Portfolio Choice with Information-Processing Limits*

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Abstract

We study the portfolio decision of an agent with limited information-processing capacity in the sense of Shannon (1948).

1. Introduction

Agents in standard models are assumed to observe the realization of the state with certainty, requiring that agents have unlimited information-processing capacity. With infinite processing capacity, agents can respond to innovations in economic variables immediately and without error. Unlimited capacity is a very strong assumption for ordinary people; individuals encounter much more information that is in principle relevant to their economic behavior than they actually seem to use. Sims (2003) argues agents have limited information-processing capacity, based on evidence about how information flow actually occurs. A capacity constraint on information flow prevents agents from digesting aggregate or individual information and market signals immediately, meaning that agents' responses to shocks may be delayed by the need to slowly absorb just how the state of the world has changed. Sims (2003) introduces information processing constraint in a standard linear-quadratic control problem and examines the implications.

Since then several papers in macroeconomics and finance use same approach within the linear-quadratic control framework. Luo (2007) and Luo and Young (2009b) study consumption dynamics, while Luo and Young (2009a) examine aggregate fluctuations. Sims (2006) examines consumption and saving decisions in a two-period model with preferences that lie outside the quadratic

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class. Lewis (2006) extends the rational inattention framework of Sims (2006) to a finite-horizon dynamic setting, and Tutino (2007) the model to an infinite horizon. A common feature of all these models is the presence of only one (risk free) asset available for saving.

Our interest in this paper lies in the implications of rational inattention for portfolio choice. Luo (2008) considers the same question in a setup with quadratic utility and Gaussian shocks; due to certainty equivalence, the precautionary savings motive generated by rational inattention is completely absent. In contrast to that paper, here we study a model that permits precautionary effects; unfortunately, computational constraints limit us to a two-period model.

2. A Standard Two Period Portfolio Choice Model

Our main interest in this paper lies with the portfolio problem under limited information-processing capacity. However, it is convenient to draw distinctions between solutions with and without these limitations, so here we present a standard two-period portfolio problem; Samuelson (1969) and Merton (1969) provide complete analyses of this problem in the case of HARA-class utility functions. Consider an agent with a CRRA utility function $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$; $\gamma \geq 0$ equals the Arrow-Pratt measure of relative risk aversion and γ^{-1} is the intertemporal elasticity of substitution. This agent faces stochastic current income e^1 and stochastic future income e^2 with distributions of $g_1\left(e^1\right)$ and $g_2\left(e^2\right)$, respectively.

There are two tradable financial assets available to the household, one risky and one risk-free. The return on the risk free asset is r^f and the return on the risky asset over the period is r^e. We consider the risky asset to be a market portfolio of equities with return distribution $\varphi\left(r^e\right)$. Letting r be the one period gross return to invested wealth, we obtain

$$r = sr_d^e + (1-s)\, r^f = s\left(r_d^e - r^f\right) + r^f,$$

where s is the proportion of wealth invested in the risky asset. In period 1, wealth w^1 is simply e^1 as the initial wealth is assumed to be 0 and saving (borrowing) is $e^1 - c^1$. In period 2, wealth consists of the return on savings and future income $\left(w^1 - c^1\right) r + e^2$. Following Sims (2006), we assume that second period consumption is equal to wealth.[1] For reasons we will elaborate on more

[1] In the standard model this assumption is without loss of generality provided utility is increasing. In the model with limited information-processing capacity, households may leave accidental bequests because they are uncertain about their exact wealth. Adding a consumption choice in the second period is computationally costly and unlikely to add any insight.

completely later, we discretize both the state and control space.

The maximization problem for this agent can be written as

$$\max_{\{c(w_i^1)\},\{s(w_i^1)\}_{i=1}^I} \left\{ \begin{array}{c} \sum_{i=1}^I u\left(c\left(w_i^1\right)\right) g_1\left(e_i^1\right) + \\ \beta \sum_{i=1}^I \sum_{d=1}^D \sum_{j=1}^J u\left(\left(e_i^1 - c\left(w_i^1\right)\right)\left(s\left(w_i^1\right)\left(r_d^e - r^f\right) + r^f\right) + e_j^2\right) \varphi\left(r_d^e\right) g_1\left(e_i^1\right) g_2\left(e_j^2\right) \end{array} \right\}.$$

In period 1, before income is realized the agent makes a contingent plan for consumption and savings that depends on the realization of e^1. This specification is equivalent to the usual timing in which the agent makes consumption-savings plan after the realization of e^1, but it will turn out to be easier to formulate the rational inattention model with this timing. The plan for period 1 then determines consumption in period 2 based on the realizations of e^2 and r^e.

We assume that agents can borrow and that consumption in each date and state must be nonnegative:

$$c\left(w_i^1\right) \geq 0 \quad \text{and} \quad \left(e_i^1 - c\left(w_i^1\right)\right)\left(s\left(w_i^1\right)\left(r_d^e - r^f\right) + r^f\right) + e_j^2 \geq 0 \quad \forall\, i = 1, ..., I;\ \forall j = 1, ..., J;\ \ \forall d = 1, ..., D.$$

Since this problem has a continuous and concave objective function and a convex opportunity set, the first-order conditions are necessary and sufficient:

$$c_1\left(w_i^1\right)^{-\gamma} = \beta \sum_{d=1}^D \sum_{j=1}^J \left(\left(e_i^1 - c\left(w_i^1\right)\right)\left(s\left(w_i^1\right)\left(r_d^e - r^f\right) + r^f\right) + e_j^2\right)^{-\gamma} \left(s\left(w_i^1\right)\left(r_d^e - r^f\right) + r^f\right) \quad \forall\, i = 1, ..., I.$$

$$\beta \sum_{d=1}^D \sum_{i=1}^I \sum_{j=1}^J \left(\begin{array}{c} \left(\left(e_i^1 - c_1\left(w_i^1\right)\right)\left(s\left(w_i^1\right)\left(r_d^e - r^f\right) + r^f\right) + e_j^2\right)^{-\gamma} \times \\ \left(e_i^1 - c_1\left(w_i^1\right)\right)\left(r_d^e - r^f\right) \varphi\left(r_d^e\right) g_1\left(e_i^1\right) g_2\left(e_j^2\right) \end{array} \right) = 0 \quad \forall i = 1, ..., I.$$

The first condition is the condition on optimal consumption over time: marginal discounted utilities are equalized. The second condition is the condition for optimal (additional) risk taking. We cannot get a closed-form solution to this problem except in special cases that are not of interest to us (quadratic or CARA utility). As Samuelson (1969) noted, one expects the following properties would hold: (i) optimal portfolio choice s^* is decreasing in γ, (ii) second period consumption and s^* are higher when r^f is low relative to β, and (iii) a mean-preserving spread in the risky asset return decreases s^*.

Following Campbell (1993) we could use a log-linearization method to solve the problem. However, because that method will not work when information-processing constraints are imposed, we solve this problem numerically; numerical solutions are provided in section 4. In the next section, we impose these constraints and study the portfolio decisions of agents with rational inattention.

