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Generalized quasi-geometric discounting ☆

Eric R. Young *

Department of Economics University of Virginia, PO Box 400182, Charlottesville, VA 22904, United States

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Abstract

This paper derives the 'generalized Euler equation' for an agent with multi-period deviations from geometric discounting. The functional equation that describes optimal consumption-savings decisions involves manipulation of future selves and indirect manipulation of more distant selves through intervening selves.

Keywords: Quasi-geometric discounting; Time-inconsistent preferences; Hyperbolic discounting

JEL classification: D90; E21

1. Introduction

Krusell et al. (2001) (hereafter KKS) explore the set of differentiable solutions to a consumption-savings problem undertaken by an infinitely lived agent with quasi-geometric discounting. Their agent has a standard geometric discount factor $\delta \in (0, 1)$ and applies a "future discount factor" $\beta > 0$ to all future periods. The resulting flow of utility can be written down as

$$U_0 = u_0 + \beta\delta u_1 + \beta\delta^2 u_2 + \beta\delta^3 u_3 + \ldots$$

in the current period,

$$U_1 = u_1 + \beta\delta u_2 + \beta\delta^2 u_3 + \ldots$$

☆ I would like to thank the helpful comments of Per Krusell and an anonymous referee. Any remaining errors are my own.
* Tel.: +1 434 924 3811; fax: +1 434 982 2904.
E-mail address: ey2d@virginia.edu.

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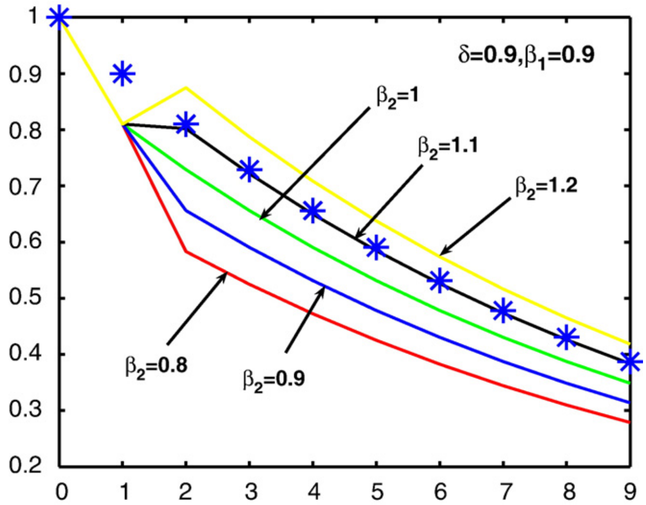

Fig. 1. Discount factors.

in the next period, and so on. It is well-known that such an individual will have time-inconsistent behavior; plans announced today regarding saving in the next period would not be carried out, because the marginal rate of substitution between periods 1 and 2 is not the same for the "current self" and the "future self." KKS characterize differentiable solutions to this problem using a "generalized Euler equation," which notes that the current self attempts to manipulate the future self in order to get an appropriate amount of savings in the future; this manipulation manifests itself as an additional term involving the derivative of the policy function which disappears if $\beta=1$. The differentiable solution is unique; other, non-differentiable equilibria also exist, as discussed in Krusell and Smith (2003).

In this paper I characterize the differentiable and continuous Markov solutions of a generalized version of this consumption-savings problem. In particular, I assume that there are n different β terms, leading to a flow of utility given by

$$U_0 = u_0 + \beta_1\delta u_1 + \beta_1\beta_2\delta^2 u_2 + \beta_1\beta_2\beta_3\delta^3 u_3 + \ldots + \left(\Pi_{i=1}^{n}\beta_i\right)\delta^n u_n + \left(\Pi_{i=1}^{n}\beta_i\right)\delta^{n+1} u_{n+1} + \ldots$$

in the current period.[1] This extension permits much more flexibility in matching consumption-savings behavior than the '(β, δ)' preferences do. Fig. 1 presents five different sequences of discount factors, with common values for $\delta=0.9$ and $\beta_1=0.9$; the '$*$' curve is the geometric discounting sequence. Note that β_1 and β_2 can reinforce each other – if both are on the same side of 1 – or work against each other, creating non-monotonic discounting sequences. In fact, it is possible here to have a sequence of discount factors which disagrees with geometric discounting only in a finite number of periods; this is the case where $\beta_2=1.1$, so that the effects of β_1 and β_2 exactly cancel each other out.

[1] Related to this paper is Karp (in press), who studies a continuous-time problem with non-constant discounting, and naturally the initial work on quasi-geometric discounting in Phelps and Pollak (1968) and Laibson (1994). Karp (2005) discusses some related issues of multiplicity.

