Frege, Mill, and the Foundations of Arithmetic

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IN his *System of Logic* John Stuart Mill claims that

> Each of the numbers two, three, four, etc., denotes physical phenomena, and connotes a physical property. . . . Two, for instance, denotes all pairs of things . . . connoting what makes them pairs.
>
> What then is that which is connoted by a name of number? Of course, some property belonging to the agglomeration of things which we call by the name; and that property is, the characteristic manner in which the agglomeration is made up of, and may be separated into, parts.**

Mill is here suggesting that, e.g., the number 2 applies to an aggregate (e.g., a pair of dice) just in case the aggregate exemplifies (or "possesses") a certain property. This property concerns the "characteristic manner" in which the aggregate is made up of its parts. Number words, for Mill, thus connote what we might call "structural properties." That is, they are properties exemplified by aggregates in virtue of their structure or composition. After presenting his account of the nature of number Mill goes on to suggest that this analysis is compatible with an account of mathematical knowledge. According to Mill, his analysis does not simply tell us what numbers are. It also explains how it is possible for us to know those mathematical truths which we in fact do know (Book III, Chapter 24).

Mill's proposal is considered and criticized in Gottlob Frege's *Foundations of Arithmetic*.† These criticisms are usually taken to

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** New York: Harper, 1874, p. 430. All page references concerning Mill are from this volume.

† Translated by J. L. Austin; Evanston, Ill.: Northwestern UP, 1974. All page references to Frege are to this volume.

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show that Mill's analysis is fundamentally misguided. I shall maintain that this attitude is mistaken. Although Frege's remarks do argue against certain specifics of Mill's proposal, they by no means show that his general approach is incorrect. In support of this claim I will present (what I take to be) a variant of Mill's proposal which avoids the standard Fregean objections. Further, the analysis of number I present, like Mill's, appears to be compatible with an account of mathematical knowledge. It provides the framework for an epistemological, as well as a metaphysical, foundation for mathematics.

I begin with a brief consideration of Frege's criticisms against Mill's view of number. Frege is concerned primarily with Mill's suggestion that numbers are properties of agglomerations, or properties of aggregates. More generally, he denies that statements of number predicate properties of objects (or "external things") of any kind. Frege offers two basic arguments in support of this. The first argument might be labeled the "relativity argument." Consider a normal property such as *being red*. Given any object, the question whether this object exemplifies this property has a determinate answer. Any object is either red or non-red. This, Frege argues, is a characteristic feature of all properties. Numbers, however, do not share this feature. If I present someone with a complete deck of cards and ask "Does this have the number 1?", there is no determinate answer that can be given. The deck has the number 1 if we are counting decks, the number 4 if we are counting suits, and the number 52 if we are counting cards. Frege notes that the number 1, or any other number, "cannot be said to belong to the pile of playing cards in its own right, but at most to belong to it in view of the way in which we have chosen to regard it" (29). In short, an ascription of number is always a relative matter. Such an ascription is always relative to "a way of regarding" the object in question. The predication of a property is not relative in this sense. Numbers are not properties of objects.

Mill assumes that a number applies to an aggregate in virtue of "the characteristic manner" in which the aggregate is made up of, and may be separated into, parts. Frege's relativity argument turns on the assumption that, from the point of view of the aggregate, it makes no sense to speak of "*the* characteristic manner" in which the aggregate is composed. An aggregate can be constructed in many different ways. According to Frege, Mill's failure to realize this crucial point is what led him to propound his mistaken analysis (30). Had Mill noticed that there are different ways in which the very same aggregate can be divided into parts, he would also have seen that his property view of number was fundamentally incorrect.

The relativity argument is, in my opinion, an important argument. It does make a significant point about the application of numbers. However, it does not seem to me to constitute a major criticism of Mill's position. In the first place, Frege simply assumes that Mill's notion of aggregate is such that the method of construction of an aggregate is not relevant to the individuation of that aggregate. On Frege's view, the aggregate of all the counties in Virginia would be identical with the aggregate of all the election districts in Virginia, since they both cover exactly the same area. The aggregate consisting of all the pieces of straw in some bundle would be identical with the aggregate consisting of all the cells in these pieces of straw. Aggregates, for Frege, are simply "heaps of stuff" (cf. 30). Mill actually says a number of things that appear to be inconsistent with this view of aggregates. In fact, he seems to suggest that, in individuating aggregates, we have to consider the *kind* of parts they comprise. Consider Mill's defense of his claim that aggregates do indeed have a "characteristic manner" of composition:

> When we call a collection of objects *two*, *three*, or *four*, they are not two, three, or four in the abstract; they are two, three, or four things *of some particular kind* . . . . What the name of number connotes is, the manner in which single objects *of the given kind* must be put together, in order to produce that aggregate (430, emphasis added).