3. The Model with Limited Information-Processing Capacity

We now assume agents have limited information processing capacity in the sense of Sims (2003). Agents choose the optimal joint probability distribution of consumption, portfolio allocation, and wealth to maximize their lifetime utility. Hence the choice variables with respect to which we maximize is the joint probability distribution function $f(\cdot)$ of consumption c^1 and the share of savings held in the risky asset s with current income e^1. The objective function for the agent is

$$\max_{\{f(s_k,c_r^1,e_i^1)\}} \sum_{k=1}^{K}\sum_{r=1}^{N}\sum_{i=1}^{I} \left(\begin{array}{c} u\left(c_r^1\right) + \\ \sum_{d=1}^{D}\sum_{j=1}^{J}\beta\left[u\left(\left(w_i^1 - c_r^1\right)\left(s_k\left(r_d^e - r^f\right) + r^f\right) + e_j^2\right)\right]\varphi\left(r_d^e\right) g_2\left(e_j^2\right) \end{array} \right) f\left(s_k, c_r^1, w_i^1\right). \tag{3.1}$$

As above, we assume that agents can borrow and the consumption in each period must be nonnegative. Budget constraints in this model enforce the nonnegativity requirement; they are satisfied automatically if agents are saving ($c_r^1 > w_i^1$ for grid points r and i) but impose restrictions whenever agents are borrowing ($c_r^1 > w_i^1$). In the borrowing case the budget constraints for period 2 take the form

$$\begin{aligned} f\left(s_k, c_r^1, e_i^1\right) &= 0 \;\; if \;\; \left(w_i^1 - c_r^1\right)\left(s_k(r_d^e - r^f) + r^f\right) + e_j^2 \le 0 \\ \forall d &= 1, ..., D;\ \forall r = 1, ..., N; \forall i = 1, ..., I; \forall j = 1, ..., J;\ \forall k = 1, ...K. \end{aligned} \tag{3.2}$$

The choice set is also restricted by the requirement that the probability density must be well-defined:

$$0 \le f\left(s_k, c_r^1, w_i^1\right) \le 1 \quad k = 1, ..., K,\ r = 1, ..., N,\ i = 1, ..., I. \tag{3.3}$$

Since income is exogenous, the marginal probability of w^1 chosen by the household must be equal to the probability distribution function of e^1:

$$\sum_{k=1}^{K}\sum_{r=1}^{N} f\left(s_k, c_r^1, w_i^1\right) = g_1\left(e_i^1\right) \qquad i = 1, ..., I. \tag{3.4}$$

The final constraint is the information processing constraint. To formulate the IPC we need to define the mutual information between random variables. The mutual information between current consumption c^1, the share of the risky asset s, and current income e^1, is defined as

$$I\left(w^1; c^1, s\right) = H\left(w^1\right) + H\left(s, c^1\right) - H\left(s, c^1, w^1\right).$$

$I\left(w^1; c^1, s\right)$ measures the reduction in uncertainty about w^1 after observing c^1 and s and is always nonnegative. The assumption of limited information processing capacity $\overline{C}$ requires mutual information not exceed capacity; that is, $I\left(w^1; c^1, s\right) \leq \overline{C}$. Therefore, the information processing constraint is given by the nonlinear inequality (**??**):

$$\sum_{k=1}^{K}\sum_{r=1}^{N}\sum_{i=1}^{I} f\left(s_k, c_r^1, w_i^1\right) \log\left(f\left(s_k, c_r^1, w_i^1\right)\right) - \sum_{i=1}^{I} g_1\left(e_i^1\right) \log\left(g_1\left(e_i^1\right)\right) - \sum_{i=1}^{I}\sum_{k=1}^{K}\sum_{r=1}^{N} f\left(c_r^1, s_k, w_i^1\right) \log\left(q\left(c_r^1, s_k\right)\right) \leq \overline{C}, \tag{3.5}$$

where the marginal distribution of consumption in period 1 is $q\left(c^1, s\right)$ and defined by

$$\sum_{i=1}^{I} f\left(s_k, c_r^1, w_i^1\right) = q\left(c_r^1, s_k\right) \qquad r = 1, ..., N, \quad k = 1, ..., K.$$

The derivation of (**??**) is given in Appendix (**??**).[2] The following proposition is straightforward.

Proposition 1. *The objective function is continuous and concave and the constraint set is convex. Therefore,(3.1) has a solution.*

The problem for the information-constrained household is to maximize (3.1) with respect to (3.2)-(**??**). For many points in the discrete outcome space $f\left(s_k, c_r^1, w_i^1\right) = 0$ may be binding (but obviously not for all of them), meaning that (3.1) is a highly-nonlinear problem; furthermore, even for relatively coarse discretizations the number of choice variables and constraints is very large. Linearization techniques are not applicable, so we solve the problem directly through a "brute force" method (albeit a highly sophisticated one). Before proceeding to the numerical solutions we examine a special case: $\overline{C} = 0$. A second special case, $\overline{C} = \infty$, is equivalent to the standard model solved in the previous section.

[2] The rational expectation model is the special case with unlimited information processing capacity.

3.1. Zero Information Processing Case

We now consider the case $\overline{C} = 0$. As noted above, $I\left(w^1; s, c^1\right)$ measures the reduction in uncertainty of w^1 after observing c^1 and s and it is always nonnegative. In other words, knowledge about $\left(c^1, s\right)$ can not increase uncertainty of w^1:

$$I\left(w^1; s, c^1\right) = H\left(w^1\right) - H\left(w^1 | c^1, s\right) \geq 0.$$

On the other hand, zero information processing capacity requires

$$I\left(w^1; s, c^1\right) = H\left(w^1\right) - H\left(w^1 | c^1, s\right) \leq 0.$$

Hence combining these two inqualities we get

$$I\left(w^1; s, c^1\right) = H\left(w^1\right) - H\left(w^1 | c^1, s\right) = 0.$$

Therefore, zero information processing capacity implies that current wealth is independent of current consumption and the risky asset share: $f\left(s_k, c_r^1, w_i^1\right) = q\left(s_k, c_r^1\right) g\left(e_i^1\right)$. Agents cannot condition the distribution of the current wealth on the realization of current consumption and the risky asset share, since those observations carry no usable information. Therefore, the objective function becomes

$$\max_{\{c^1, s\}} \left\{ u\left(c^1\right) + \sum_{i=1}^{I} \sum_{d=1}^{D} \sum_{j=1}^{J} \beta \left[u\left(\left(e_i^1 - c^1\right) \left(s\left(r_d^e - r^f \right) + r^f \right) + e_j^2 \right) \right] \varphi\left(r_d^e\right) g_2\left(e_j^2\right) g_1\left(e_i^1\right) \right\}$$

The constraints on current and future consumption are

$$q\left(s_k, c_r^1\right) = 0 \ \text{ if } \ \left(e_i^1 - c_r^1\right) \left(s_k \left(r_d^e - r^f \right) + r^f \right) + e_j^2 \leq 0 \text{ and } c_r^1 > e_i^1$$

for each d, r, i, j, and k. It turns out that the model displays sharp changes in behavior as we approach the case where $\overline{C} = 0$.

3.2. Computation

Our computational method is similar to those used in Sims (2005) and Lewis (2006). As noted above, we discretize the state and outcome spaces and permit the agent to attach probabilities to each of those outcomes, subject to the appropriate restrictions. We assume that first period income has 16 grid points with values ranging from 0.01 to 0.16 and second period income has 4 grid points ranging from 0.02 to 0.08. The risky return has 8 grid points ranging from 0.79 to 1.35. For simplicity we assume that both second period income and the risky return are uniform, while we vary the distribution of first period income (the benchmark distribution is normal with mean 0.085 and standard deviation is 0.023). We also assume no correlation between labor income and the risky return in the second period; aggregate data shows little correlation between stock returns and wages at business cycle frequencies, so this assumption seems a natural benchmark. For the outcome space, current consumption has 32 grid points between $[0.005, 0.16]$ and the risky asset share has 21 grid points between $[0, 1]$. The risk-free rate is $r^f = 1.02$, the expected excess return is 0.05 and the standard deviation of the risky return is 19.6 percent; these values are consistent with the premium of US equities over T-bills. We set $\beta = 0.97$ and consider several different values for γ.[3]

The resulting problem has a large number of choice variables and a large number of constraints. To solve this problem, we use the AMPL programming environment as a gateway to the KNITRO solver (a commercial solver used for large nonlinear problems that combines automatic differentiation with sophisticated Newton-based iterations and active set methods to handle the constraints). We then upload our program to the NEOS Server for Optimization, a publicly-available resource that implements the AMPL-KNITRO program (it greatly increases the size of the problem that we can consider as it uses idle supercomputer resources). In total, our problem features $14,226$ choice variables, 1328 linear equality constraints, and 1 nonlinear inequality constraint.[4]

4. Results

In this section, numerical results of rational expectation models will be discussed and then numerical results of the model with limited information processing capacity (IPC) will be analyzed. The

[3]The value for β guarantees that agents will save in some parts of the state space and borrow in others.

