Journal of Economic Dynamics & Control 34 (2010) 36–41

Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Solving the incomplete markets model with aggregate uncertainty using the Krusell–Smith algorithm and non-stochastic simulations

Eric R. Young *

Department of Economics, University of Virginia, PO Box 400182, Charlottesville, VA 22904, USA

ARTICLE INFO

Article history:
Received 29 August 2008
Accepted 26 November 2008
Available online 8 August 2009

JEL classification:
C63
E21
E32

Keywords:
Idiosyncratic risk
Business cycles
Numerical methods

ABSTRACT

This article describes the approach to computing the version of the stochastic growth model with idiosyncratic and aggregate risk that relies on collapsing the aggregate state space down to a small number of moments used to forecast future prices. One innovation relative to most of the literature is the use of a non-stochastic simulation routine.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

The solution to models that incorporate both idiosyncratic and aggregate risk is of central interest to any researcher seeking, for example, a quantitative model of the macroeconomy that can capture the effect of policy on inequality. Two pioneering papers demonstrated that these models—which feature a state vector that is typically not finite-dimensional, or even countably dimensional—are computable: den Haan (1997) and Krusell and Smith (1998). In particular, the second paper has opened a large literature that applies their result—that predicting future prices is possible using only the mean of the wealth distribution—to a wide variety of environments.[1] I describe in this paper a solution algorithm very close to Krusell and Smith (1998) with an important difference in the simulation procedure.

Before proceeding to the solution method, I want to make several points regarding terminology. The first is in regards to the term *approximate aggregation*, which is used by Krusell and Smith (1998) to describe their results. There are two ways to interpret this term. First, one could read approximate aggregation as implying that the economy's statistical properties are similar to an economy that can be solved using a stand-in household that chooses aggregates directly; that is, approximate aggregation means that there exists an agent whose decisions approximately coincide with aggregates. Second, one could make the weaker statement that approximate aggregation only implies that forecasting prices only requires first moments. Many economies approximately aggregate in the second sense but generate dynamics that are not

* Tel.: +1 434 924 3811; fax: +1 434 982 2904.
E-mail address: ey2d@virginia.edu

[1] Good surveys can be found in Ríos-Rull (2001), Young (2007a), and Krusell and Smith (2006). The number of papers applying this insight is now very large.

0165-1889/$ - see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.jedc.2008.11.010

at all similar to an economy with a stand-in household, at least not one with the usual preferences.[2] Therefore, when one states that an economy approximately aggregates one must supply evidence that additional state variables do not change the dynamics of interest.

A second point involves the interpretation of the algorithm itself. The literature seems to have adopted a *bounded rationality* interpretation, in which the agents in the model are assumed to ignore some information about the economy. Presumably, this information would be useful but nevertheless ignored. However, Young (2007a) demonstrates that the information that is ignored (namely, all the higher-order moments of the wealth distribution) is simply not useful for forecasting prices; that is, agents are not boundedly rational at all. Thus, an equally acceptable (or perhaps more acceptable) interpretation of the algorithm is based on projection: the true doubly infinite-dimensional operator is approximated by projecting it onto a space of low dimension. This interpretation may be preferable because it implies that agents are not behaving suboptimally in any sense, so welfare calculations are well-defined.

2. Algorithm

The difficulty with solving a model with incomplete markets and aggregate uncertainty is that the entire distribution is part of the state of the world. While the current prices are only functions of the mean and the current aggregate shock, future prices could depend nontrivially on higher-order moments through nonlinearities in the individual decision rules. Since a continuous distribution is impossible to represent completely on a computer except in special cases, and even a discrete distribution might suffer severely from the curse of dimensionality, one must in general reduce the state space. As noted above, the approach here is to assume that households do not use all the information contained in the current state space to forecast future prices; however, in the model computed here (den Haan et al., 2009) no information that is useful for predicting future prices is being ignored.