Thus it might appear that, for Mill, the identity of an aggregate is sensitive to the way in which we choose to regard it.[1] When we say that an aggregate has the number 2 we really mean that an aggregate composed in a certain way from a certain kind of part has the number 2. Any aggregate that differs in number from this aggregate will, of necessity, be composed of different parts and, hence, will be a different object.

If we wanted to put all this in a more modern setting we could do so by noting that Frege-style aggregates are individuated in terms of their "atomic parts." That is, they are individuated in terms of those parts which themselves have no proper parts. This is the kind of aggregate that Nelson Goodman favors over classes.[2] On the alternative reading I have just considered, a given aggregate can have as parts only certain *kinds* of things. Anything that is not of the appropriate kind will not be a part of the aggregate and will therefore

[1] This view of aggregates is even more plausible against the background of Mill's phenomenalism. A complete exposition of the view would have to consider it in this light. For reasons which will soon become obvious, such a complete exposition is unnecessary here.

[2] See for example, *The Structure of Appearance* (Indianapolis: Bobbs-Merrill, 1966), ch. 1.

not figure into the identity conditions of the aggregate. In a sense, it is still true that aggregates are individuated in terms of their parts. However, the parts in terms of which an aggregate is individuated may themselves have (proper) parts.[3]

If we understand aggregates in the way I have just suggested, then Mill's analysis does avoid Frege's relativity argument. The same aggregate cannot have different numbers. The way in which the parts of the aggregate are counted goes to determine which aggregate is being considered. However, this reading must be rejected on other grounds. Mill maintains that the basic or fundamental truth upon which all of number theory as a deductive science rests is the familiar axiom "The sums of equals are equals." He paraphrases this basic truth as

> Whatever is made up of parts, is made up of the parts of those parts (431).

The problem is that this fundamental axiom is false on the proposed understanding of aggregates. Let x be an aggregate composed of certain pieces of straw. We noted that, on this view of aggregates, only things of the appropriate kind are parts of an aggregate. Since x is, by hypothesis, composed of pieces of straw, its parts are just those pieces. The cells in the pieces of straw cannot be parts of the aggregate x. They are not things of the "appropriate kind." However, every such cell is presumably a part of one of the pieces of straw which go to compose x. That is, each cell is a part of a part of x. The above axiom would thus entail that each cell is a part of the aggregate x. The proposed understanding of aggregates is thus inconsistent with Mill's basic axiom. On the other hand, the axiom is obviously true on Frege's notion of aggregate. Any decomposition of a Fregean aggregate is a decomposition of the aggregate into its parts. Any way we slice it, the aggregate remains the same. In fact, Mill's basic truth is generally taken as one of the axioms of the calculus of individuals, which provides an axiomatic treatment of this type of aggregate. In summary, a consistent interpretation of Mill's position cannot incorporate both his fundamental axiom and the suggested understanding of aggregates. If Mill's fundamental principle is indeed fundamental to his system, he cannot answer Frege's relativity argument simply by appealing to the suggested construal of aggregates. A solution to the problem that Frege raises must lie in a different direction.[4]

[3] For a discussion of aggregates of this general kind, see Tyler Burge, "A Theory of Aggregates," *Noûs*, XI, 2 (May 1977): 97–117.

[4] Although this argument is sufficient to show that, given his fundamental axiom, the suggested understanding of aggregates is not open to Mill, there is yet

I mentioned earlier that Frege's relativity argument contains a significant insight. This insight has been overlooked in the above discussion. Frege notes that when we ask whether an object has some particular color we ask a determinate question. When we ask whether an object has some particular number we do not. He concludes from this that numbers don't apply to (external) objects. There is, however, quite another explanation for the fact that we fail to ask a determinate question when we ask whether an object has some particular number. Consider the following question: Is Charlottesville west? Here, too, there is no determinate answer. Following Frege's strategy, one might be inclined to conclude that direction does not apply to objects. This is clearly the wrong conclusion to draw. The above question lacks a determinate answer because 'west' refers to a relation and not to a simple property. The problem about determinateness arises not from applying 'west' to objects, but from treating 'west' as a monadic rather than a relational predicate. Frege's relativity argument may be understood as establishing the same point with respect to numbers. The question, "Does 52 apply to this pack of cards?" lacks a determinate answer not because we have applied a number to an external object, but because we have mistaken a relation for a simple property. Mill came close to the truth in treating numbers as properties. Frege's relativity argument may be taken to show that a satisfactory analysis would treat numbers as relations.