[4]In the absence of a binding information-processing constraint, the problem can be written as a linear program and solved using methods described in Trick and Zin (2000). We are currently exploring the possibility of using "approximate dynamic programming" tools – see Powell (2007) – to break the severe curse(s) of dimensionality that RI problems pose and explore longer horizon problems.

benchmark model has $\gamma = 2$ and $\beta = 0.97$, but we will explore alternative parameter values.

4.1. Standard (RE) model

In a standard (rational expectations) model agents observe the realization of current wealth with certainty; since their objective functions are strictly concave, their decisions become deterministic functions of this realization. For comparability reasons we solve the rational expectation models using same numerical approach that we apply to the limited information-processing settings.

The expected risky asset share is 0.95 for the agent with $\gamma = 0.5$ and 0.97 for the agent with $\gamma = 2$. An agent with a higher intertemporal elasticity of substitution has larger current consumption and therefore borrows in more states with lower current wealth.[5] Hence, the $\gamma = 0.5$ agent chooses to invest in the risk-free asset at higher levels of wealth. Figure (1) shows the risky asset share over the grid of current period wealth. For each value of γ, agents choose to invest all their saving in the risky asset in states with relatively high wealth and choose to invest in the risk-free asset in states with low wealth. As the risky asset has significantly higher expected return than the risk-free asset, agents like to enjoy this high expected return in states with positive saving. However, agents want a lower return at the states with borrowing. The resulting decision rule is essentially discontinuous; a related but (slightly) less-extreme result can be found in Krusell and Smith (1997) that is driven by the absence of insurance markets.

Table 1 presents expected (average) consumption in each period for both $\gamma = 0.5$ and $\gamma = 2$. The difference between first and second period consumption is smaller for the agent who is less risk averse ($\gamma = 0.5$).

Table 1

Expected Consumption

γ	0.5	2
$E(c_1)$	0.0633	0.0624
$E(c_2)$	0.0732	0.0742

Figure (2) shows the consumption allocations on the grid points for current period wealth. First period consumption increases as current period wealth increases for both types of agent. However, $\gamma = 0.5$ agent allocates slightly larger consumption on grid points with lower wealth than the $\gamma = 2$

[5] Standard portfolio choice models with no uncertainty in current wealth will predict higher expected risky asset share for the $\gamma = 0.5$ agent.

agent and allocates slightly lower consumption on grid points with higher current wealth.

4.2. Model with limited IPC

Note that as $\overline{C} \to \infty$ the limited IPC model will converge to the RE model; we therefore expect that the benchmark model will accurately represent choices of households whose capacity constraints are large. How much capacity is sufficient is model-dependent. For this model, the exogenous distribution for the current income is centered on its mean 0.085, with the entropy of this process being $H(e^1) = -\sum_{i=1}^{I} g_1(e_i)\log(g_1(e_i)) = 2.074$ nats. From the definition of the mutual information we know that the knowledge of current consumption and share of risky asset can decrease the uncertainty about current wealth at most by this amount. Hence for $IPC \geq 2.074$ nats the model should provide similar results as the model with unlimited information processing capacity.

4.2.1. Joint distribution of current wealth and current consumption

The information processing constraint changes the optimal distributions. Shaded plots of the joint densities of c_1 and w_1 are shown in Figures (3) and (4). The darker the box, the higher the probability weight placed on the corresponding grid point. Sufficiently large $\overline{C}$ (Figure 3) allows the agent to observe every realization of the current wealth and hence he chooses distribution with perfect correlation between current wealth and current consumption and put probabilities along the rational expectation solutions. However, as $\overline{C}$ decreases the agent allocates probability on fewer grid points with low consumption. Small $\overline{C}$ does not allow the agent to learn every realization of the current wealth. Thus he wants to be well informed about the grids with lower wealth to prevent future consumption close to zero. $\gamma = 2$ agent is willing to sacrifice consumptions at the grids with high current wealth to avoid zero future consumption because of the infinitely high utility cost of the zero consumption. The smaller the IPC the more difficult to decide on which state occurred and agents respond with smoother consumption distribution over the less informed states. Therefore, the agent becomes highly risk averse as the information processing capacity gets smaller and allocates similar consumption levels on possible states of the current wealth.

Figure (4) shows the joint probability distribution of $c_1$and $w_1$of an agent with $IPC = 0.1$, which allows him to solve only about 5% of the total uncertainty in the current wealth. The agent puts probability on fewer grids with lower consumption and puts probability 0.64 on the grid with consumption of 0.055 (Figure 3b). When IPC becomes zero the distribution of current wealth becomes independent from the joint distribution of consumption and risky asset share and the agent chooses a degenerate distribution.

4.2.2. The expected current and future consumptions

For the model with sufficiently large IPC, we expect similar results as in the model with rational expectation. Agents with $\gamma = 2$ wants consumptions in different states to be highly similar. Moreover, they want consumptions in 2 periods to be highly similar. The following table shows the expected consumptions of the model with unlimited information processing capacity[6]. The expected consumptions are similar to the rational expectation solutions as expected.

γ	0.5	2
$E(c_1)$	0.0625	0.0623
$E(c_2)$	0.0741	0.0743

The consumption behavior changes with IPC. The expected current consumption falls and the expected future consumption increases with smaller IPC. Initially, the expected current consumption falls gradually and then starting from a certain low capacity it decreases sharply. As IPC gets smaller, agents become extremely tolerant towards the changes in consumption over time. Hence the intertemporal elasticity of substitution increases and we see a separation of the risk aversion and the intertemporal elasticity of substitution. Figure (5) presents the expected current and future consumption for agents with different risk aversion parameter. Starting from the capacity $\overline{C} = 0.3$, there are substantial changes in expected consumptions. At the capacity $\overline{C} = 0.3$ agents can solve about 15% of the uncertainty in the current wealth and growth rate of the expected current consumption is 36% as opposed to the growth rate of 19% for unlimited IPC.

In the previous section, we observed that the agent with limited information processing capacity allocates probabilities over fewer grids with lower values resulting in smoother consumption distribution. When IPC is smaller than 0.2 the conditional expectation of consumption over the wealth is almost flat and very low (Appx. Figure 1A). As a result of the consumption smoothing over the current states with higher wealth, the expected "precautionary saving", the difference between the expected current income and the expected current consumption, increases substantially with smaller IPC.

[6]IPC is greater than the entropy of the current income process.

IPC	∞	2	1	0.5	0.3	0.2	0.1	0.05
$\gamma = 2$	0.27	0.27	0.28	0.30	0.32	0.33	0.39	0.45
$\gamma = 0.5$	0.27	0.26	0.26	0.27	0.28	0.30	0.36	0.43

Table shows the saving rate, ratio of the expected saving and the expected current income. The expected saving rate is 27% for $IPC > 1$. However, as IPC becomes smaller the saving rate increases dramatically. The future wealth depends on saving, return on savings and future income. Hence, increased precautionary saving due to the limited information processing capacity results in increased future wealth and thus increased future consumption. The growth rate of the expected consumption is slightly smaller for relatively less risk-averse agents as it was expected.