2. Model

As noted above, the agent has quasi-geometric discount factors; I assume that there are $n<\infty$ different "future discount factors" so that the sequence of discount factors $\{1, b_1, b_2, \ldots\}$ converges to a geometric sequence in the tail. That is, the particular sequences I admit take the form

$$\left\{1, \beta_1\delta, \beta_1\beta_2\delta^2, \beta_1\beta_2\beta_3\delta^3, \ldots, \left(\Pi_{i=1}^{n}\beta_i\right)\delta^n, \left(\Pi_{i=1}^{n}\beta_i\right)\delta^{n+1}, \ldots\right\};$$

while all results will be stated and interpreted for the case $\beta_i \leq 1$, they would hold with appropriate modifications for $\beta_i > 1$ for some i as well. The resource constraint for the agent is very simple:

$$c + k' \leq f(k),$$

where $f(k)$ is weakly concave, strictly increasing, and at least once continuously differentiable. The utility function $u(c)$ which delivers period utility satisfies the same properties. I explore only differentiable Markov equilibria, just as in KKS, and abstract from trigger–strategy equilibria. Denote by $g(k)$ the decision rule used by future "selves." The problem of the current self for fixed n is

$$V_0(k) = \max_{k'}\left\{u(f(k)-k') + \sum_{j=1}^{n}\left(\Pi_{i=1}^{j}\beta_i\right)\delta^j u(f(g^{n-1}(k'))-g^n(k')) + \left(\Pi_{i=1}^{n}\beta_i\right)\delta^{n+1}V(g^n(k))\right\}; \tag{1}$$

g^n is g composed n times. Because the sequence of discount factors becomes geometric after n periods, the value function $V(k)$ must satisfy the subgame perfect equilibrium requirement

$$V(k) = u(f(k)-g(k)) + \delta V(g(k)), \tag{2}$$

where δ is the geometric discount factor. A subgame perfect Markov equilibrium for the "multiple selves" game is when $k' = g(k)$.

3. Example: $n=2$

For $n=2$ this problem is

$$V_0(k) = \max_{k'}\{u(f(k)-k') + \beta_1\delta u(f(k')-g(k')) + \beta_1\beta_2\delta^2 V(g(k'))\}.$$

It is easy to show that optimal behavior leads to a path $\{c_t, k_t\}_{t=0}^{\infty}$ that satisfies the "generalized Euler equation"

$$\begin{aligned} u'(c_t) = {} & \delta\beta_1 u'(c_{t+1})f'(k_{t+1}) + \delta\beta_1\left(\frac{1}{\beta_1}-1\right)u'(c_{t+1})g'(k_{t+1}) \\ & -\delta^2\beta_1\beta_2\left(\frac{1}{\beta_2}-1\right)u'(c_{t+2})(f'(k_{t+2})-g'(k_{t+2}))g'(k_{t+1}). \end{aligned} \tag{3}$$

The first-order condition for the current self at time t is

$$u'(f(k_t)-k_{t+1}) = \delta\beta_1 u'(f(k_{t+1})-g'(k_{t+1}))(f'(k_{t+1})-g'(k_{t+1})) + \delta^2\beta_1\beta_2 V'(g(k_{t+1}))g'(k_{t+1}). \tag{4}$$

Taking the derivative of Eq. (2) yields

$$V'(k_{t+1}) = u'(f(k_{t+1})-g(k_{t+1}))(f'(k_{t+1})-g'(k_{t+1})) + \delta V'(g(k_{t+1}))g'(k_{t+1}).$$

Solving for V' $(g\ (k_{t+1}))$ and inserting into the first-order condition evaluated at $k_{t+1}=g\ (k_t)$ yields

$$\begin{aligned} u'(f(k_t)-k_{t+1}) = {} & \delta\beta_1 u'(f(k_{t+1})-g'(k_{t+1}))(f'(k_{t+1})-g'(k_{t+1})) \\ & + \delta\beta_1\beta_2(V'(k_{t+1})-u'(f(k_{t+1})-g(k_{t+1}))(f'(k_{t+1})-g'(k_{t+1}))). \end{aligned}$$

The last steps are to solve this equation for V' (k_{t+1}), advance it forward one period to obtain V' (k_{t+2}), insert it back into Eq. (4), and impose $k_{t+1}=g\ (k_t)$ to obtain Eq. (3).