On the model I am suggesting, a number is to be understood as a special sort of relation which holds between aggregates and properties that pick out parts of those aggregates. For example, in claiming that a certain aggregate x contains 52 cards we are claiming that the numerical relation 52 obtains between the aggregate x and the property of *being a card*. We might write this as '52(x, *being a card*).' When we say that the aggregate x contains one complete deck we are saying that x bears the numerical relation 1 to the property of being a complete deck; i.e., 1 (x, *being a complete deck*). In general, to say that an aggregate x has n p-parts (i.e., n parts that exemplify the property p) is to say that x stands in the numerical relation n to

another alternative. One might maintain that although an aggregate is constructed from, and individuated in terms of, a certain kind of part, the parts of these parts are still parts of the original aggregate. That is, an aggregate can have many different parts only some of which are relevant to its individuation. This would force us to reject the intuitively plausible idea that an aggregate is individuated in terms of its "most basic" parts. It requires that we treat the individuation of aggregates in a property-relative way. This treatment seems to be entirely compatible with the approach to number which is outlined below.

the property *p*. Putting the proposal in a slightly different way, the numeral '*n*' is understood to denote that numerical relation which holds between an aggregate *x* and a property *p* just in case *x* has *n* *p*-parts.

It should be noted that not all properties succeed in picking out determinate parts of aggregates. For example, the properties of *being a sample of gold* and *being red* do not. Frege was well aware of this general point. He makes an analogous claim about concepts:

> Only a concept which isolates what falls under it in a definite manner, and which does not permit any arbitrary division of it into parts, can be a unit relative to a finite number (66).

Let us call a property that satisfies this condition an *individuating property*.[5] An individuating property is a property that is the referent of a sortal predicate, or a predicate that divides its reference in the appropriate way. Then the suggested analysis of number understands numbers to be a special sort of relation between aggregates and individuating properties.

This view of number is similar to Mill's proposal in several respects. In the first place, statements of number are still statements about aggregates or objects. (Numbers do apply to "external things.") However, on this analysis, unlike Mill's, they apply to "objects" only with respect to some individuating property. We also noted that Mill treated numbers as a kind of "structural property" of aggregates. This too has its analogue in the suggested analysis. The fact that an aggregate *x* has a part that is a *p* is a fact about the structure of *x*. No less so is the fact that, e.g., *x* has a part *y* that is a *p*, a part *z* that is a *p*, $y \neq z$, and *y* and *z* are the only *p*-parts of *x*. But this is just to say that *x* bears the numerical relation 2 to the individuating property *p*. Numerical relations are thus exemplified by aggregates and properties in virtue of facts about the structure of the aggregates. On this analysis numbers turn out to be a certain type of "structural relation."

These informal remarks can be made more precise by utilizing the calculus of individuals. I assume familarity with the axioms of this system and the related notions of atomicity, disjointness, and nominalistic sum (denoted here by the symbol '+').[6]

[5] See, for example, W. V. Quine, *Word and Object* (Cambridge: MIT Press, 1960), p. 90. A more detailed exposition would have to make explicit the conditions that individuating properties must satisfy.

[6] I rely on the following axiomatization of the calculus of individuals:

$$(x)(y)(z)(x < x \,\&\, (x < y \,\&\, y < z \cdot \supset x < z))$$
$$(x)(y)(\exists w)(\exists z)(\mathrm{atom}(w) \,\&\, \mathrm{atom}(z) \,\&\, z < x \,\&\, w < y \,\&\, (x \neq y \supset z \neq w))$$
$$(\exists x)Px \supset (\exists y)(z)(\mathrm{atom}(z) \supset (z < y \supset (\exists w)(z < w \,\&\, Pw)))$$