2.1. Forecasting functions

To begin, I assume that the set of moments used to forecast future prices is given by M; denote the cardinality of this set by $\#M<\infty$. Since aggregates are only states because they help forecast prices—either current or future—the set of aggregate state variables is effectively reduced to only M. Note that m_1, the first moment of the capital distribution, must be an element of M since the current prices are functions of it. I then seek an "approximate equilibrium" law of motion $\widehat{H}:\mathbb{R}^{\#M}\rightarrow\mathbb{R}^{\#M}$ which maps current elements of M into their values next period, because the true law of motion H (which maps distributions into distributions) is infinite-dimensional and therefore not computable. Whether this equilibrium law of motion is adequate then depends on how well agents could do by using additional information into their forecasting functions or by using a different functional form; from the perspective of the households this additional information is freely available. It has been found that m_1 is itself nearly sufficient to predict the future evolution of prices if the law of motion is assumed to be log-linear:

$$\widehat{H}(m_1,a)=\exp(A(a)+B(a)\log(m_1)).$$

Note that the coefficients depend on the current aggregate shock a.[3] The Inada conditions for the firm ensure that aggregate capital remains positive and the logarithms stay real-valued.[4]

The success of the finite forecasting approach is discussed in Krusell and Smith (1998) and Young (2007a). The reason that the mean is nearly sufficient by itself is that agents have linear savings rules over most of the state space, and these rules are shifted in parallel by movements in the aggregate state (m_1,a). For the small measure of households who have nonlinear decision rules, their lack of wealth makes them irrelevant for determining aggregate capital. Thus, the mean actually is a nearly sufficient statistic for next period's return to capital. It is important to note that the mean is not a sufficient statistic for all aggregates, as higher-order moments of the wealth distribution are orthogonal to the mean and

[2] See Krusell and Smith (1998) for a demonstration of this result using an economy with idiosyncratic shocks to the discount factor. Young (2007a) shows that estimating the Euler equation of a stand-in household on artificial data generated from this model would produce estimates for risk aversion that are strongly biased downward, even in large samples.

[3] An alternative is to assume that a is simply another linear term:

$$\widehat{H}(m_1,a)=\exp(A+B\log(m_1)+Ca).$$

For this model it makes no difference which specification is used because B turns out not to vary with a. For some models that may not hold, in which case the more general version in the text would be needed.

[4] I also compute the model using the log-polynomial function

$$\widehat{H}(\mu_1,a)=\exp\begin{pmatrix}A(a)+B(a)\log(m_1)\\+C(a)[\log(m_1)]^2\end{pmatrix}.$$

For several variants, this formulation was numerically unstable due to collinearity between $\log(m_1)$ and $[\log(m_1)]^2$ and was not computed. In any case, the presence of the quadratic term did not seem to affect the solution. An implementation using Chebyshev polynomials and/or increasing the number of observations is likely to mitigate the collinearity problem.

the distribution of consumption is not generally well-predicted by the mean either; Young (2007a) contains a complete discussion.

2.2. Computing the value function

To solve the consumer problem, I use value function iteration with Howard's improvement algorithm on a finite grid in (k, m_1) space. I implement a hybrid cubic spline and polynomial interpolation routine to evaluate the value function at points off the grid. Cubic splines are used in the k direction since the value function has more curvature along that axis and I can use fast quasi-Newton methods to solve for the policy function. Since the value function has little curvature in the m_1 direction, either linear or polynomial interpolation in this direction is sufficient.[5] During the simulation linear interpolation is used to evaluate decision rules in the m_1 direction; the simulation procedure presented in the next subsection is such that decision rules are never evaluated at k points that are not on the grid.

To compute the optimal choice for k', I use Newton–Raphson on the first-order condition whenever the borrowing constraint is not expected to bind, and Brent's method whenever it is.[6] That is, I solve the Kuhn–Tucker condition

$$\begin{aligned} -u'(c) + \beta \sum \pi_{a',\varepsilon'|a,\varepsilon} v_1(k', \varepsilon', m_1', a') &\leq 0 \\ &= 0 \quad \text{if } k' > 0 \end{aligned} \tag{2.1}$$

along with the budget constraint

$$c = (r(m_1, a) + 1 - \delta)k + ((1 - \tau(m_1, a))\bar{l}\varepsilon + \mu(1 - \varepsilon))w(m_1, a) - k', \tag{2.2}$$

where

$$\begin{aligned} r(m_1, a) &= \alpha a m_1^{\alpha-1}(\bar{l}L(a))^{1-\alpha}, \\ w(m_1, a) &= (1 - \alpha) a m_1^{\alpha}(\bar{l}L(a))^{-\alpha}, \\ \tau(m_1, a) &= \frac{\mu(1 - L(a))}{\bar{l}L(a)}, \end{aligned}$$

the derivative term is evaluated using the cubic splines. I then check *ex post* whether the Euler equations are satisfied within numerical tolerance:

$$-u'(c) + \beta \sum \pi_{a',\varepsilon'|a,\varepsilon} u'(c')(r(m_1', a') + 1 - \delta) = 0 \tag{2.3}$$

for any (k, ε, m_1, a) in which the first-order condition held with equality; for this calculation I again assume that the expectations regarding m_1' do not use the entire distribution. Values for c' are computed using cubic splines over the consumption function. Speed improvements are available over this method, such as more aggressive value iteration steps or the endogenous grid point method of Carroll (2006), but the algorithm is quite fast already.