For interpretation of Eq. (3) it is instructive to start with the special case $\beta_1=1$ and $\beta_2\neq 1$:

$$\begin{aligned} u'(c_t) = {} & \delta u'(c_{t+1})f'(k_{t+1}) + \delta^2\beta_2\left(\frac{1}{\beta_2}-1\right)u'(c_{t+2})g'(k_{t+2})g'(k_{t+1}) \\ & -\delta^2\beta_2\left(\frac{1}{\beta_2}-1\right)u'(c_{t+2})f'(k_{t+2})g'(k_{t+1}). \end{aligned} \tag{5}$$

The RHS of Eq. (5) has three terms. The first two terms are standard and appear in KKS. The only difference is that the term which reflects the added benefit of savings appears in period '$t+2$' rather than '$t+1$'; it reflects the desire of 't' to alter the savings of '$t+2$', which can only be achieved by first altering the savings of '$t+1$'. If $\beta_2<1$, this term is positive, so that the cost of saving more today is smaller, because saving more today induces '$t+1$' and '$t+2$' to save more as well.

The third term is new and is negative whenever $\beta_2<1$. This term reflects the conflicting desires of 't' and '$t+1$' regarding '$t+2$'; since '$t+1$' will not want '$t+2$' to oversave, this desire induces a change in g' (k_{t+1}) that tends to reduce the value of savings to 't', who does want more savings out of '$t+2$' — by delivering more to '$t+1$' person 't' will tend to reduce '$t+2$' savings. Since this is exactly what 't' does not want to do, the benefit of saving is reduced by an amount proportional to the change in output at time $t+2$; the proportionality is the amount of change in savings at $t+1$.

The case $\beta_1\neq 1$ and $\beta_2\neq 1$ yields Eq. (3). Now 't' and '$t+1$' disagree about the value of savings in '$t+1$', so a term appears that reflects the added benefit of additional savings in period $t+1$. There is no term that corresponds to the additional cost of savings due to the behavior of intervening selves, since no selves exist between 't' and '$t+1$'.

Inspection of the generalized Euler equation when $\beta_i=1$ for some $i>1$ would make it appear to be a difference equation of higher order than 2. Given that there are only two boundary conditions – namely the current capital stock k_t and the requirement that transversality be respected, which translates to an initial condition on the marginal utility of consumption – it would appear that the equation has indeterminacy. Furthermore, it appears that the order of indeterminacy would be $n-1$, so it would rise with the number of discount factors that differ from 1. But this interpretation is not correct, for the following reason.[2] The above equation would have indeterminacy only if $g\ (k_{t+1})$ could be evaluated at arbitrary k_{t+1}; however it cannot, because $g\ (k_{t+1})$ is something determined by the Euler equation,

[2] I would like to thank Per Krusell for pointing out the correct argument here.

rather than embedded within it — k_{t+1} must equal $g(k_t)$. The correct way to consider this equation is as a functional equation that must hold for all k_t; k_{t+1} is then identified uniquely by the unknown function $g(k)$, provided this function is itself unique. Transversality is respected provided $g'(k)<1$ around some steady state value.

3.1. 'Log-Cobb' parametric example

I now present the parametric example solved by KKS: $u(c)=\log(c)$ and $f(k)=Ak^{\alpha}$ with $\alpha\in[0, 1]$. In the case $\beta_1=\beta_2=1$ one can solve explicitly for the savings rule:

$$g(k) = \alpha\delta Ak^{\alpha};$$

the agent saves a constant fraction of output each period and that fraction is increasing in δ. When $\beta_2=1$ but $\beta_1\neq 1$ the solution is

$$g(k) = \frac{\alpha\delta\beta_1}{1-\alpha\delta(1-\beta_1)}Ak^{\alpha};$$

savings rates are strictly increasing in both δ and β_1. When $\beta_2\neq 1$ make a guess $k_{t+1}=\pi Ak_t^{\alpha}$ for some unknown π and substitute into the Euler equation, which reduces to one equation in the unknown π. The unique solution is

$$\pi = \frac{\alpha\delta\beta_1-\alpha^2\delta^2\beta_1(1-\beta_2)}{1-\alpha\delta(1-\beta_1)-\alpha^2\delta^2\beta_1(1-\beta_2)}.$$

Thus, any combination $(\widehat{\delta}, \widehat{\beta}_1, \widehat{\beta}_2)$ that satisfies

$$\alpha\widehat{\delta}= \frac{\alpha\widehat{\delta}\,\widehat{\beta}_1}{1-\alpha\widehat{\delta}(1-\widehat{\beta}_1)} = \frac{\alpha\widehat{\delta}\widehat{\beta}_1-\alpha^2\widehat{\delta}^2\widehat{\beta}_1(1-\widehat{\beta}_2)}{1-\alpha\widehat{\delta}(1-\widehat{\beta}_1)-\alpha^2\widehat{\delta}^2\widehat{\beta}_1(1-\widehat{\beta}_2)}$$

would generate the same observed savings rates and the same stationary capital stocks.