Consider first the number 0. Frege felt that the number 0 presented a special problem for Mill's analysis. According to Mill, numbers apply to aggregates in virtue of the characteristic manner in which they may be separated into their parts. It follows that if some aggregate has the number 0, then this aggregate simply has no parts into which it can be characteristically separated. However, every aggregate is a part of itself. So it looks as if the number 0 is never exemplified. Frege suggests that this poses an epistemological problem for Mill (11). If we construe numbers as relations rather than properties, this problem is entirely avoided. 0 is that numerical relation which holds of an aggregate x and an individuating property p just in case x contains no p-parts. More formally,

(1) $$0(x, p) \text{ iff } (w)(w < x \supset {\sim}Pw)$$

(I adopt the convention that if the predicate letter 'P' refers to any property at all, it refers to the property p.) The number 1 is that numerical relation which an aggregate bears to a property p just in case x contains exactly one p-part. We eliminate the circularity from this definition in the standard way:

(2) $$1(x, p) \text{ iff } (\exists y)(y < x \,\&\, Py \,\&\, (z)(z < x \,\&\, Pz. \supset z = y))$$

The definition of successor needs a bit more motivation. Let 'm'' denote the successor of the number m. Intuitively, an aggregate x bears the numerical relation m' to a property p just in case removing exactly one p-part from x leaves an aggregate that bears the relation m to p. Any aggregate that bears m' to p contains one more p-part than any aggregate that bears m to p. On this basis we might be led to define the successor relation as follows:

(3) $$m'(x, p) \text{ iff } (\exists y)(\exists z)(x = y + z \,\&\, m(y, p) \,\&\, 1(z, p))$$

This definition is inadequate because of the "problem of fusion."

Suppose that a and b are gold spheres. Let y be an aggregate consisting of a plus half of b. Let z be the aggregate consisting of that part of b not contained in y. Then we have that 1 (y, *being a gold sphere*) and 0 (z, *being a gold sphere*). z contains no gold-sphere-parts and the only such part of y is a. By (3), we would have that

I also make use of the following defined terms:

$$\text{atom}(x) \text{ iff } (w)(w < x \supset x = w)$$
$$x \bigcirc y \text{ iff } (\exists z)(z < x \,\&\, z < y) \quad \text{(overlap)}$$
$$x/y \text{ iff } {\sim}(x \bigcirc y) \quad \text{(disjointness)}$$
$$z = x + y \text{ iff } (w)(w \bigcirc z \equiv w \bigcirc x \cdot \vee \cdot w \bigcirc y)$$

Note, however, that a commitment to atomicity (as embodied in the second axiom above) is not essential to the account of number that I offer.

$0'$ ($y + z$, *being a gold sphere*). However, since $y + z$ contains both a and b, we should have that 2 ($y + z$, *being a gold sphere*). Something has clearly gone wrong. The problem here is that a part of y and a part of z, neither of which is itself a gold sphere, have "fused" in $y + z$ to yield a new gold-sphere-part. The following restriction on the bound variables y and z of (3) solves this problem.

(4a) $$(w)(w < y + z \,\&\, Pw \cdot \supset \cdot w < y \cdot \vee \cdot w < z)$$

This says that any p-part of $y + z$ must be either a p-part of y or a p-part of z. In the example just considered the aggregates y and z do not satisfy (4a) with respect to the property of *being a gold sphere*. Hence, we cannot conclude that $0'$ ($y + z$, *being a gold sphere*).

There is yet another problem with (3). Suppose that Willow and Bree are cats. Consider an aggregate y that consists of both Willow and Bree, and an aggregate z that consists of just Willow. Then 2 (y, *being a cat*) and 1 (z, *being a cat*). By (3) we can derive that $2'$ ($y + z$, *being a cat*). But this is incorrect. '$2'$ ($y + z$, *being a cat*)' is true iff there are parts w and t of $y + z$ such that 2 (w, *being a cat*) and 1 (t, *being a cat*), where w and t *have no cat-parts in common*. The parts w and t must be disjoint with respect to their cat-parts. We take this into consideration by placing the following additional restriction on the bound variables of (3):

(4b) $$(w)(w < y + z \,\&\, Pw \cdot \supset \cdot -(w < y \cdot \,\&\, \cdot w < z))$$

(4b) is a nominalistic analogue of set-theoretic disjointness. It says that no p-part of $y + z$ is a p-part of both y and z.

