Mathematics and Modality

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It has been suggested in a number of places that a platonist account of mathematical truth raises difficult problems for a theory of mathematical knowledge.[1] A "platonist theory of mathematical truth" is, roughly, a theory of mathematical truth which contains explicit reference to peculiarly mathematical objects such as sets or numbers. In this paper I consider several "alternatives" to a platonist account of mathematical truth. They can all be called "modal views of mathematics" in that they purport to analyze mathematical statements in such a way that reference to abstract mathematical objects is eliminated in favor of talk about necessity and possibility. The details of this elimination vary significantly from case to case. They all appear to agree, however, on the claim that elementary number theory (and, indeed, all of mathematics) need not be understood as a theory about a special domain of abstract mathematical objects.

One proponent of a modal approach to mathematics is Putnam. He is quite explicit about what he takes to be the point of viewing mathematics from a modal perspective. In "Mathematics Without Foundations" he suggests that we are too much in the grip of the "mathematics as set theory" picture ([7]: 11). What is notable about this picture is not just that it identifies numbers with sets. Rather "the important thing about the picture is that [according to it] mathematics describes 'objects' " ([7]: 9). Putnam feels that a "modal view of mathematics" provides us with an "equivalent description" of the same mathematical facts as are described by a "mathematics as set theory" view. However, by focussing on a modal picture, we see that the emphasis on mathematical objects (be they numbers or sets) is not essential. We need not analyze the content of mathematics in terms of abstract mathematical objects. In another paper Putnam suggests that this wedding of mathematics to mathematical objects in the "mathematics as set theory" picture is actually pernicious ([8]: 60-1, 72). It blocks the way to a theory of mathematical knowledge which can only be developed when we take seriously a modal view.

In the sequel I consider several different modal views of mathematics. I argue that none of them succeeds in providing an analysis of mathematical truth which avoids appeal to abstract mathematical objects.[2] As a consequence I believe (*contra* Putnam) that none of these modal views avoids the epistemological problems engendered by a platonist theory of mathematical truth. Since there are other modal views which I do not examine in detail here these remarks are not conclusive. They do not show that no modal view could be adequate. I do not argue for this stronger claim in this paper.

I

I understand a "modal view of mathematics" to be a view about the "logical from" of the sentences of mathematics. Any position which maintains that a certain type of (nonstandard) theory of logical form is the correct semantic analysis of mathematical assertions will be classified as a "modal view." This characterization is broad enough to include as modal radically different accounts of the nature of mathematics.

I assume that a *theory of logical form* is a translation function. The domain of this function comprises the sentences in the language whose form is being analyzed. The range of the function consists of the sentences of some interpreted formal language. By an interpreted formal *language* F I understand an ordered pair (L, I). L is the *language* of F. It contains the primitive symbols and formulae of F. I is an *interpretation* for L. It is the basis of a (recursive) specification of truth-conditions for every sentence (i.e. closed formula) of L. A theory of logical form for elementary number theory (ENT) is thus a function M from sentences of elementary number theory to an interpreted formal language (L, I), where L is a syntactical specification of a language and I is an interpretation for L. If

L is a first-order language, then M is a first-order theory of logical form. If L contains among its primitive symbols a modal operator, then M is a *modal theory of logical form*. A *modal view of mathematics* is a view which maintains that some modal theory of logical form is the correct semantic analysis of mathematical sentences. (In this paper we limit ourselves to sentences of ENT.)

This framework accommodates a wide variety of modal views of mathematics. Let M and M′ be two modal theories of logical form for ENT. Let their associated formal languages be (L, I) and (L, I′), respectively. Suppose that I and I′ contain "possible world" analyses of the modalities which are based upon different accessibility relations. (See [4]: 63.) In this case, even if M and M′ translate each sentence S of ENT into syntactically identical sentences M(S) and M′(S) of L, they are still different theories of logical form. M might, for example, analyze ENT in terms of "logical necessity" while M′ analyzes it in terms of "physical necessity." I consider the former view in this section and the latter in the next.

In "Mathematics Without Foundations" Putnam presents a modal view which is based upon "logical necessity" by way of an example. I take it that this suggestion is meant to be no more than a hint of things to come and perhaps should not, on that account, be criticized. However, I think that this type of example is to a large extent responsible for the plausibility of Putnam's overall position. I would like to undercut this plausibility right at the outset.

We are asked to consider some finitely axiomatizable first-order sub-theory of arithemtic in which all true atomic formulae and all true negations of atomic formulae (in the language of ENT) are derivable. Let Q be the conjunction of the axioms of this system.[3] Then, for example, any true formula in the language of ENT which has the form

$$\exists x_1 \ldots \exists x_n T(x_1, \ldots, x_n)$$

is derivable from Q. Let S be a true formula of this form. On the "mathematics as set theory" view we understand S as saying something like: there exist numbers $x_1, \ldots, x_n$ such that $T(x_1, \ldots, x_n)$.