4.2.3. Optimal share of risky asset

The joint distribution of the current wealth and the risky asset share changes with IPC. When information processing capacity is large the agent can learn about the current wealth realization and he allocates positive probabilities over the states with low wealth and zero risky asset share and high wealth and risky asset share of 1. At the state with very low current wealth the agent is likely to borrow and hence he wants to borrow at the lowest possible rate. However, at the state with sufficiently large wealth the agent is likely to save. Thus he wants to earn highest possible returns on his savings. There is only a very slight change in distribution of the risky asset share in response to a reduction in IPC until it gets very small. To choose the optimal distribution of the risky asset share the agent needs to know if the current wealth is higher or lower than a certain threshold level.[7] This information is available until IPC becomes very small. When IPC becomes too small even this information is difficult to obtain and the agent responds to it allocating probabilities over the grids with lower risky asset share to decrease the probability of borrowing with high interest rate.

Unlimited information processing capacity ($IPC \longrightarrow \infty$) allows the agent to observe the current wealth realization accurately. Hence he chooses to invest in the risky asset similarly as in the rational expectation model. However, limited information processing capacity changes the agent's consumption and investment behavior substantially as he can not observe the state realization with certainty. Limited information capacity increases overall uncertainty about current wealth

[7] At the states with current income below the threshold level the agent will borrow and at the states with current income above the threshold level the agent will save.

and hence uncertainty about saving. Therefore, the agent faces a risk of a high interest payment on borrowing when he chooses to invest in the risky asset and a risk of a low return on saving when he chooses to invest in the risk free asset. More risk averse agent is willing to sacrifice high returns on states with saving to reduce the probability of borrowing with high interest rate and therefore to avoid close to zero consumption in the future. Thus the agent allocates larger probabilities over the states with smaller risky asset share when information processing capacity becomes very small (Figure 5b).

Figures (6) and (7) present the joint distribution of the risky asset share and the current wealth for information processing capacity of 2 and 0.1 respectively. At the capacity $\overline{C} = 0.1$, agents can solve only about 5% of the uncertainty in the current wealth and the expected risky asset share is 0.37.

Figure (**??**) shows the expected value of the optimal share of risky asset for different level of information processing capacities. The agent with full information processing capacity has the expected risky asset share, $ES = 0.975$. Initially, the expected optimal share of risky asset falls gradually and then starting from the capacity level, $\overline{C} = 0.1$ it decreases sharply.

The agent's investment behavior changes significantly with respect to their risk aversion level. The expected risky asset share of the low γ agent decreases slightly from $ES = 0.96$ as capacity decreases and converges to a significantly large number (i.e. in this model it goes to around 0.92). As IPC gets very small, the less risk averse agent will ignore the grids with low wealth and allocates probabilities along one risky asset share grid with a significantly high value (Figure 9).

4.2.4. Welfare

The agent's welfare is expected to decrease as the constraint on information processing capacity gets tighter. Figure 2A in the appendix3 shows the change in the expected utility at different IPC levels as the percentage of the expected utility at the full capacity level. For both types of agents the welfare decreases as IPC gets smaller, but the magnitude of the change is different. The less risk-averse agent is more tolerant to consumption fluctuations and his welfare decreases slowly, while the more risk-averse agent's welfare decreases dramatically.

4.2.5. Income distribution and IPC

In this section, we examine the effects of the current income distribution on the agent's behavior. The benchmark model has current income with a mean centered distribution with standard deviation of 0.02 (Figure 3A in the appendix3). First, we increased the standard deviation of the distribution to .033, which results in increase in entropy with the overall uncertainty of 2.59. When we increase uncertainty in the current income, the expected current consumption decreases faster than the expected current consumption in the benchmark model. It is shown in Figure 4A and Figure 5A in the appendix3.The expected risky asset share is lower at each capacity level and the expected current consumption decreases earlier than the model with less uncertainty in the current wealth. We also examined effects of the correlation between the future income and the risky asset return on the agent's behavior. We imposed a joint distribution of future income and risky asset share with a correlation coefficient of -0.2. Thus the risky asset is relatively more valuable than the one in the benchmark model. Model setting changes slightly as the future income and the risky asset share have joint distribution. The agent is more reluctant to reduce the investment in the risky asset in response to decrease in IPC as was expected.

4.2.6. Beta

The benchmark model has $\beta = 0.97$. However, for the model with 2 period the smaller β may be appropriate. So we set β equal to 0.54 and solved for the agent's optimal distribution. Main results are similar as the results from the benchmark model. The expected risky asset share decreases with limited IPC and the agent invests less in the risky asset at each capacity level than in the $\beta = 0.97$ model. The agent in this model is less patient and consumes more in the first period and hence he borrows on more current wealth grids. In fact, the expected current consumption is greater than the expected future consumption for sufficiently high IPC. But smaller capacity pushes the agent to consume less and eventually the expected current consumption gets smaller than the expected future consumption.

5. Conclusion

Computational results of the model with rational inattention are very similar to the results of the model with rational expectation when there is unlimited information processing capacity as we expected. The analysis shows that the idea of rational inattention is crucial to understand individual investor's behavior on the asset market. Rational inattention provides sufficiently low investment rate in the risky asset for lower capacity level and for reasonable risk-aversion parameter in an otherwise standard portfolio choice model.

As information processing capacity gets smaller the agent becomes highly risk averse and allocates similar consumption level on possible states of the current wealth, which in turn increases precautionary saving. Moreover, the agent becomes extremely tolerant towards the changes in consumption over time for a given risk aversion and intertemporal elasticity of substitution increases. The simple portfolio choice model with rational inattention provides higher risk aversion and higher intertemporal elasticity (IES) of substitution simultaneously. Therefore, it sheds some light on explaining well known "equity premium puzzle" and "risk free rate puzzle". These results are robust to changes in income distribution, risk aversion parameter γ and discount factor β.

In this model, to keep it simple we implicitly assumed the uncertainty to be fully solved in the future. Thus it may have dampened the impact of the rational inattention. To fully understand the impact of RI on individual's and macroeconomic behavior we need more general model with many periods.

6. Appendix 1: Information Theory

The size of the information produced by the stochastic process is measured by its entropy $H(\cdot)$, following Shannon (1948). $H(X)$, the entropy of a discrete random process X (which is a measure of the expected uncertainty in X), is defined by

$$H(X) = -\sum_x p(x)\log(p(x)).$$

Entropy measures the information provided by a random process about itself.[8] Entropy has the following properties:

1 $H(X) \leq \log(n)$ with equality if and only if $p(x) = 1/n$ for all x (that is, uniform random variables have maximum entropy);

2 $H(X) \geq 0$ with equality if and only if X is deterministic (that is, all nondegenerate random variables have positive entropy);

3 The conditional entropy of the jointly-distributed X, Y is denoted

$$H(X|Y) = E[-\log(p(X|Y))] = \sum_y H(X|Y=y)\,p(y).$$

[8] See Gray (1990).

$H(X|Y) \leq H(X)$ with equality if and only if X and Y are independent (that is, conditioning on a second random variable can never increase entropy);

4 $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y) \leq H(X) + H(Y)$ with equality if and only if X and Y are independent (the joint entropy of two random variables is highest when they provide no information about each other).