I call aggregates y and z which satisfy conditions (4a) and (4b) with respect to a property p *p-disjoint* (written 'yp/z'). We thus have

(5) $$yp/z \text{ iff } (w)(w < y + z \,\&\, Pw \cdot \supset \cdot (w < y \cdot \vee \cdot w < z) \,\&\, \sim (w < y \cdot \&\cdot w < z))$$

We are now in a position to define the successor relation.[7] It is defined in two stages:

(6a) $0'(x, p)$ iff $1(x, p)$

(6b) $m'(x, p)$ iff $(\exists y)(\exists z)(x = y + z \,\&\, yp/z \,\&\, m(y, p) \,\&\, 1(z, p))$

[7] Strictly speaking, I am not entitled to treat the successor relation as a function until I have established that each natural number has at most one successor. This is in fact provable in our system.

(6b) holds when $m \neq 0$.[8] We can also define

(7) $$2\ (x, p) \text{ iff } 1'\ (x, p)$$
$$3\ (x, p) \text{ iff } 2'\ (x, p), \text{ etc.}$$[9]

It is now a simple matter to express the facts that (i) 0 is a natural number, (ii) any number generated from a natural number by application of the successor relation is a natural number, and (iii) these are the only natural numbers. Letting 'Nn' denote the property of being a natural number we define

(8) $$\text{Nn}(m) \text{ iff } (F)(F(0)\ \&\ (n)(F(n) \supset F(n')) \cdot \supset F(m))$$

This is just the usual Frege-style definition of natural number.[10]

Finally, we must consider the problem of providing truth-conditions for identities of pure number theory. On the view being considered, an identity statement of the form '$m = n$' is true just in case 'm' and 'n' refer to the same numerical relation. The problem

[8] It might appear that we could define the successor relation by the single sentence

$$m'(x, p) \text{ iff } (\exists\, y)(\exists\, z)(x = y + z\ \&\ yp/z\ \&\ m(y, p)\ \&\ 1(z, p))$$

However, this is not the case. Let x be a gold sphere that weighs exactly 17 ounces. Let p be the property of being a 17-ounce gold sphere. Then, by the above, we have that $0'(x, p)$ iff, for some a and b,

(i) $x = a + b$
(ii) ap/b
(iii) $0\ (a, p)$
(iv) $1\,(b, p)$

By (i), both a and b must be parts of x. By (ii), $a \neq b$. Hence, $b \neq x$; i.e., b is a proper part of x. But if x weighs exactly 17 ounces and b is a proper part of x, then b cannot weigh exactly 17 ounces. Hence, (iv) is false.

This problem can be avoided either by positing the existence of a "null aggregate" or by dropping the above definition in favor of those cited in the text. I can find no argument favoring the first alternative.

[9] It is also a simple matter to define addition and multiplication. I consider only addition. If m and n are numbers, then $m + n$ is also a numerical relation that holds between aggregates and individuating properties. (I use the symbol '+' to denote both nominalistic and arithmetic sum. The context should make clear which is intended in any given case.) In particular, if n is not equal to 0, it is that numerical relation which holds between an aggregate x and a property p just in case x can be divided into p-disjoint parts y and z such that y has m p-parts and z has n p-parts.

$$m + 0\ (x, p) \text{ iff } m(x, p)$$
$$m + n(x, p) \text{ iff } (\exists\, y)(\exists\, z)(x = y + z\ \&\ yp/z\ \&\ m(y, p)\ \&\ n(z, p))$$

On the basis of these definitions it is possible to prove that $m + 0 = m$ and $m + n' = (m + n)'$.

[10] A more thorough exposition of this position would consider the nature of second-order quantification. For a discussion of this question, see my Ph.D. dissertation, *Numbers, Truth and Knowledge*, Princeton, 1976, secs. 1–3 and 2–3.

of giving truth conditions for identities of pure number theory thus reduces to the problem of giving truth conditions for identities of numerical relations. It is fairly clear that the condition

$$(9) \qquad (x)(p)(m(x, p) \equiv n(x, p))$$

although necessary, is not sufficient to establish the identity of numerical relations m and n. Coextensional predicates need not refer to the same relation. We need a stronger condition in order to establish that 'm' and 'n' denote the same numerical relation. Note, however, that if we can establish that, for any individuating property p, the property of having n p-parts is identical with the property of having m p-parts, then we will have shown that m and n are identical. This is precisely what we establish when we show that an equivalence of the above form (9) is valid in any model of the calculus of individuals and the definitions (1)–(8). Suppose that (9) is true in every such model. Then it follows that both $m(x, p)$ and $n(x, p)$ are provably equivalent to