By our choice of S we have that

(1-1) $\vdash(Q \supset S)$,

where '$\vdash$' denotes provability in first-order logic. So we know that

(1-2) '$(Q \supset S)$' is valid.

Suppose we have a modal formal language in which '$\Box$' is interpreted as "logical necessity" or truth in all logically possible interpretations. Given this interpretation of '$\Box$' we could translate (1-2) as

(1-3) $\Box(Q \supset S)$.

Let L be a first-order modal language which contains among its primitive symbols the individual constant 'a_0', the dyadic predicate symbol 'S', and the triadic predicate symbols 'A' and 'M'. There is a way to replace the mathematical symbols '0', '′', '+', and '·' by the symbols 'a_0', 'S', 'A', 'M' of L (respectively) without effecting the validity of '$Q \supset S$'. (1-3) can thus be transformed into the sentence '$\Box(Q^* \supset S^*)$' of pure modal logic. Then (1-2) is equivalent to

(1-4) $\Box(Q^* \supset S^*)$,

which is now a sentence of the modal language L.

These remarks can be seen as providing a translation procedure which takes sentences of ENT which are provable in Q to sentences of the form (1-4) and sentences of ENT which are refutable in Q to sentences of the form

(1-4′) $\Box(Q^* \supset -S^*)$,

That is, they amount to a modal theory of logical form with the translation function M defined as follows

(1-5) $M(S) = \Box(Q^* \supset S^*)$,

where S is a sentence of ENT provable from Q. The interpreted modal language (L, I) associated with this translation function M might contain a modal interpretation I = (A, R, G) of the following form. A, the collection of "worlds" of I, comprises all logically possible interpretations of the non-modal sub-part of L. By a "logically possible interpretation" I simply mean an interpretation in which the logical constants $-$, &, $\supset$, $\forall$, and $\exists$ receive the standard interpretation. R, the accessibility relation of I, is the universal relation on $A \times A$. For a predicate symbol or individual constant K in L and a world i in A we let G(i, K) be K's interpretation in i. We can then show that for a closed formula S of ENT, S is true-in-I at i just in case $i \models S$. (See [4].) A statement of the form (1-4) will be true-in-I at a point just in case '$(Q^* \supset S^*)$' is true-in-I at all logically possible interpretations of the non-modal part of L. That is, '$\Box(Q^* \supset S^*)$' is true-in-I (at a point) just in case '$Q^* \supset S^*$)' is logically valid.

Let us see what this is supposed to show. Putnam claims that "The mathematical content of assertion (1-4) is certainly the same as that of the assertion that *there exist numbers* $x_1, \ldots,$ x_n such that $T(x_1, \ldots, x_n)$." He goes on to say that "the mathematical equivalence is so obvious that they might as well be synonymous as far as the mathematician is concerned" ([7]: 10). There is no need, we are told, to understand S as a statement describing "eternal objects." (1-4) simply says that Q entails

$$\exists x_1 \ldots \exists x_n T(x_1, \ldots, x_n)$$

and this does not appear to be about objects at all.

I believe that Putnam's claim that (1-4) has the same mathematical content as the statement that there exist numbers $x_1, \ldots, x_n$ such that $T(x_1, \ldots, x_n)$ is clearly false. (1-4) is equivalent to the statement in question, *if at all, only under the assumption that no contradiction can be derived from* Q^*. Alternatively, they are equivalent only under the assumption that Q^* has a model. The sentence S and its proposed modal translation (1-4) do not provide an "equivalent description" of the same mathematical fact(s) at all. For suppose that Q^* is not consistent. Then both

(1-5) $\Box(Q^* \supset S^*)$

and

(1-6) $\Box(Q^* \supset -S^*)$

are true in the modal interpretation just sketched. But if statements of this form are equivalent to statements about the existence of numbers, then it follows that both

$$\exists x_1 \ldots \exists x_n T(x_1, \ldots, x_n)$$

and

$$-\exists x_1 \ldots \exists x_n T(x_1, \ldots, x_n)$$

are true. *This* is a consequence that no mathematician (nor anyone else) could accept. Hence, we must reject the claim that the sentences of ENT have the same mathematical content as their proposed modal translations.

This objection is sufficient to show that the modal view under consideration does not provide *logically* equivalent modal translations for the sentences of ENT. Nevertheless, one might maintain that, if Q* is consistent, the view still provides "mathematically equivalent" translations. Putting the matter in a slightly different way, S and its modal translation (1-4) might be "*mathematically* equivalent descriptions" of the same fact(s). Putnam himself seems to endorse this point in the remarks quoted two paragraphs back. However, even if such a position could be coherently filled out it is of little help to the modalist.