2.3. Simulation

Rather than use Monte Carlo simulation to generate an updated cross-sectional distribution (as is done in Krusell and Smith, 1998), I use a histogram over a fixed grid to construct tomorrow's distribution. This method has three advantages. One, it does not introduce additional sampling error that must be removed for the households' aggregate behavior to be consistent with the law of large numbers; as noted in Algan et al. (2007), sampling error may be significant for certain conditional moments, so avoiding Monte Carlo approaches for the cross-section can be important. Two, it can be applied to economies in which the mean of the idiosyncratic shock depends on individual decisions and is thus not known in advance, such as a search model of unemployment; imposing the law of large numbers would not be feasible without knowledge of the mean. Three, it turns out to be considerably faster than the Monte Carlo approach. Since this algorithm is the main difference between my method and that used in Krusell and Smith (1998), I discuss it in detail.

Consider a generic period t during the simulation, with current distribution $\Gamma_t(k, \varepsilon)$. I first locate the policy function $k'(k, \varepsilon, m_1, a)$ within the grid; that is, I find the index J such that $k'(k, \varepsilon, m_1, a) \in [k_J, k_{J+1}]$. For values of m_1 that do not lie on the grid, I use linear interpolation to generate approximate decision rules. I then redistribute the current mass $\Gamma_t(k, \varepsilon)$ to the points (k_J, e), (k_J, u), (k_{J+1}, e), and (k_{J+1}, u) according to the weights $\omega \cdot \pi_{a'e|a\varepsilon}$, $\omega \cdot \pi_{a'u|a\varepsilon}$, $(1 - \omega) \cdot \pi_{a'e|a\varepsilon}$, and

[5] In high dimensions linear splines are preferable to polynomials, although for very high dimensions (more than 4) orthogonal polynomials are probably necessary.

[6] An alternative is to use a more sophisticated solution method that deals with the inequality constraint, such as a feasible sequential quadratic programming approach (FSQP). I found that I did not need sophisticated methods for dealing with the constraint since the points where it binds are relatively easy to identify and the solution method is fast enough already. For economies with multiple choice variables (bonds, elastic labor supply) FSQP has proven to be fast and robust.

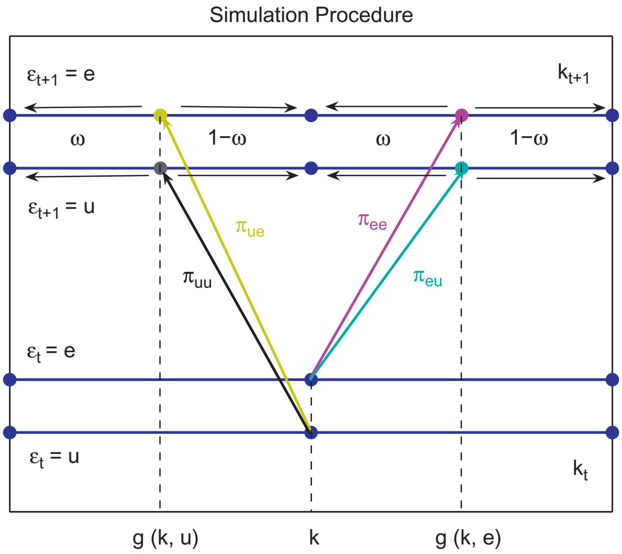

Fig. 1. Simulation procedure.

$(1-\omega)\cdot\pi_{a'u|a\varepsilon}$, where

$$\omega = 1-\frac{k'(k,\varepsilon,m_1,a)-k_J}{k_{J+1}-k_J},$$

the transition probabilities determine the measure of agents who move to the discrete realizations of l, and ω is then a reassignment of individuals to the end points of the interval, where the measure is determined by how close the true decision rule lies to each endpoint. Provided that the grid is sufficiently dense, the relocation of mass does not materially affect the outcomes for any agents. The procedure requires a uniformly spaced grid for k to avoid spikes in the distribution. Fig. 1 illustrates the method—the bottom two lines are the capital grid today for both employed (top) and unemployed (bottom) agents. The arrows show the location of the optimal choice of k' for each type and the measure of agents who transition between each state ε_t and ε_{t+1}.