$$\begin{aligned}(10) \quad (\exists x_1)\ldots(\exists x_r)\ (x_1 < x \ \&\ldots\&\ x_r < x \\ \&\ x_1 p/x_2 \ \&\ldots\&\ x_{r-1} p/x_r \\ \&\ Px_1 \ \&\ldots\&\ Px_r \\ \&\ (y)(Py \ \&\ y < x \cdot \supset \cdot y = x_1 \vee \ldots \vee y = x_r))\end{aligned}$$

for some r. This means that m and n apply to an aggregate x (with respect to a property p) in virtue of the same aspect of that aggregate's structure. This is the characteristic of having a particular number of p-parts. If we let '$\vdash$' stand for truth in every model of the calculus of individuals and definitions (1)–(8), we have

$$(11) \qquad \text{`}m = n\text{' is true iff } \vdash (x)(p)(m(x, p) \equiv n(x, p))$$

This completes a rough outline of the theory of numerical relations. We have seen that this analysis successfully avoids Frege's first major criticism of Mill, the relativity argument. A number is still exemplified by an aggregate in virtue of the aggregate's structure. However, numbers are no longer taken to be simple properties of aggregates.

At this point it would be useful to consider the above proposal in a somewhat broader setting. It is standard practice to impose two constraints upon any adequate analysis of number. Such an analysis must (i) be compatible with an account of the extramathematical applications of number theory, and (ii) verify the Peano axioms. A number of recent articles have suggested yet another constraint that must be placed upon any adequate analysis. This is the requirement that any adequate theory of mathematical truth must be compatible with an account of how mathematical knowledge is possible.[11] Whatever numbers are, it must be possible to explain how one could come to have knowledge, and hence beliefs, about them.[12] In treating numbers as numerical relations we reduce the problem of accounting for our beliefs about numbers to the problem of explaining the possibility of beliefs about certain sorts of structural relations. In fact, in view of the above definitions, an account of beliefs about numerical relations ultimately reduces to an account of beliefs about the part-whole relation (and the identity relation). For this reason I would argue that the suggested analysis of number seems much more likely to be compatible with the above epistemological constraint than does, e.g., a set-theoretic account of mathematical truth. Attempts to provide a sound epistemological foundation for set theory have, I believe, been remarkably unsatisfying. We do not even have the beginnings of a plausible theory of belief (and knowledge) about sets. The analysis of numbers as numerical relations looks more promising from this epistemological standpoint. This is a claim that I will not support here.[13]

Aggregates have played a fairly central role in our development of the theory of numerical relations. In spite of this, very little has been said about what an aggregate is and what kinds of objects go to comprise aggregates. This omission leaves the proposed theory open to Frege's second major criticism of Mill. I refer to this as the "generality argument":

> Mill maintains that the truth that whatever is made up of parts is made up of parts of those parts, holds good for natural phenomena of every sort, since all admit of being numbered. But cannot still far more than this be numbered?
>
> It would indeed be remarkable if a property abstracted from external things could be transferred without any change of sense to events, to ideas, and to concepts. The effect would be just like speaking of fusible events, or blue ideas, or salty concepts or tough judgements (31).

When we understand these remarks as a criticism against the proposed theory of numerical relations, they amount to the follow-

[11] See, for example, Paul Benacerraf, "Mathematical Truth," this JOURNAL, LXX, 19 (Nov. 8, 1973): 661–679.

[12] It has been suggested by Hilary Putnam and others that this epistemological constraint is easily satisfied if we abandon a platonistic approach to mathematical truth in favor of a "modal view" of mathematics [e.g., Putnam, "Mathematics without Foundations" and "Mathematical Truth," reprinted in his *Mathematics, Matter, and Method* (New York: Cambridge, 1976)]. Elsewhere I have argued that there are reasons for thinking that this is not the case ["Mathematics and Modality," *Noûs*, XII, 4 (November 1978): 421–441].

[13] For a complete discussion see my *Numbers*, ch. 1.

ing. The part-whole relation is "abstracted from" physical aggregates. Talk of nonphysical aggregates, such as aggregates of numbers, proofs, or ideas is, therefore, without sense. This kind of talk embodies a category mistake. However, the proposed analysis would reduce all statements of number to statements about aggregates. It thereby renders meaningless a wide range of numerical statements that are perfectly intelligible. The proposed analysis thus conflicts with the intuitive generality of number theory.