The above remarks show that the adequacy of the modal view under consideration presupposes the truth of the sentence

(1-7) Q* is consistent.

If (1-7) is false, then the sentences of ENT are neither logically, nor "mathematically," equivalent to the suggested modal translations. Now the point of proposing a modal view of mathematics is to avoid tying the truth of mathematical sentences to some special domain of abstract mathematical objects. If it should turn out that the truth condition for (1-7) makes essential reference to such a domain, then the modal view would presuppose for its adequacy the very thing that it was introduced to avoid. If the modalist cannot explain how one could come to know that (1-7) is true, then he cannot support his claims about the adequacy of the proposed modal translations. Hence, unless the modalist can provide an analysis of (1-7) which itself foregoes appeal to abstract mathematical objects, he must undertake the very project he forswore at the outset; *viz.*, an explanation of the possibility of knowledge about abstract mathematical objects. This would undercut the whole motivation for a modal approach to mathematics.

It might appear that the required analysis of (1-7) is readily available to the modalist. The claim that Q* is consistent can be understood as the claim that there is some logically possible interpretation of Q*. This amounts to saying that there is some world i (in A) such that i $\models$ Q*. Thus, (1-7) can be mapped into the modal sentence

(1-8) $\Diamond Q^*$

(1-8) contains no explicit reference to mathematical objects. Indeed, it appears to avoid reference to any object at all. It might thus seem that the suggested modal view is a genuine alternative to the "mathematics as set theory" picture.

However, in order to establish this conclusion more argumentation is required. It is certainly true that the modal translation '$\Diamond Q^*$' of (1-7) contains no explicit reference to, or quantification over, abstract mathematical objects. What is not clear is whether this fact is sufficient to establish that the adequacy of the modal view under consideration does not depend upon the existence of such objects. Let us consider in a bit more detail the translation '$\Diamond Q^*$' of (1-7). '$\Diamond Q^*$' will be true in the modal interpretation I just in case there exists a model of the sentence 'Q*'. Such a model consists of a domain, an interpretation of the predicates 'A', 'M', 'S', and an interpretation of the individual constant 'a_0'. Further, the domain of any such model must be of cardinality $\aleph_0$ or greater. No interpretation of Q* with a finite domain will validate all of its component axioms.

Let us suppose that our collection A (of non-modal interpretations) is such that for every i in A, the domain of interpretation i is a collection of actual physical objects. It then may very well be the case that '$\Diamond Q^*$' is false in the modal interpretation I. There simply may not be enough actual physical objects to form a model of the axiom Q*.[4] If we limit ourselves to *physical interpretations* of Q* there is no guarantee that Q* has a model and some reason to think that it does not. In this case, however, the suggested modal translation of (1-7) may turn out to be false. Since its truth is a precondition for the adequacy of the modal view under consideration this would entail that the modal view itself is inadequate. But surely the question of whether a modal analysis of number theory is correct does not turn on the question of how many physical objects happen to exist. We must therefore either reject the supposition that A is a collection of physical interpretations or reject the proposed modal translation of (1-7).

Suppose that we accept the first alternative. One way in which to do this is obvious. We could argue for the existence of certain *sets* and allow such sets to be elements of the interpretations in A. For example, we might maintain that the worlds in A contain the finite ordinals. Alternatively, we might argue for the existence of the infinite collection of numbers and include these in our universe. It is clear that either of these proposals would suffice to establish the truth-in-I of '◇Q*' and perhaps the adequacy of our modal view. What is not clear is how we could establish the truth-in-I of '◇Q*' without some such proposal. That is, it is difficult to see how we could establish the adequacy of our modal view without arguing that the interpretations in A contain at least one such collection of "eternal objects." If we adopt this alternative our modal translations seem to be no less in the grip of a special domain of abstract objects than do the unmodalized originals.[5]

The other alternative open to the modalist is to reject the proposed translation of (1-7). He might, for example, suggest a more proof-theoretic interpretation of the claim that Q* is consistent. Let T be some standard system of first-order logic. Then (1-7) might be translated as

(1-9) There exists no proof in T of the sentence 'Q* ⊃ P & −P',

where 'P' is a propositional variable of L. Sentence (1-9) is true just in case no finite sequence of formulae of L which constitutes a proof in T has 'Q* ⊃ P & −P' as its last line. It should be clear that we cannot limit the range of the quantifier here to *actualized* proof-tokens of T. If (1-9) is to serve its purpose we must either understand the quantifier as ranging over possible but unactualized proof-tokens or proof-types. In either case, however, the quantifier range is understood to contain some sort of non-physical object. Indeed, some such interpretation seems unavoidable if (1-9) is to capture the full import of the claim that Q* is consistent. But since proof-types (or possible proof-tokens) themselves qualify as abstract mathematical objects we once again find that the modal view presupposes for its adequacy the very sort of entity that it was introduced to avoid.[6] A modalist might try to bypass this conclusion by recasting (1-9) in the following form:

(1-10) It is not possible to construct a proof in T which has as its last line the sentence '$Q^* \supset P \ \& \ -P$'.