To begin the simulation, I use an initial distribution derived from the solution to the steady state problem; specifically, the algorithm used is similar to that used in Aiyagari (1994). In Young (2007a), it is shown that the steady state distributions computed using Aiyagari's approach and the Krusell–Smith approach are essentially indistinguishable, meaning that the aggregate shocks do not induce much in the way of additional precautionary saving.[7] Then I draw a long series of aggregate shocks and update the cross-sectional distribution in each period according to the above procedure. During the simulation, I always check that the bounds on m_1 and the upper bound on k are never reached.

2.4. Checking fit

Given the simulation, I must check two things. One, the implied law of motion for the elements in M must be consistent with $\widehat{H}(M)$ so that it constitutes a rational expectations equilibrium. This consistency is ensured by iterating on the coefficients until they converge using a fixed point approach; convergence is typically very rapid, as decisions do not seem to depend too much on the forecasting functions.[8] Two, I must check that the forecasters are not doing "too badly" relative to what they could do if they used additional information. To check this, the first step is to compute measures of fit. Some measures of fit that can be employed here include R^2, the one-step ahead forecast error of an OLS regression $\widehat{\sigma}$, and the maximum deviation in the simulation divided by the average capital stock $e_{\max}$. Finally, the accuracy check proposed in den Haan (2009) is conducted, a more demanding test that explores the coherence between simulations conducted using the entire model versus ones computed using only the aggregate law of motion for capital.

I next compute the model using additional elements in M: I use the cross-sectional standard deviation, the cross-sectional absolute deviation, and the fraction of agents with less than 1 unit of capital as three additional forecast variables. Each additional variable requires another dimension to the value function and a full set of coefficients for its law of motion, so clearly parsimony is important. If the fit measures do not change significantly (up or down, as there is no reason that including additional forecasting variables will improve the fit of the equilibrium laws of motion because behavior changes)

[7] This result is hardly surprising, given the size of the aggregate shocks relative to the idiosyncratic ones. The important implication is that no periods of the simulation need to be discarded to ensure the economy is inside the ergodic set.

[8] Provided they imply reasonable values for K'. The initial guess is taken from an economy with complete markets.

and a battery of aggregate implications are largely unchanged, I consider the quest for an equilibrium to be complete. Note that there does not currently exist a proof that this procedure will converge as $\#M \to \infty$ for some class of approximating functions.[9] For brevity the results of these computations are omitted; see Young (2007a) for details. A quick summary of these experiments is that they do not matter; the agents' predictions regarding the laws of motion for all moments are insensitive to the number of elements in M, even moments that are never used in M explicitly (such as the Gini coefficient for wealth).

2.5. Technical details

The programs run here were compiled in Intel Digital Fortran 9.0 and executed on a Pentium 4—3.0 GHz with 2 GB of RAM and Windows XP Pro operating system. For convergence, I choose an irregularly spaced k grid of 150 points and a uniformly spaced m_1 grid of four points.[10] After the value function has converged, I resolve once over a k grid of 5000 points and a m_1 grid of 30 points, both evenly spaced, for use in the simulation. The upper bound for k chosen is 1200, the convergence tolerance for policy functions chosen is 10^{-10}, and the value function is converged to eight decimal places. Finally, simulations are of length 10,000 with the first 500 observations dropped to eliminate transitional dynamics, starting from the stationary distribution of the model without aggregate shocks. Hot starts for the coefficients of the forecasting rules are exploited to speed up convergence, although good initial conditions do not appear to be critical provided the initial guess implies stationary dynamics. One iteration on the coefficients of the forecasting function takes less than 30 s, including the simulation; there is a tremendous increase in speed relative to the Monte Carlo approach used in Krusell and Smith (1998). The speed could probably be increased by reducing the number of grid points, as the grids have not been chosen to maximize efficiency but rather accuracy.