The crucial premise in Frege's argument is the claim that the fact that the part-whole relation is "abstracted from" external or physical objects precludes its extension to other domains. It is fairly clear that the sense of 'abstraction' Frege has in mind here is epistemological (cf. 61). The idea is that if we acquire, or come to have beliefs about, a notion on the basis of physical instances, then it cannot have nonphysical instances. Phrased in this way, however, Frege's principle seems far from obvious. Consider, for example, the relation of identity. It is not implausible to assume that we do initially come to have beliefs about the identity relation on the basis of contact with physical objects. If Frege is right, it would then be "remarkable" if we could transfer the notion or identity without change of sense to events, ideas, and other "nonphysical" items. But this does not seem to me to be remarkable in the least.[14] I would argue that the fact that a certain notion generally has a particular kind of "epistemological heritage" does not settle the question of the range of its applicability. In particular, the fact that we initially come to have beliefs about the part-whole relation via its physical instances does not establish that nonphysical aggregates do not exist.

Frege's criticism notwithstanding, we do appear to speak of "nonphysical" aggregates of various kinds. Consider the following sentences.

The proposed alternatives met all the objections.

The proofs of the theorem took an hour to review.

It is plausible to understand the subject terms in these sentences as referring to pluralities, or aggregates of alternatives and proofs, re-

[14] It might be objected that, given the close connection between number and identity, a Fregean would simply deny that we do come to have beliefs about the identity relation in the manner suggested. However, the general point remains. The transfer is "remarkable" only if the spatiotemporality of those objects on the basis of which we come to have beliefs about the part-whole relation is somehow essential to the application of this relation. The next paragraph presents a reason for thinking that this is not the case.

spectively. It is clear that these aggregates differ from aggregates of physical objects in a number of important respects. In particular, aggregates of physical objects are spatiotemporally extended but these are not.[15] On the view I am suggesting, this particular virtue of aggregates of physical objects is important only from an epistemological standpoint. We may very well come to have beliefs about the nature of the part-whole relation through reasoning about physical aggregates. This does not entail, however, that all aggregates must be spatiotemporally extended. A term of the form 'the aggregate of *x*s', where the *x*s are physical objects, can be treated as referring to the objects that exhaust some space-time region. We obviously cannot understand a term of the form 'the aggregate of *p*s', where (for example) the *p*s are properties, as referring to the objects contained in some space-time region. But this does not mean that the term fails to refer. The abstractness (i.e., nonspatiotemporality) of property aggregates presents no more of a problem for reference to aggregates than does the abstractness of properties for reference to individuals. I am thus suggesting that any object to which it is possible to refer (and about which it is possible to have beliefs) can be a constituent or a component of an aggregate. If there are problems with aggregates of ideas, properties, or angels, the problems arise from the nature of ideas, properties, or angels themselves and not from the aggregates that contain them.[16]

One consequence of these remarks is that every (finite) numerical relation is exemplified. It is clear that the number 0 is exemplified. Now consider the number 1. This numerical relation is exemplified by the number 0 with respect to the individuating property of *being identical with 0*. The number 2 is exemplified in that aggregate which contains just the numerical relations 0 and 1 with respect to the property of *being identical with 0 or 1*.[17] The existence of an

[15] For a discussion of some related points see Burge, *op. cit.*

[16] This question is dealt with in more detail in my *Numbers*, sec. 2–3.

[17] I should note that not all numerical statements that involve disjunctive individuating properties of this kind can be analyzed in the style suggested by (6). For such an analysis to be adequate the disjuncts must display a certain sort of independence from each other. Suppose that *p* and *q* are individuating properties that satisfy the condition

$$(x)(Px \supset (\exists\, y)(y < x \,\&\, y \neq x \,\&\, Qy)$$

In this case we say that *p includes q*. If *p* includes *q*, then a numerical statement of the form *n* (*x*, *p* or *q*) cannot be treated in the way suggested by (6). In this case we could interpret the statement in the following way:

$$(\exists\, m)(\exists\, r)(m(x, p) \,\&\, r(x, q) \,\&\, m + r = n)$$

where the operation denoted by '+' is addition. For a more detailed discussion of this point see *Numbers*, sec. 3–1.