However, until a more detailed discussion of the notion of "possibility of construction" is provided it cannot be decided whether the truth of (1-10) turns on the existence of abstract mathematical objects. I will not pursue this issue any further here. For even if it could be shown that an unobjectionable construal of (1-10) is possible, the modal view under consideration is open to an even more serious objection.

It follows from Gödel's incompleteness result that mathematical truth cannot be reduced to derivability from a recursive set of first-order axioms. But this is precisely what the modal view under consideration purports to do. Hence, even if the consistency problem can be avoided, there will be truths of ENT whose modal correlates are false.[7] There is no apparent reason, however, to limit ourselves to systems such as Q and its axiomatizable extensions. Appeal to a set of first-order axioms serves only to delimit the class of logically possible interpretations of L in which we are interested. We need not restrict ourselves to first-order axioms in order to accomplish this.

Let us expand our first-order modal language L to a new interpreted second-order language L^+. In L^+ we can express a second-order induction axiom

(1-11) $\forall Y(Y(0) \ \& \ \forall x \forall y(Y(x) \ \& \ S(x, y) \supset Y(y)) \ .\supset \forall x Y(x))$

(where 'Y' is a predicate variable). Let Q_2^* be formed from (Q & (1-11)) in the same way that Q^* was formed from Q. Let the interpretation I of the modal language L^+ be the same as above. We can translate any sentence S of ENT as

(1-12) $\Box(Q_2^* \supset S^*)$.

Suppose S is a true closed formula of ENT. That is to say, S holds in the "standard model" of number theory. It is a theorem of second-order logic that every model of Q_2^* is isomorphic to this standard model. (See [2]: Chap. 18.) Hence, S holds in every model of Q_2^* and '$\Box(Q_2^* \supset S^*)$' is true-in-I.

Putnam himself makes a proposal which can be understood in this way ([7]: 12). He suggests that sentences of ENT be translated into modal statements with a component of the form

(1-13) (If α is an ω-sequence, then . . . α. . .).

This may be taken as a second modal view. It is significantly different from the first-order modal view we considered at the beginning of this section. It is different, in part, because the property of being an ω-sequence is not a (general) first-order property. There is no way, if we restrict ourselves to a standard (i.e. non-modal) first-order language, to single out those structures which form ω-sequences. This is of course possible if we allow ourselves unrestricted second-order quantification.

A second-order modal view analyzes the logical form of sentences of ENT in terms of modality and second-order logic. Our attitude towards such a view should be tempered by our attitude towards second-order logic. If, as Quine has claimed, second-order logic is really "set theory in sheep's clothing," then this modal view sees mathematics in terms of modality and sets. ([9]: 66-8.) The epistemological gain over the "mathematics as set theory" picture would then be negligible. The modal picture might still be taken to show that *any* ω-sequence of sets enters into the truth conditions of statements of elementary number theory. However, given the view of second-order logic under consideration, it still bases mathematical truth on the properties of sets.

The modalist could question the supposition that second-order logic is set theory in disguise by arguing that it is a mistake to identify sets and properties. If such an argument can be made out, then we need not understand the predicate variables as ranging over sets. ([3]: Chap. I-III.) However, in this case, the adequacy of the second-order modal view depends upon the theory of properties that we adopt. If there do not exist properties such as that of *being a natural number* or that of *being an ω-sequence*, then the second-order modal view will be inadequate. In this case a model of Q_2^* need not be isomorphic to the "standard model" of number theory. The modal translations of sentences of ENT would then involve not only ω-sequences of objects but other sequences as well.[8] This is contrary to Putnam's stated intentions (cf. (1-13) above). Since properties such as being an ω-sequence seem to qualify as abstract mathematical entities the adequacy of the modalist's program once again appears to depend upon a commitment to such objects.

These arguments suggest that the modal views so far considered do not provide an analysis of the content of ENT which is essentially "non-platonistic." in each case abstract mathematical objects appear to play a fundamental role. For this reason I believe that an analysis of ENT in terms of logical necessity and logical possibility is not likely to be of much help to the modalist in securing a coherent epistemology for mathematics.