The mutual information is a measure of the information contained in one process regarding another process. Suppose $\{X, Y\}$ is a random process. The average mutual information between X and Y is defined by

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

We can also write the mutual information in a more intuitive way using the conditional entropy as

$$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X).$$

Both definitions are equivalent. Mutual information between the current consumption, savings and the random vector of income in period 1 is given by

$$I\left(e^1; c^1, s\right) = H\left(e^1\right) - H\left(e^1|c^1, s\right)$$

or equivalently

$$I\left(e^1; c^1, s\right) = H\left(e^1\right) + H\left(c^1, s\right) - H\left(e^1, c^1, s\right).$$

According to Property 4, the entropy of independent random variables is equal to the sum of the entropy of each random variable. Thus we get

$$I\left(e^1; c^1, s\right) = H\left(e^1\right) + H\left(c^1, s\right) - H\left(e^1, c^1, s\right).$$

Writing the entropy explicitly yields

$$\sum_{k=1}^{K}\sum_{r=1}^{N}\sum_{i=1}^{I} f\left(s_k, c_r^1, e_i^1\right) \log\left(f\left(s_k, c_r^1, e_i^1\right)\right) - \sum_{i=1}^{I} g_1\left(e_i^1\right) \log\left(g_1\left(e_i^1\right)\right) - \sum_{k=1}^{K}\sum_{r=1}^{N} q\left(c_r^1, s_k\right) \log\left(q\left(c_r^1, s_k\right)\right). \tag{6.1}$$

This expression is constrained to be smaller than the channel capacity, giving rise to (**??**). Since entropy is a concave function, the resulting constraint set is convex.

7. Appendix2: First-order Conditions

The first-order conditions for $f(\cdot) \in (0,1)$ are

$$\frac{U_{kri} - v_i}{1 + \log\left(\frac{f\left(s_k, c_r^1, w_i^1\right)}{q\left(c_r^1, s_k\right)}\right) - \frac{f\left(s_k, c_r^1, w_i^1\right)}{q\left(c_r^1, s_k\right)}} = \frac{U_{mns} - v_s}{1 + \log\left(\frac{f\left(s_m, c_n^1, w_s^1\right)}{q\left(c_n^1, s_m\right)}\right) - \frac{f\left(s_m, c_n^1, w_s^1\right)}{q\left(c_n^1, s_m\right)}} \quad \forall k = 1, ..., K, r = 1, ..., N, i = 1, ..., I,$$

where $U_{kri} = u\left(c_r^1\right) + \beta \sum_{d=1}^{D} \sum_{j=1}^{J} u\left(\left(w_i^1 - c_r^1\right)\left(s_k\left(r_d^e - r^f\right) + r^f\right) + e_j^2\right) \varphi\left(r_d^e\right) g_2\left(e_j^2\right)$ is the expected lifetime utility at the state with consumption c_r^1, risky asset share s_k, and wealth w_i^1. The agent chooses optimal probabilities over different states such that the utility per marginal mutual information are equated across different states.

Proposition 2. *The agent allocates higher relative probability over the states with higher expected utility if $v_i = v_s$.*

Suppose $v_i = v_s$ and $U_{kri} > U_{mns}$. This implies the following

$$\frac{f\left(s_k, c_r^1, w_i^1\right)}{q\left(c_r^1, s_k\right)} - \log\left(\frac{f\left(s_k, c_r^1, w_i^1\right)}{q\left(c_r^1, s_k\right)}\right) < \frac{f\left(s_m, c_n^1, w_s^1\right)}{q\left(c_n^1, s_m\right)} - \log\left(\frac{f\left(s_m, c_n^1, w_s^1\right)}{q\left(c_n^1, s_m\right)}\right)$$

Let $\varphi(x) = x - \log(x)$. Then $\varphi'(x) < 0$ if $0 < x < 1$, so $\frac{f\left(s_k, c_r^1, w_i^1\right)}{q\left(c_r^1, s_k\right)} > \frac{f\left(s_m, c_n^1, w_s^1\right)}{q\left(c_n^1, s_m\right)}$.

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Figure 1: Risky Asset Share, Rational Expectations