aggregate containing just 0 and 1 as parts follows informally from the remarks of the preceding paragraph. Formally speaking, the existence of this aggregate is established by the "comprehension axiom" of the calculus of individuals once we allow as an individual any object to which it is possible to refer. In general we can see that, for any finite number *n*, there is some aggregate in which *n* is exemplified. Given this fact, it is possible to derive the Peano postulates from definitions (1)–(8) plus the axioms of the calculus of individuals.[18]

A familiar criticism of the standard *set-theoretic* accounts of mathematical truth is that there are uncountably many set-theoretic models of number theory.[19] That is, there are many different set-theoretic interpretations of number theory all of which satisfy at least the nonepistemological conditions of adequacy [i.e., (i) and (ii), p. 74]. In view of this, there appears to be no reason to assume that any one of these reductions provides the "correct" analysis of number. This is sometimes expressed by saying that if we have any simple infinite series of objects (under some appropriate ordering relation), then the elements of this sequence "can play the role of the natural numbers." The analysis of number we have been considering not only is compatible with this important observation but actually explains why it is the case. Consider any simple infinite series of objects. These can be sets, inscription types, possible inscription tokens, stars, or whatever. Given any finite number *n*, there will be some portion of this sequence which actually exempli-

[18] We can also extend the above account in a fairly straightforward way, to provide an analysis of integers and rational numbers. This extension parallels the standard treatment that takes integers and rationals to be sets of ordered pairs of natural numbers. An integer can be understood as a certain type of relation that obtains between natural numbers. In particular, $-m$ and $+m$ are definable as follows:

$$-m(x, y) \text{ iff } y = m + x$$

$$+m(x, y) \text{ iff } x = m + y$$

where m, x, and y are all natural numbers. Similarly, a rational number m/n will be understood as that relation between natural numbers defined as follows

$$m/n(x, y) \text{ iff } nx = my$$

where m, n, x, and y are natural numbers and $n \neq 0$. It is a simple matter to define positive and negative rationals in the manner suggested above. Finally a real number can be construed as a certain kind of aggregate of rationals. This parallels the standard set-theoretic reduction. There is no need to develop extensions of our basic analysis here.

[19] For example, see Benacerraf, "What Numbers Could Not Be," *Philosophical Review*, LXXIV, 1 (January 1965): 47–73; Michael Jubien, "Ontology and Mathematical Truth," *Noûs*, XI, 2 (May 1977): 133–151; Nicholas P. White, "What Numbers Are," *Synthese*, XXVII, 1/2 (May/June 1974): 111–124.

fies *n* with respect to the appropriate property (i.e., *being a set*, *being an inscription type*, etc.). That is, each numerical relation is actually exemplified by some segment of the sequence. This segment can therefore go proxy for, or play the role of, the numerical relation itself. This is most easily seen when our sequence consists of numerals of some kind. For example, suppose we are working in stroke notation. Then the number *n* is exemplified in any sequence of *n* strokes with respect to the property of *being a stroke*. When we establish that there are *n p*'s by showing that these items correspond one–one to the numerals '1' to '1...1' (*n* '1's), we have thus indirectly shown that the aggregate of the items themselves exemplifies the numerical relation *n* with respect to the appropriate property *p*. Our account thus makes clear why it is that the elements of any simple infinite series of objects can go proxy for the numbers themselves.

A complete exposition of the theory of numerical relations would have to consider a number of problems and criticisms which we have so far been able to avoid. For example, is this analysis of number compatible with the extramathematical applications of number theory? The remarks in the last paragraph indicate that it is, but a more thorough consideration of this matter would be desirable. Another apparent difficulty stems from the fact that number-words seem to function much like the quantifiers 'all' and 'some'. The standard first-order analysis of, e.g., 'There are 52 cards in this pile' supports this intuition by actually analyzing away the apparent reference to a number via quantifiers. Our analysis of this sentence is '52 (this pile, *being a card*)'. It is natural to ask about the relation between the two analyses. In fact, given some rather plausible assumptions, these two analyses turn out to be quite closely related to each other. This, too, is a topic I will not discuss here. In spite of these omissions, I hope I have said enough to persuade the reader that Mill's analysis of number, or something very much like it, deserves more careful consideration than it usually receives. The suggested modification of Mill's proposal may well provide the basis for an epistemological, as well as a metaphysical, foundation for mathematics.

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