II

The attempt to analyze mathematical truth in terms of logical modalities led us back to an infinite domain of abstract mathematical objects. The leading idea of the present section is that we might be able to avoid this consequence by appeal to the notions of "physical necessity" and "physical possibility." We try to analyze ENT in terms of physically possible structures. If this program can be carried through, then ENT can be understood as the study of actual physical structures and their physically possible extensions.

Let us consider a set of first-order Peano axioms *PA*. *PA* is the set containing

(1) $\forall x - S(x, x)$

$\forall x \forall y (S(x, y) \supset -S(y, x))$

$\forall x \forall y \forall z (S(x, y) \,\&\, S(x, z) \,.\!\supset y = z)$

(2) $\forall x \forall y \forall z (S(x, z) \,\&\, S(y, z) \,.\!\supset x = y)$

(3) $\forall x \forall y (S(x, y) \supset y \neq 0)$

(4) $\forall x \exists y S(x, y)$

and any instance of the schema

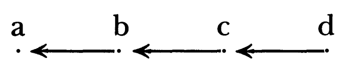
(5) $(Q(0) \mathbin{\&} \forall x \forall y(Q(x) \mathbin{\&} S(x, y) \,.\!\supset Q(y))) \supset \forall x Q(x)$

obtained by replacing 'Q' by a predicate in the language of *PA*.

A physical interpretation of the language of *PA* is a structure of the form (D, 0, s). D, the domain of the structure, is a collection of physical objects. 0 is an element of D and s is a subset of D $\times$ D. If s satisfies the axioms (1) - (3) and the condition

(6) $\forall x(x \neq 0 \supset \exists y S(y, x))$

I say that the interpretation (D,0,s) is a *concrete structure for PA*. An example of a concrete structure for PA is the diagram

(2-1) 

The domain D of the structure is the collection of labelled points. Point a is the 0-element of the structure and the relation s (indicated by the arrows) is the (physical) relation of *being-to-the-immediate-left-of*.[9]

The aim of the modal view I am constructing is to analyze sentences of *PA* in terms of statements about physically possible concrete structures for *PA*. To this end I define the notion of a concrete modal interpretation for *PA*. A *concrete modal interpretation for PA* is an ordered triple I = (A, R, G). A is a collection of physically possible concrete structures for *PA* which satisfies the following two conditions:

(2-2) For every i, j in A, either $i \subset j$ or $j \subset i$,

where '$\subset$' denotes the substructure relation.

(2-3) For every i in A, cardinality(d(i)) is finite,

where d(i) is the set of individuals which exist, or are actual, in i. R is the accessibility relation on the worlds in A. If iRj holds for i, j in A we can say that 'j is physically possible from the standpoint of i'. If, in addition, cardinality (d(j)) > cardinality (d(i)), we say that 'j is a physically possible extension (from the standpoint) of i'. For any i in A, i has the form (D, 0, s). We define the valuation function G as follows:

(2-4) G(i, 0) = 0

(2-5) G(i, S) = s.

This completes the definition of a concrete modal interpretation for *PA*. Note that such an interpretation contains only finite concrete structures.

Let us now impose one further condition upon concrete modal interpretations for PA. We stipulate that in any such interpretation the accessibility relation must be transitive. That is,

(2-6) For any i, j, k in A, iRj & jRk ⊃ iRk.

If we want to distinguish such interpretations from others in which (2-6) fails we can call them *transitive* concrete modal interpretations for *PA*. A modal view which purports to analyze *PA* in terms of physical possibility can now be defined. Let I be a (transitive) concrete modal interpretation for *PA*. Let L be a first-order modal language which contains '0' and 'S' among its primitive symbols. The modal view is defined by the following translation function M from *PA* to the interpreted modal language (L, I):

(2-7) M(P) = P, if P is an atomic formula.

(2-8) M(−R) = −M(R)

(2-9) M(R & T) = M(R) & M(T)

(2-10) M(R ⊃ T) = M(R) ⊃ M(T)

(2-11) M(∃xR) = ◇∃xM(R).[10]

On this modal view an existentially quantified sentence of *PA* is translated into a statement concerning the existence of certain concrete physical objects in some physically possible structure. In order to validate the modal translations of the *PA* axioms we need not assume that a completed domain of mathematical objects exists in any one world. We need only assume that it is physically possible to extend any finite concrete structure for *PA*. On this modal analysis the truth of *PA* rests upon the possibility of an unbounded sequence of ever-larger finite concrete structures for *PA*.

This modal view can be criticized in two quite different ways. The first is based upon problems with that notion of physical possibility to which mathematics is purportedly reduced. The second turns upon the structure of the semantic theory in which the reduction is carried out. I take these points in turn.