3. Endowment economy

In this final section I present an extension that solves an endowment economy with a risk-free bond (see den Haan, 1997). The same approach is used to solve for the decision rules as in the previous model, with the obvious notational differences. Here, it is important to allow agents to respond to actual current prices; having agents choose their decision rules based on prices that are forecasted using only limited information means that markets are not clearing in the current period.[11] To address this problem, I iterate on the value function using a two-step procedure similar to the one proposed in Ríos-Rull (2001). To set notation, let y denote an idiosyncratic endowment shock, a denote an aggregate endowment shock, b denote current bond holdings, and q be the current bond price. $\underline{b}<0$ is an exogenous limit on debt.

First, given a guess for the value function $v^n(b,y,a)$ compute

$$\widehat{v}(b,y,a,q) = \max_{b' \geq \underline{b}} \{u(ya+b-qb') + \beta E[v^n(b',y',a')]\} \tag{3.1}$$

for a grid of points in the q direction. The resulting demand functions

$$g_b(b,y,a,q) = \operatorname*{argmax}_{b' \geq \underline{b}} \{u(ya+b-qb') + \beta E[v^n(b',y',a')]\} \tag{3.2}$$

depend explicitly on the current bond price q. I then update this function by setting

$$v^{n+1}(b,y,a) = \widehat{v}(b,y,a,Q(a)), \tag{3.3}$$

using linear interpolation to evaluate $\widehat{v}(b,y,a,q)$, where $Q(a)$ is the predicted value of the bond price given the current aggregate shock. This procedure, which is very similar to that employed in Reiter (2003), is inspired by Kydland (1989) and ensures that the value function reflects the fact that agents will respond to actual prices both today and in the future. With appropriate modifications, this approach can be used to solve any economy in which some prices (or aggregate quantities) are unknown functions of the state of the world.

To simulate the model and construct an approximate bond pricing function $Q(a)$, I need a method to calculate the bond price q_t that clears the bond market at each point in time:

$$\int g_b(b,y,a_t,q_t)\Gamma_t(b,y) = 0, \tag{3.4}$$

[9] This is in contrast to approaches that collapse the state space in the spirit of parameterized expectations; a proof of convergence for PEA can be found in Marcet and Marshall (1994). The difficulty is that, as mentioned in this paper, the target keeps moving.

[10] Specifically, 30 points for k are located on the interval $[0,3]$, 70 on the interval $[3,90]$, and 50 on the interval $[90,3600]$.

[11] Also, in cases such as the bond economy considered in Krusell and Smith (1997), market clearing at each date and state requires solving for market-clearing prices at each date, since bond demand will follow an AR(1) with high persistence; thus, clearing this market "on average" would produce a time series for the bond price that does not look similar to the one that clears markets at each date and state. The size and behavior of deviations from market clearing in models with elastic labor supply (particularly Chang and Kim, 2006, where the labor market does not clear at every date and state) remains an open question. Given that the model where markets are cleared is computationally somewhat more burdensome, a careful study of this issue is warranted.

where $\Gamma_t(b,y)$ is the density of individual agents over bond holdings and idiosyncratic endowments. Using the fact that the decision rules depend explicitly on q, I can use linear interpolation to make the LHS a continuous function of q_t, given a realization for a_t and the current distribution $\Gamma_t(b,y)$. During the simulation I solve this equation using Brent's method and then update the distribution as in the previous section.[12] The resulting time series for q can be projected on the aggregate state to obtain $q_t = Q(a_t)$, which is used to update the value function.

A quick summary of the results for this model is given here; more details can be found in Young (2007b). First, bond pricing is quite good when no cross-sectional moments are used (this setup is the one detailed above). Agents make mistakes estimating the current bond price (which they do not use in their decisions, since they are responding to actual prices) only in two or three periods after a change in the aggregate state, and these mistakes are quite small. Using two past values of the aggregate shock as additional moments turns out to perfectly price bonds. Introducing cross-sectional moments into the set of forecasting moments turns out to greatly increase the computational burden without changing the bond price implications, because accurate representations of the laws of motion are difficult; the moments of the distribution of wealth are not independent in this model. One important difference between the models with and without capital is the large number of constrained agents that arise in the endowment economy; whether this difference accounts for the interdependence of cross-sectional moments is not known.[13]