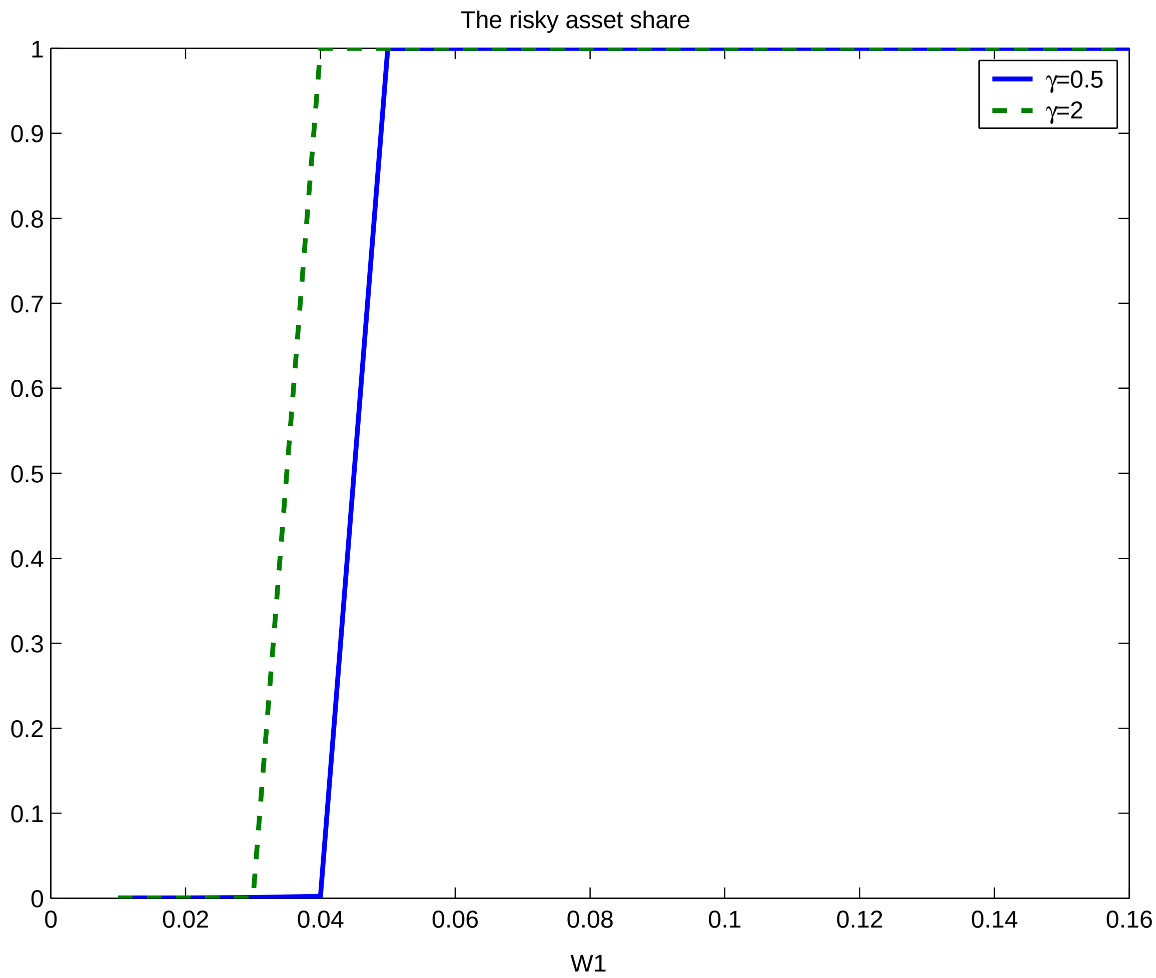

Figure 2: Consumption, Rational Expectations

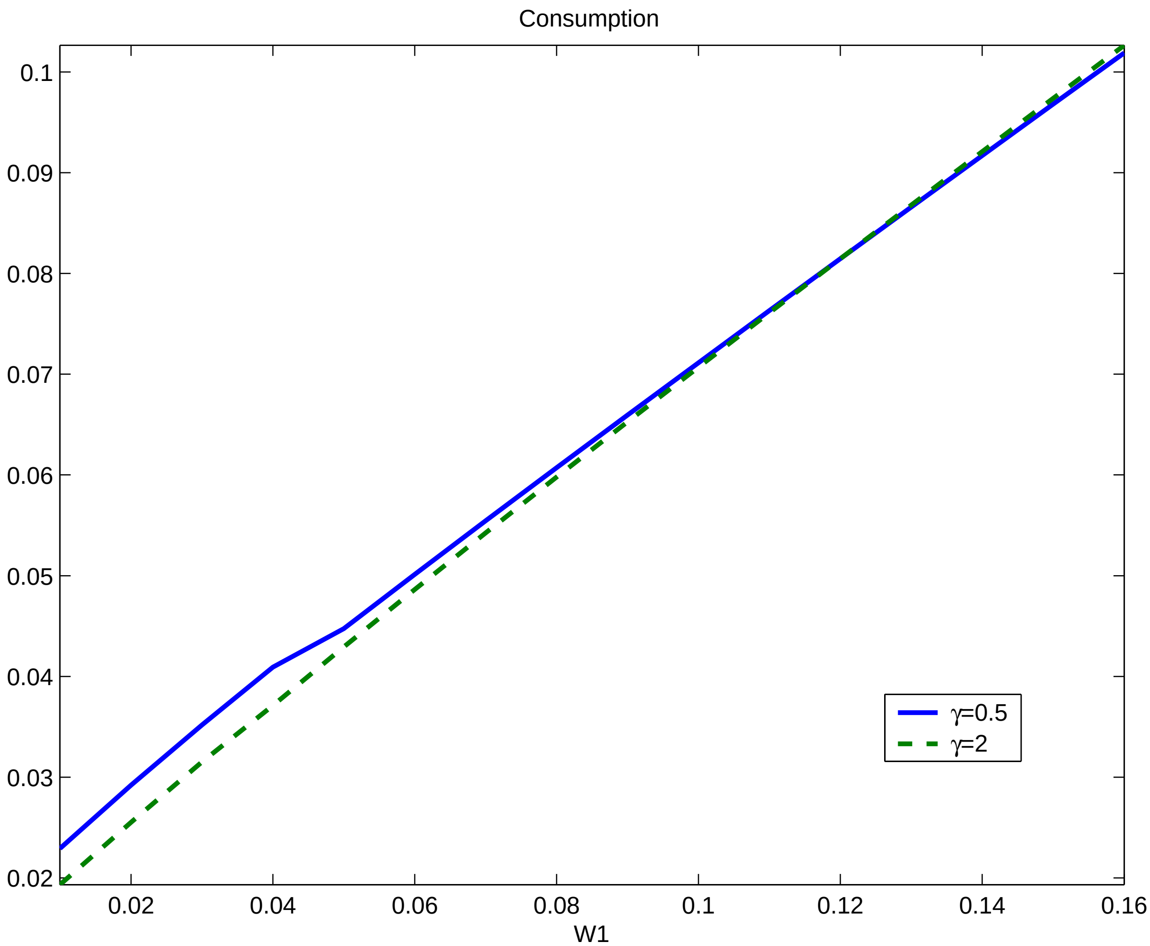

Figure 3: Consumption and Wealth, Rational Inattention

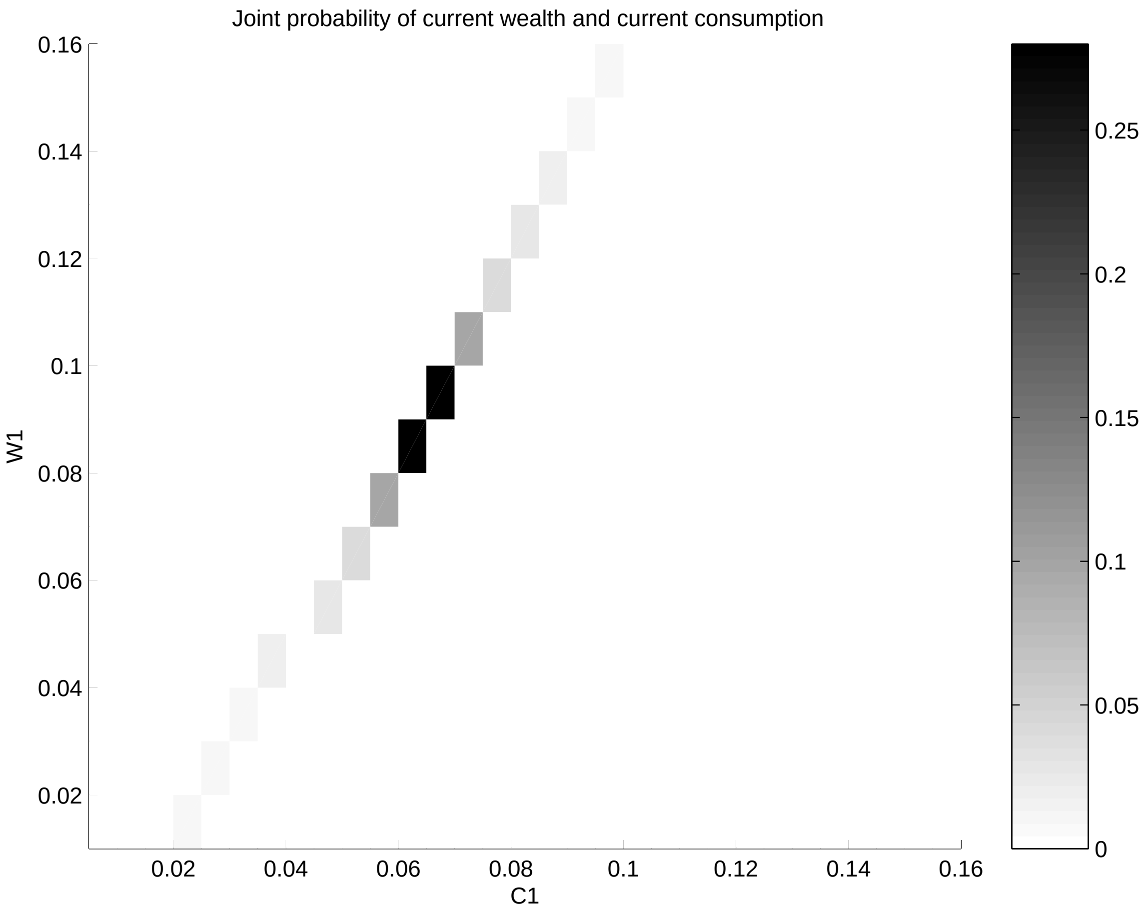