It is not difficult to show that the transitivity restriction (2-7) is necessary for the adequacy of the modal view under consideration.[11] We know from modal logic that this accessibility relation is transitive only if physical necessity satisfies the following modal axiom:

(2-12) $\Box S \supset \Box\Box S$.[12]

Hence, unless it is possible to explain how we could know that (2-12) is true, the modalist will not be able to explain how he can know that the proposed modal view is adequate. Let us see what might be required in order to possess such knowledge.

One obvious way to understand the notions of physical necessity and possibility is in terms of derivability from physical laws. We say that S is physically necessary just in case S follows from the prevailing physical laws, and S is physically possible just in case S is compatible with the prevailing physical laws. That is,

(2-13) $\Box S =_{df} \vdash L_i \supset S$.

L_i is the set of physical laws which prevail throughout the world i at which S is supposed to be necessary.

In our modal interpretation for *PA* the worlds were simply concrete structures for *PA*. We made no reference to laws which prevail at these structures. If we are to make any sense of physical possibility in this setting we must now suppose that these structures are themselves embedded in more "complete" possible worlds. These must be thought of as having enough structure to contain physical laws. Let 'L_i' denote the set of laws associated with the world in which a concrete structure i (in A) is situated. If we think of all concrete structures as embedded in worlds which obey the laws prevailing at the actual world @, then $L_i = L_@$ for every i in A.

It is extremely difficult to make sense of the transitivity axiom (2-12) given the above interpretation (2-13) of physical necessity. The difficulty is just the general problem of understanding iterated modalities when the modal operators have been interpreted in terms of derivability. Consider the claim that '□□ S' is true at a world i. Under the interpretation (2-13) this amounts to the claim that it follows from L_i that □S; i.e., it follows from L_i that S follows from L_i. More concisely we have

(2-14) $\vdash L_i \supset (\vdash L_i \supset S)$.

But (2-14) is not even a syntactically well-formed sentence. It can, of course, be turned into one if we Gödel number all our sentences, introduce a proof predicate into our language, and take care to allow ourselves enough number theoretic apparatus. However, this is not at all desirable in the case at hand. The knowledge that the proposed modal view is adequate rests upon the knowledge that (2-12) is true. If, in order to explain how one could know that (2-12) is true, appeal must be made to our knowledge of ENT, then the suggested modal view does not really solve any problems about mathematical knowledge. At best it fails to consider the central problem at issue.

We can avoid all of this entirely by viewing the notion of physical possibility in a less formal way. One might do so by taking a more realistic attitude towards the possible worlds. We have assumed that there is a set of prevailing physical laws L_i associated with each world i in a concrete modal interpretation for *PA*. To say that '◇S' is true at i can be taken to mean that there is some possible world j, which is compatible with the laws of L_i, such that S is true at j. Similarly, to say that '◇◇S' is true at i is to say that there are worlds j and k such that j is physically possible with respect to L_i, k is physically possible with respect to L_j, and S is true at k. However, while this understanding of the modalities renders the transitivity requirement coherent, it also undercuts whatever plausibility the requirement might appear to have. There is no reason to think that physical possibility, so understood, is transitive.

Let us suppose that j is possible with respect to L_i and that k is possible with respect to L_j; i.e. i R j and j R k. We need for transitivity that k is permitted by the laws L_i. But what reason do we have to suppose that this must be the case? Let us consider a particularly appropriate example. Suppose it turns out to be a physical law of some world i that there is an upper bound on the number of elementary particles that can exist. Let this upper bound be N. Then there is some possible world j, which is compatible with L_i, such that j contains N elementary particles. In order for transitivity to hold it must be a law at j, as well as at i, that at most N such particles can exist. But there seems to be no reason to believe that this will be the case. It might well turn out that the N-particled world j is so different from the world i that many of its laws are different from those which prevail at i. It might even be the case that L_j contains the statement that up to N + 1 elementary particles can exist. At least, there appears to be no argument which rules out this possibility. In such a case, however, transitivity fails.

If it is true that the physical laws which prevail at the actual world prevail at every physically possible world, then physical possibility is a transitive relation. Since the same laws prevail at every world, each world is permitted by every other. To this end one could, of course, simply define the physically possible worlds as those at which the actual physical laws prevail. This is tantamount to proclaiming the necessity of our own physical laws. Alternatively, it amounts to maintaining that the actual world could not have obeyed physical laws which are different from those which in fact prevail. This seems to me to be an implausible claim, but I will not defend my position here.[13] For even if we could settle the issue in favor of the transitivity of physical possibility more serious problems remain for the modal view under consideration.