3.2. 'AK-CRRA' parametric example

A second parametric example that has a closed-form solution under geometric discounting has $u(c)=(1-\sigma)^{-1}c^{1-\sigma}$ and $f(k)=Ak$. When $\beta_1=\beta_2=1$ the savings function is

$$g(k) = \delta^{\frac{1}{\sigma}}A^{\frac{1-\sigma}{\sigma}}k_t;$$

for this solution to make sense $\delta^{\frac{1}{\sigma}}A^{\frac{1-\sigma}{\sigma}}<1$. When $\beta_1\neq 1$ and $\beta_2=1$ the savings rate is a solution

$$1 = \delta\beta_1A^{1-\sigma}\pi^{-\sigma} + \delta(1-\beta_1)A^{1-\sigma}\pi^{1-\sigma}.$$

There may be more than one π that solves this equation, as previously noted in Phelps and Pollak (1968). Provided this savings rate is less than 1, there will exist observational equivalence again. For the case $\beta_1 \neq 1$ and $\beta_2 \neq 1$ the savings rate solves

$$1 = \delta\beta_1 A^{1-\sigma}\pi^{-\sigma} + \delta(1-\beta_1)\pi^{1-\sigma}A^{1-\sigma} + \delta^2\beta_1(1-\beta_2)A^{2-2\sigma}\pi^{2-2\sigma} - \delta^2\beta_1(1-\beta_2)A^{2-2\sigma}\pi^{1-2\sigma}; \tag{6}$$

again, observational equivalence will obtain. I can find cases where there are two savings rates that satisfy Eq. (6) if $\sigma > 1$.[3] The RHS converges to $\delta A^{1-\sigma} < 1$ as $\pi \to 1$; whether there are two solutions depends on the limit as $\pi \to 0$. When σ is large enough, this limit is negative and there exist 2 solutions; otherwise, there is only 1.

3.3. Steady states

The steady state for the geometric discounting problem is unique (on the interior) and satisfies the equation

$$1 = \delta f'(\bar{k}).$$

When $\beta_2 = 1$ and $\beta_1 \neq 1$ the steady state determination is more difficult, as it depends on the derivative of the unknown policy function:

$$1 = \delta\beta_1\left(f'(\bar{k}) + \left(\frac{1}{\beta_1} - 1\right)g'(\bar{k})\right).$$

Thus, solving for the steady state of the model with quasi-geometric discounting requires solving for the decision rule, at least locally around a steady state point. The solution to this equation is unique if g' is monotone decreasing.[4]

When $\beta_2 \neq 1$ the steady state equation is

$$1 = \delta\beta_1(1-\delta(1-\beta_2)g'(\bar{k}))f'(\bar{k}) + \delta(1-\beta_1 + \delta\beta_1(1-\beta_2)g'(\bar{k}))g'(\bar{k}). \tag{7}$$

It does not appear to be possible to prove that Eq. (7) has a unique solution. I therefore solved the model numerically for a wide range of parameter values.[5] It is possible to find multiple solutions for $g(k)$ when $\sigma > 1$ and α is very close to 1, and these solutions tend to have dramatically different implications for the resulting (apparently unique) steady state $\bar{k}$.[6] In all cases considered the decision

[3] Set $A=1$, $\sigma=2$, $\delta=0.95$, $\beta_1=1.0$, and $\beta_2=0.5$. The two solutions are $\pi=0.3536$ and $\pi=0.9667$.

[4] Steady states are indeterminate if non-differentiable decision rules are permitted. See Krusell and Smith (2003) for explicit construction of step–function equilibria.

[5] The model is solved using a projection method with Chebyshev polynomials. To solve the model I use the method of KKS to obtain a good initial condition to start the search for the projection coefficients. Judd (2005) contains a discussion of the merits and pitfalls associated with the KKS procedure.