4. Conclusion

The method used here has been applied in a wide range of environments and holds up well; Young (2007a) details many experiments designed to test the limits of approximate aggregation. The experiment conducted by Porapakkarm and Young (2009) merits particular mention—how well does an economy aggregate when agents have different information sets? In that paper, some agents observe only the prices $(r_t\eta_t, w_t\varepsilon_t)$ and use the Kalman filter to extract estimates of $(a_t, m_{1t}, \varepsilon_t, \eta_t)$, where ε_t is an idiosyncratic shock to the individual wage and η_t is an idiosyncratic shock to the individual return, while other agents observe $(r_t, w_t, \varepsilon_t, \eta_t)$ and can therefore determine (a_t, m_{1t}) exactly. Approximate aggregation holds in that environment as well; the fit of the function that forecasts aggregate capital is very high. One interesting characteristic of that economy is heterogeneity in beliefs among the uninformed, a feature which might help account for the extremely high trading volume observed in asset markets.

Acknowledgment

I would like to thank Yann Algan, Chris Carroll, Daniel Carroll, Ken Judd, Poe Porapakkarm, Michael Reiter, Ayşegül Şahin, and Tony Smith and especially Wouter den Haan for helpful discussions. All remaining errors are my responsibility.

References

Algan, Y., Allais, O., den Haan, W., 2007. Solving heterogeneous-agent models with parameterized cross-sectional distributions. Journal of Economic Dynamics and Control 32 (3), 875–908.
Aiyagari, S.R., 1994. Uninsured idiosyncratic risk and aggregate saving. Quarterly Journal of Economics 109 (3), 659–684.
Carroll, C.D., 2006. The method of endogenous grid points for solving dynamic stochastic optimization problems. Economics Letters 91 (3), 312–320.
Chang, Y., Kim, S.-B., 2006. From individual to aggregate labor supply: a quantitative analysis based on a heterogeneous-agent macroeconomy. International Economic Review 47 (1), 1–27.
den Haan, W., 1997. Solving dynamic models with aggregate shocks and heterogeneous agents. Macroeconomic Dynamics 1 (2), 355–386.
den Haan, W., 2009. Assessing the accuracy of the aggregate law of motion in models with heterogeneous agents. Journal of Economic Dynamics and Control, this issue, doi:10.1016/j.jedc.2008.12.009.
den Haan, W., Judd, K.L., Juillard, M., 2009. Computational suite of models with heterogeneous agents: incomplete markets and aggregate uncertainty. Journal of Economic Dynamics and Control, this issue, doi:10.1016/j.jedc.2009.07.001.
Krusell, P., Smith Jr., A.A., 1997. Income and wealth heterogeneity, portfolio choice, and equilibrium asset returns. Macroeconomic Dynamics 1 (2), 387–422.
Krusell, P., Smith Jr., A.A., 1998. Income and wealth heterogeneity in the macroeconomy. Journal of Political Economy 106 (6), 867–896.
Krusell, P., Smith Jr., A.A., 2006. Quantitative macroeconomic models with heterogeneous agents. In: Blundell, R., Newey, W., Persson, T. (Eds.), Advances in Economics and Econometrics, Ninth World Congress, Cambridge University Press, Cambridge.
Kydland, F.E., 1989. Monetary policy in models with capital. In: van der Ploeg, F., de Zeeuw, A.J. (Eds.), Dynamic Policy Games in Economics. North-Holland Press, Amsterdam.
Marcet, A., Marshall, D.A., 1994. Solving nonlinear rational expectations models by parameterized expectations: convergence to stationary solutions. Working paper, Northwestern University.
Porapakkarm, P., Young, E.R., 2009. Information heterogeneity in the macroeconomy. Working paper, University of Virginia.
Reiter, M., 2003. Recursive computation of heterogeneous agent models. Working paper, Universitat Pompeu Fabra.
Ríos-Rull, J., 2001. Computation of equilibria in heterogeneous agent models. In: Marimon, R., Scott, A. (Eds.), Computational Methods for the Study of Dynamic Economies. Oxford University Press, Cambridge.
Young, E.R., 2007a. Approximate aggregation. Working paper, University of Virginia.
Young, E.R., 2007b. Pricing assets in endowment economies with incomplete markets. Working paper, University of Virginia.

[12] This approach requires that markets clear for every distribution that appears in a simulation and thus depends on some notion of continuity to argue that markets are clearing for distributions that are "near" those that appear in the simulation.

[13] Other experiments suggest all properties also hold for pricing Lucas trees (an asset with a risky dividend) and land (an asset that pays no dividend and produces nonmarketable consumption flows).