The modal view I have suggested purports to analyze number theory by appeal to the notion of physical possibility. Physical possibility is meant to do the work that abstract mathematical objects are usually employed to do. Physical possibility, in its turn, is analyzed in terms of physical laws. We usually understand by physical laws those laws which are basic according to our best theories. We hope that these theories are true as well as useful. But surely this characterization of physical law will not do for *our* purposes. The "laws" of mathematics are no less essential to our physical theories than any others. There can be no laws more basic to physical theory than these.[14] If mathematical axioms are counted among the physical laws it is no surprise that mathematics can be analyzed in terms of physical possibility. What is surprising is that we went to so much trouble to make the point. We should simply have said that a modal view of mathematics is bound to be correct because all mathematical truths are consequences of physical (i.e. mathematical) laws, and physical laws define physical necessity.

One might try to deny that the mathematical laws are physical laws in the relevant sense. This is a very difficult case to make out. Physical laws are couched in the language of mathematics. They use the notions of e.g. number, function, vector, derivative, force, etc. These are an essential part of physical theory. Even if we deny that the mathematical axioms play any role in the determination of physical possibility, the language of number theory is indispensible for the expression of our physical theories. Unless we can find a way to separate mathematics off from physical theory, however, a modal view which purports to understand mathematics in terms of physical possibility is bound to fail. An understanding of mathematics seems to be presupposed by an understanding of physical laws. For this reason we should not expect the notion of physical possibility to aid us in explaining how mathematical knowledge is possible. Nor should we expect an appeal to physical possibility to help us in the discovery of what numbers really are. From the point of view of physical possibility, numbers are whatever objects we need to insure the truth of certain physical-mathematical laws. Physical possibility drops out of the picture entirely.

The other line of argument against the suggested modal view turns on the metalinguistic definition of truth-in-a-(modal)-interpretation to which I have tacitly appealed. In particular, it concerns the conditions under which a modal sentence of the form '◇S' is true-in-an-interpretation. We said in effect that such a sentence is true at a world i just in case *there exists a world* in the domain of the interpretation at which S is true and which is accessible to i. A non-standard analysis of object language quantification rests upon objectual metalinguistic quantification over possible worlds.

In view of this fact the benefit to be derived from an analysis of *PA* in terms of concrete modal interpretations for *PA* seems to be illusory. The advantage of such an approach is supposed to be that it provides an analysis which does not appeal to an infinite domain of existing mathematical objects. Such a domain is replaced instead with a sequence of ever larger possible worlds. In our metalanguage, however, we objectually quantify over these worlds. It could thus be argued that we are committed to the existence of this infinite domain of possible worlds.

The present modal view appears simply to have traded an infinite collection of abstract mathematical objects for an infinite collection of possible worlds. If we take possible worlds to be abstract objects of some sort (e.g., sets of some appropriate kind), we are back to where we started. Modal views were suggested as a means of avoiding an analysis of mathematical truth in terms of abstract mathematical objects in view of the problems such an analysis raises for an account of mathematical knowledge. If possible worlds are themselves abstract objects it is difficult to see why they bring us any closer to such an account. Alternatively, we can take a realistic view towards possible worlds. We then face the problems discussed above.

III

I have by no means canvassed all the avenues open to the modalist. One could still try to take the requisite modalities to be *primitive* notions which require no further analysis. Such a modal view would deny that the modal translations of the sentences of ENT need be interpreted in terms of metalinguistic quantification over some collection of "possible worlds." While mathematical truth may be analyzed in terms of, e.g., mathematical possibility, the notion of mathematical possibility need not itself be further analyzed. A modal view along these lines is immune to much of the criticism that I have advanced in this paper. In order to turn this into a positive account, however, one would have to explain how it is possible to acquire the relevant beliefs and knowledge about the requisite modalities. If this cannot be done, then this modal view will also violate the epistemological constraint which must be placed upon any adequate analysis of mathematical truth.[15]

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NOTES

[1]This problem is discussed in [1], [3], and [8].

[2]It is true that, according to Putnam's "now you see them, now you don't" approach to mathematical objects, any modal analysis of mathematical truth can be transformed into an analysis which is based upon mathematical objects. My point will be that an account of mathematical truth which is based upon necessity and possibility requires an appeal to abstract mathematical objects. Elsewhere I argue that one can give an adequate account of mathematical truth and knowledge which is based upon mathematical objects without appeal to the notions of possibility and necessity. (See [3].)

[3]We might take Q to be the conjunction of the axioms of "Robinson's system":

Q1 $\forall x \forall y(x' = y' \supset x = y)$

Q2 $\forall x(x' \neq 0)$

Q3 $\forall x(x \neq 0 \supset \exists y(x = y'))$

Q4 $\forall x(x + 0 = x)$

Q5 $\forall x \forall y(x + y' = (x + y)')$

Q6 $\forall x(x \cdot 0 = 0)$

Q7 $\forall x \forall y(x \cdot y' = (x \cdot y) + x)$

See, e.g., [2], chapter 14.