[6] One combination which leads to two solutions is $A=1$, $\sigma=2$, $\delta=0.95$, $\beta_1=1$, $\beta_2=0.5$, and $\alpha=0.99$. The resulting two steady state levels of capital are 58.19 and 194.38.

rules were monotone-increasing and the stationary capital stocks were increasing in the discount factors.[7]

4. General case

Now consider the general case described by expressions (1) and (2). Define

$$dg^2(k) = g'(g(k))g'(k)$$
$$dg^3(k) = g'(g(g(k)))g'(g(k))g'(k)$$

and so on; $dg^n(k)$ is the change in the savings behavior of the self n periods distant induced by a change in the savings behavior of the current self generated through changes in all intervening selves. The Euler equation is

$$u'(c_t) = \delta\beta_1 u'(c_{t+1})f'(k_{t+1}) - \sum_{j=2}^{n} \delta^j\left(\Pi_{i=1}^{j}\beta_i\right)\left(\frac{1}{\beta_j}-1\right)u'(c_{t+j})f'(k_{t+j})dg^{j-1}(k_{t+1}) + \sum_{j=1}^{n} \delta^j\left(\Pi_{i=1}^{j}\beta_i\right)\left(\frac{1}{\beta_j}-1\right)u'(c_{t+j})dg^j(k_{t+1}). \quad (8)$$

For the 'log-Cobb' parametric case with complete depreciation, the unique savings rate satisfies

$$\pi = \frac{\alpha\delta\beta_1 - \sum_{j=2}^{n} \alpha^j\delta^j\left(\Pi_{i=1}^{j}\beta_i\right)\left(\frac{1}{\beta_j}-1\right)}{1-\alpha\delta(1-\beta_1) - \sum_{j=2}^{n}\alpha^j\delta^j\left(\Pi_{i=1}^{j}\beta_j\right)\left(\frac{1}{\beta_j}-1\right)}.$$

Observational equivalence is therefore a property of the model for arbitrary n, provided this solution makes economic sense. It does not seem generally possible to obtain restrictions for the general case that ensure $\pi \in (0, 1)$, however. Similarly, for the 'AK-CRRA' case the savings rate is a solution to

$$1 = \delta\beta_1 A^{1-\sigma}\pi^{-\sigma} + \delta A^{1-\sigma}(1-\beta_1)\pi^{1-\sigma} + \sum_{j=2}^{n} \delta^j\left(\Pi_{i=1}^{j}\beta_i\right)\left(\frac{1}{\beta_j}-1\right)A^{j-j\sigma}\pi^{j-j\sigma} - \sum_{j=2}^{n} \delta^j\left(\Pi_{i=1}^{j}\beta_i\right)\left(\frac{1}{\beta_j}-1\right)A^{j-j\sigma}\pi^{j-1-j\sigma}; \quad (9)$$

since this expression again implies constant savings rates, observational equivalence would obtain. I have found examples with 2 solutions when $\sigma > 1$.

A steady state for the general equation is a solution to

$$1 = \delta\beta_1 f'(\bar{k}) - f'(\bar{k})\sum_{j=2}^{n} \delta^j\left(\Pi_{i=1}^{j}\beta_i\right)\left(\frac{1}{\beta_j}-1\right)\left(g'(\bar{k})\right)^{j-1} + \sum_{j=1}^{n} \delta^j\left(\Pi_{i=1}^{j}\beta_i\right)\left(\frac{1}{\beta_j}-1\right)\left(g'(\bar{k})\right)^j; \quad (10)$$

it is not clear how to characterize the number of solutions to Eq. (10), but it seems obvious that there is

[7] As noted in Krusell and Smith (2003), there exist bounds on $f'(\bar{k})$ that determine an interval over which they can construct step–function equilibria. It can be shown that the upper bound does not depend on β_i for $i > 1$. Constructing step–function equilibria for cases when $n > 1$ is left for future research.

no method of guaranteeing uniqueness.[8] I again solve this case numerically for a wide range of parameter values (including $\beta_i < 1$ and $\beta_i > 1$ cases); I find that two solutions for $g(k)$ appear when $\sigma > 1$ and α is very large, and these solutions imply very different stationary capital stocks, but the capital stocks appear to be unique for a given $g(k)$. Monotonicity is always a property of the solutions as well.

5. Conclusion

This note has presented the derivation of the optimal differentiable decision rule for an agent with a multi-period deviation from geometric discounting. Krusell et al. (2001) examine the welfare properties of the equilibria for the 'log-Cobb' parametric case with ”$n=1$, finding that welfare is higher in a decentralized allocation if $\alpha < 1$; it seems likely that their proof would go through when $n > 1$ provided all β_i lie on the same side of 1, since it depends critically only on the point that planners see decreasing returns to additional capital while competitive agents see constant returns. When some $\beta_i < 1$ and some $\beta_i > 1$, general statements regarding welfare would be more difficult to obtain. I leave these considerations for future research, since they lie well beyond the scope of this paper.

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[8] It should be understood that sums and products for indices that would be negative are 0 and 1, respectively.