Figure 4: Consumption and Wealth, Rational Inattention

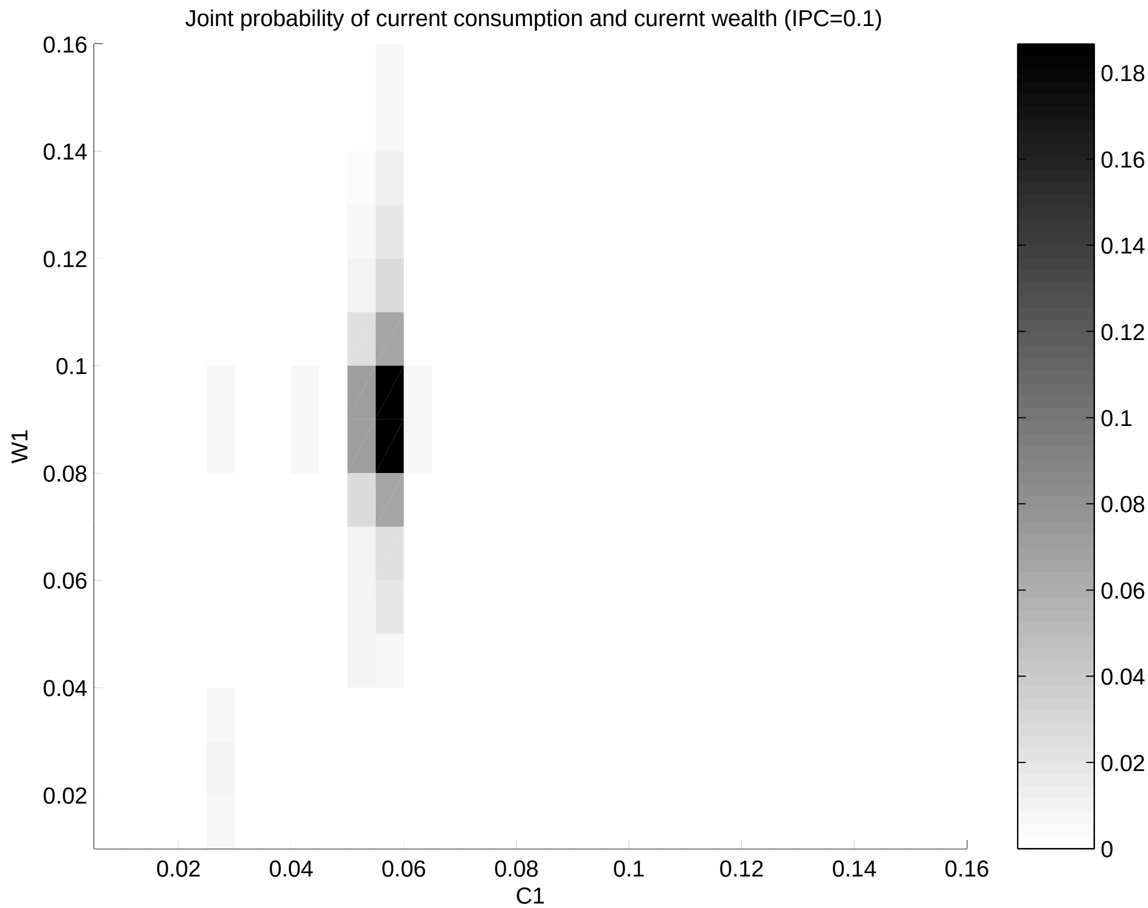

Figure 5: Expected Consumption in Period 2

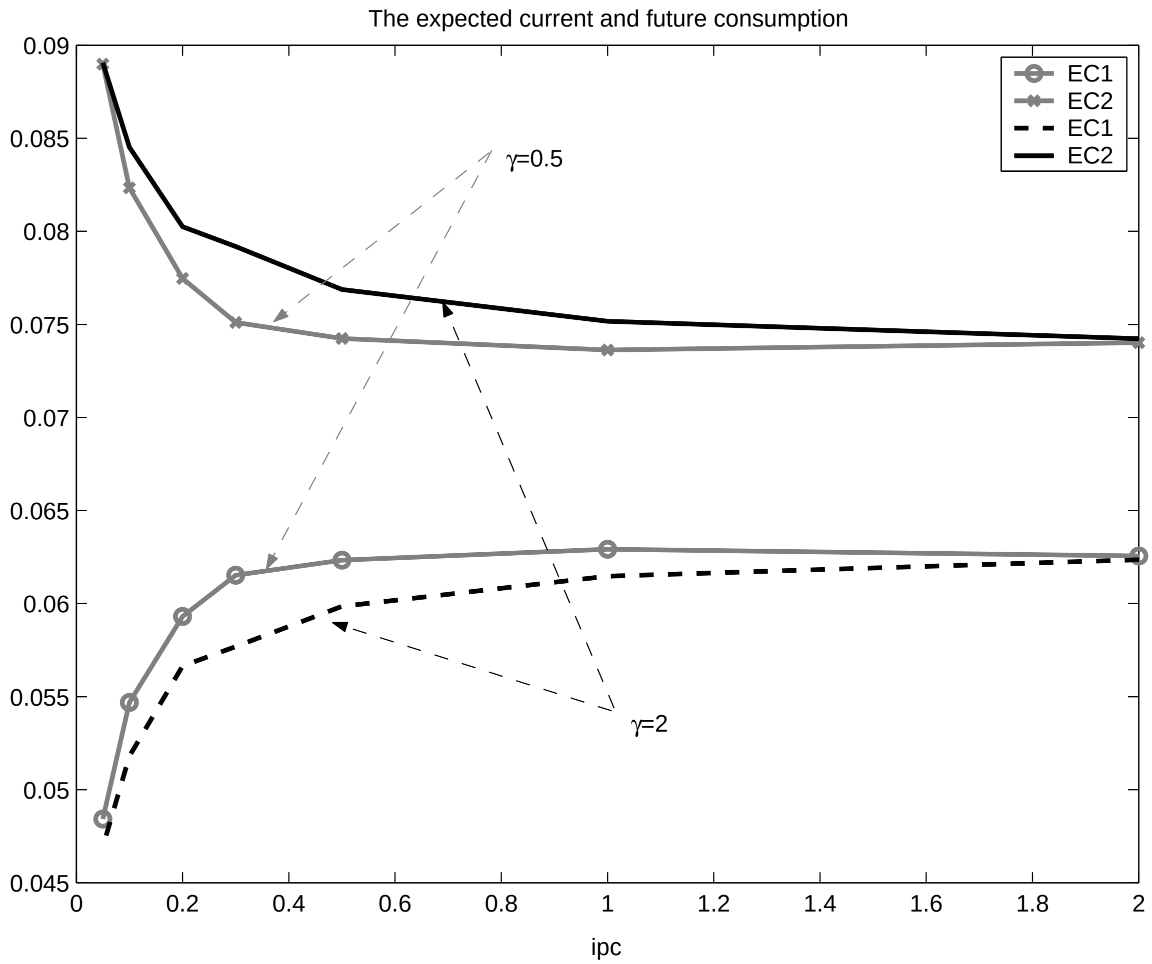

Figure 6: Risky Asset Share, Rational Inattention

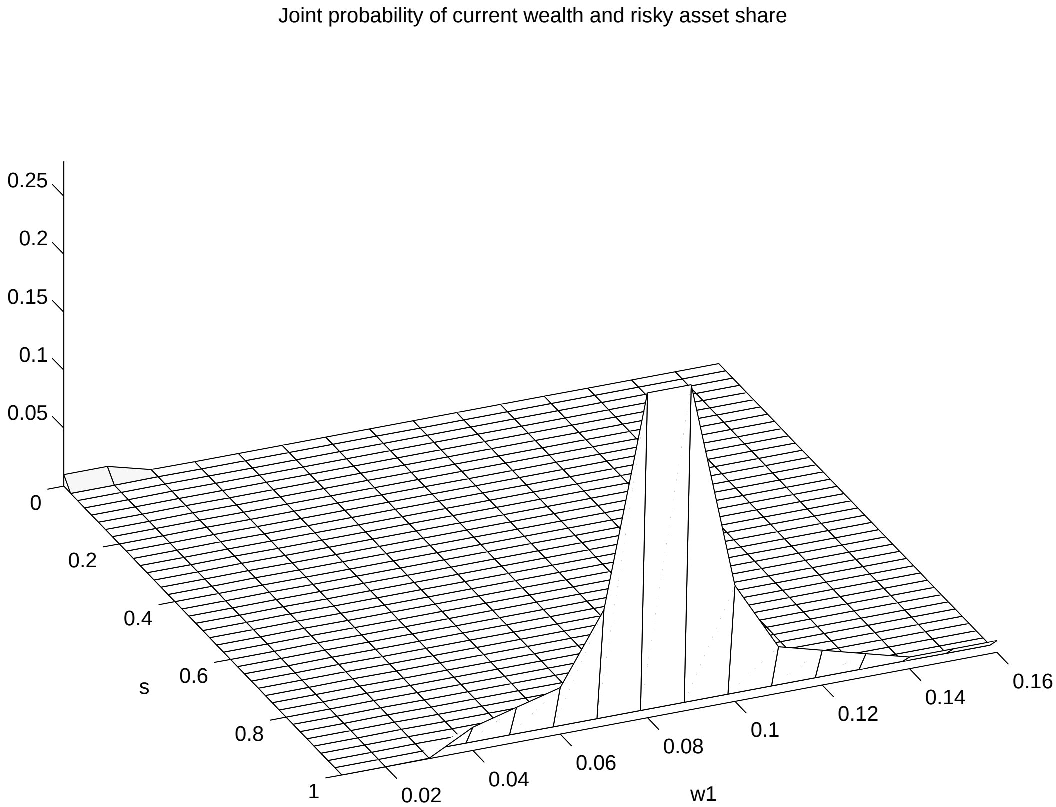

