\Diamond A$ & $-\Diamond B\in j$. By hypothesis iRj, so $\Diamond(\Diamond A$ & $-\Diamond B)\in i$. This is a contradiction since i is consistent.

Theorem 1. *If A is a theorem of* KTBM, *then A is valid in all frames which represent future branching time.*

Proof. Suppose A is a theorem of KTBM. By Lemma 2 A is valid in all reflexive, symmetric, and monotonic frames. By Lemma 1 these include the frames which represent future branching time.

Lemma 3. *The canonical accessibility relation for* KTBM *has a subrelation P that meets conditions (a)—(d).*

Proof. Define iPj iff $\forall A(\Diamond A\in i\supset\Diamond A\in j)$.

(a) First, suppose iPj. Then $\forall A(\Diamond A\in i\supset\Diamond A\in j)$ and by T $\forall A(A\in i\supset\Diamond A\in i)$. This yields $\forall A(A\in i\supset\Diamond A\in j)$, which is jRi. By symmetry iRj. Next, suppose $i\breve{P}j$. Then jPi and $\forall A(\Diamond A\in j\supset\Diamond A\in i)$. By T we have $\forall A(A\in j\supset\Diamond A\in i)$. This is iRj. Finally, suppose iRj but neither iPj nor $i\breve{P}j$. Then for some A, $\Diamond A\in i$ and $-\Diamond A\in j$ and for some B, $\Diamond B\in j$ and $-\Diamond B\in i$. Then $\Diamond A$ & $-\Diamond B\in i$ and, by M, $-\Diamond(-\Diamond A$ & $\Diamond B)\in i$. Since $\Diamond B$ & $-\Diamond A\in j$ and iRj this is a contradiction.

(b) P is reflexive since $\forall A(\Diamond A\in i\supset\Diamond A\in i)$, for all $i\in I$.

(c) P is transitive since if iPj (i.e., $\forall A(\Diamond A\in i\supset\Diamond A\in j)$ and jPk (i.e., $\forall A(\Diamond A\in j\supset\Diamond A\in k)$) then $\forall A(\Diamond A\in i\supset\Diamond A\in k)$ which is iPk.

(d) Suppose iPj & iPk. By iPj and T we get $\forall A(A\in i\supset\Diamond A\in j)$. This is jRi and by symmetry iRj, $\forall A(A\in j\supset\Diamond A\in i)$. By iPk we get $\forall A(A\in j\supset\Diamond A\in k)$ which is kRj.

Two points are *clustered* just in case iPj & $i\breve{P}j$. In the canonical frame for KTBM it is possible that the subrelation P just defined is such that $P\cap\breve{P}\neq id$. There may be non-identical clustered points. To eliminate this possibility we construct a new frame $\langle I^*, R^*\rangle$ in which the clustered points of $\langle I, R\rangle$ have been "flattened out". $\langle I^*, R^*\rangle$ represents future branching time. It is constructed by the bulldozing technique ([3], pp. 304—305).

For bulldozing we define a new space $I^*=I\times N$, where N is the set of natural numbers. Let O be a linear ordering on I. In I^* we have denumerably many copies of every point of I. We abbreviate the point $\langle i, k\rangle\in I^*$ as i_k and refer to any point of I^* with a first coordinate i as a *copy of* i in I^*. Define P^* as follows:

(i) $i, j\in I$ are not clustered.
  $\forall i_m, j_n\in I^*$, $i_mP^*j_n$ iff iPj.
(ii) $i, j\in I$ are clustered.
  (a) $\forall i_m, j_m\in I^*$ (i_m, j_m belong to the same copy of the same cluster), $i_mP^*j_m$ iff iOj.
  (b) $\forall i_m, j_n\in I^*$(i_m, j_n belong to different copies of the same cluster), $i_mP^*j_n$ iff $m<n$.

Define $R^*=P^*\cup\breve{P}^*$.

Lemma 5. *Let $i_m, j_n\in I^*$ be copies of clustered points $i, j\in I$. Then $i_mR^*j_n$ and iRj.*

Proof. If $m=n$ it follows from (iia) that $i_mR^*j_n$. If $m\neq n$ then either $m<n$ and $i_mP^*j_n$ or $n<m$ and $j_nP^*i_m$. In either case $i_mR^*j_n$ by the definition of R^*. Since iPj & jPi by hypothesis, and $R=P\cup\breve{P}$ by (a) of Lemma 3, it follows that iRj.

Lemma 6. *If iRj then for any $i_m, j_n\in I^*$ which copy i, j resp., $i_mR^*j_n$.*

Proof. By condition (a) if iRj then iPj or jPi. If iPj & jPi, $i_mR^*j_n$ holds for arbitrary copies of i and j by the previous lemma. If i, j are not clustered then by (i) iPj iff $i_mP^*j_n$ and jPi iff $i_mP^*j_n$, for arbitrary copies of i and j. By definition of R^* this gives us $i_mR^*j_n$ for arbitrary copies of i and j.

Lemma 7. *P^* is reflexive and transitive.*

Proof omitted

Lemma 8. *P^* satisfies condition (d).*

Proof. Suppose $i_m P^* j_n$ & $i_m P^* k_l$. If j_n, k_l copy clustered points in I then $j_n R^* k_l$ by Lemma 5. If they do not copy clustered points then iPk & iPj by (i) (and possibly Lemma 5). By (d) jPk or kPj and by (i) again j_n and k_l are P^*- and hence R^*-connected.

So R^* satisfies (a)—(e) and $\langle I^*, R^*\rangle$ represents future branching time.

Lemma 9. *A fails in $\langle I, R\rangle$ only if A fails in $\langle I^*, R^*\rangle$.*

Proof. Suppose A fails under V based on $\langle I, R\rangle$. Then $V(A) \neq I$. For all formulae A, let $V^*(A) = \{i_m \in I^* : i \in V(A)\}$. V^* is based upon the frame $\langle I^*, R^*\rangle$. The proof that basedness conditions (1) and (2) are satisfied is straightforward.

For (3) consider $i_m \in V^*(\Box A)$ for some $i_m \in I^*$. Suppose that for some $j_n \in I^*$, $i_m R^* j_n$ and $j_n \notin V^*(A)$. Then by definition of V^*, $j \notin V(A)$. If i_m and j_n are clustered then iRj by Lemma 5. If they are not clustered then iRj by (i). This gives a contradiction in either case.

Suppose $\forall h^* \in I^*(i_m R^* h^* \supset h^* \in V^*(A))$ (where h^* ranges over points in I^*) and $i_m \notin V^*(\Box A)$. Then by definition of V^* for some $j \in I$, iRj and $j \notin V(A)$. By Lemma 6, for some $j_n \in I^*$ which copies j, $i_m R^* j_n$. By definition of V^*, $j_n \notin V^*(A)$, which contradicts the hypothesis.

Theorem 2. *If A is valid in all frames which represent future branching time then A is a theorem of* KTBM.

Proof. Suppose A is not a theorem of KTBM. By Lemma 2 A fails in the canonical frame $\langle I, R\rangle$ for KTBM. By Lemmas 4—9 A fails in some frame which represents future branching time.

Theorem 3. *A is valid in all frames which represent future branching time iff A is a theorem of* KTBM.

Proof. Immediate from theorems 1 and 2.

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