[4]For present purposes I need not make precise the notion of "physical object." I mean something like "ordinary physical objects" in the sense in which sets and physical properties are not ordinary physical objects.

[5]One might still argue that progress has been made. The sentences of ENT appear to commit us to one particular domain of abstract mathematical objects called 'the natural numbers.' The modal translation procedure reveals that, as long as Q* has a model, ENT can be understood as making claims about any structure of objects which models Q*. This modal view still presupposes the existence of nonphysical objects and it even looks as though some of these must be abstract mathematical objects. But the modal view underscores the fact that we need not take ENT to be a theory about any particualr domain of mathematical objects. However, even if this claim is correct it is of little help to the modalist. In order to explain our mathematical knowledge we must still explain how belief and knowledge about abstract mathematical objects is possible.

[6]For a discussion of the relation between types, tokens, and modality see [6], p. 151.

[7]A striking example is the statement

$$(*) \qquad \forall x \forall y(x + y = y + x).$$

The proposed modal translation assigns to it the sentence

$$(**) \qquad \Box(Q^* \supset \forall x \forall y \forall z(A(z, x, y) \equiv A(z, y, x))).$$

Now it is a fact that (if Q is consistent) (*) is not derivable from Q. By completeness it follows that

$$Q \supset \forall x \forall y(x + y = y + x)$$

is not valid and hence that (**) is false. Of course we could strengthen Q to get a new theory Q^+; e.g. by adding (*) as an axiom. Due to incompleteness, however, this provides only the appearance of a remedy.

[8]It is well known that they would involve sequences of order-type $\omega + (\omega^* + \omega).\eta$, where ω^* is the order-type of the sequence (. . . , 3, 2, 1) and η is the order-type of the rational numbers.

[9]Another example is the sequence of numeral inscriptions

$$0', 0'', 0''', 0''''.$$

In this case the intended relation s is a bit more difficult to describe. We might define it in terms of the relation of *being structurally-isomorphic-to-a-part-of*. See [6], pp. 158-60.

[10]Given these definitions and the fact that

$$\Box S \equiv -\Diamond-S$$

we can show that

$$M(\forall xR) = \Box\forall xM(R).$$

[11]Consider axiom (4) of *PA*: $\forall x \exists y S(x, y)$. On our modal view this is translated as

$$(*) \qquad \Box\forall x\Diamond\exists y S(x, y).$$

If we relax the transitivity restriction on R, then (*) is true even in concrete modal interpretations which do not contain an unending sequence of ever larger concrete structures. That is, (*) is true in modal interpretations in which the universe is finite.

To see that this is the case consider the following non-transitive concrete modal interpretation of *PA*:

$$A = (i, j, k)$$
$$d(i) = (a); d(j) = (a, b); d(k) = (a, b, c)$$
$$R = ((i, j), (j, k))$$
$$G(1, 0) = a, \text{ where } 1 = i, j, \text{ or } k.$$
$$G(i, s) = \Lambda; G(j, s) = ((a, b)); G(k, s) = ((a, b), (b, c)).$$

This is a concrete modal interpretation for *PA* which violates (2-6), the transitivity restriction on R. It is easy to see that (*) is true-in-I at world i although there is a largest finite world k in A.

[12]Equivalently, the axiom could be expressed as

$$\Diamond\Diamond S \supset \Diamond S.$$

[13]The following fictional example indicates the direction in which my doubts lie. Suppose that our universe was formed by a "big bang." Suppose further that certain aspects of this event are responsible for the speed of light. Then, given that the big bang occurred, it may very well be necessary that the speed of light is 3×10^8 m/sec. This may be a "physical law." However, it might also be possible that the particular big bang which formed our universe did not occur. In this case, although it is necessary *given certain facts* that the speed of light is 3×10^8 m/sec, these facts themselves display a kind of contingency.

The assumption that the actual physical laws prevail at every physically possible world amounts to the supposition that there are some (perhaps undiscoverable) "rock bottom" physical laws the truth of which is not contingent upon anything else. This is the claim that I find implausible. These considerations arose from a discussion with Adam Morton. They are hinted at in [5], p. 394. Of particular relevance is his suggestion that we think of partial worlds as "embryo worlds" (p. 400).

[14]Steiner offers a similar argument for a rather different purpose. See [10], chapter 4.

[15]I would like to thank Clark Glymour, David Lewis and Adam Morton for their comments on an earlier draft of this paper. I owe a special thanks to Paul Benacerraf. His thoughtful criticism was a continuing source of insight into the problems discussed in this paper.