Figure 7: Risky Asset Share, Rational Inattention

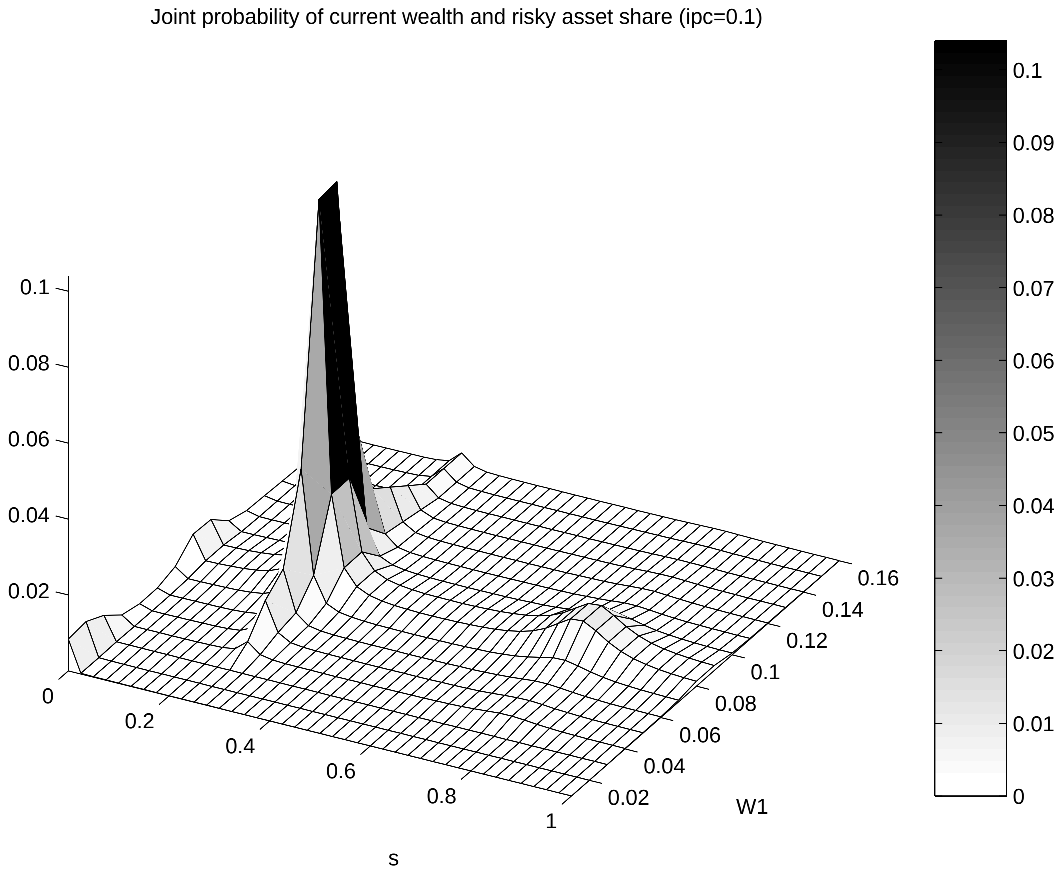

Figure 8: Expected Risky Asset Share, Rational Inattention

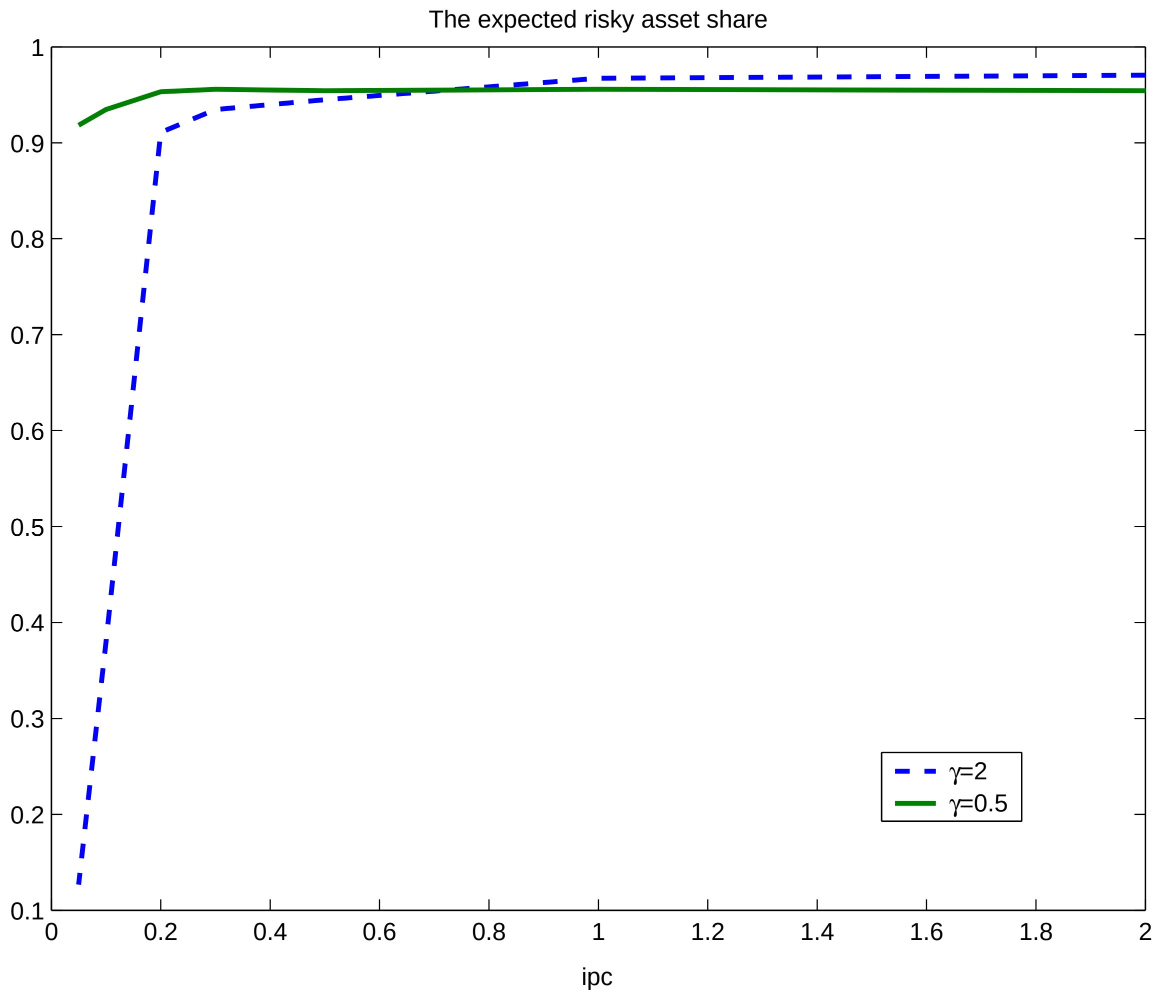

Figure 9: Wealth and Risky Asset Share, Rational Inattention