Sample Questions

Mathmatical Methods

1. Associated with the Lucas tree model is the functional equation

$$w(x) = \beta \int w(x')\, dF(x', x) + g(x),$$

where $g(x)$ is a bounded and continuous function. We assume that the distribution function $F(x', x)$ is such that for every continuous and bounded function $h(x')$ the function

$$f(x) = \int h(x')\, dF(x', x)$$

is continuous in x.

   1. Prove that the functional equation has a unique continuous and bounded solution $w(x)$ by showing that an appropriate operator is a contraction.

      The pricing equation for the Lucas model is given by

$$u'[c(x)]\, p(x) = \beta \int u'[c(x')]\, [p(x') + d(x')]\, F(dx', x)$$

      so that

$$w(x) = u'[c(x)]\, p(x)$$

      and

$$g(x) = \int u'[c(x')]\, d(x')\, F(dx', x).$$

      Let our operator be defined by

$$(Tw)(x) = \beta \int w(x')\, F(dx', x) + g(x).$$

      We must show that T maps continuous and bounded functions into themselves. Let $f(x)$ be bounded and continuous. Then

$$(Tf)(x) = \beta \int f(x')\, F(dx', x) + g(x)$$

must be continuous since it is the sum of two continuous functions. We also have

$$\begin{aligned} |(Tf)(x)| &\leq \beta \int \left|f\left(x'\right)\right| F\left(dx', x\right) + |g(x)| \\ &\leq \beta \|f\|_\infty \int F\left(dx', x\right) + \|g\|_\infty \\ &= \beta \|f\|_\infty + \|g\|_\infty . \end{aligned}$$

We know that $f(x)$ and $g(x)$ are bounded; thus $(Tf)(x)$ is bounded as well:

$$\|Tf\|_\infty \leq \beta \|f\|_\infty + \|g\|_\infty < \infty.$$

We now must verify Blackwell's conditions (setting $\phi = 1$). Let $h(x) \geq f(x)\ \forall x$. This immediately implies

$$\begin{aligned} (Th)(x) &= \beta \int h\left(x'\right) F\left(dx', x\right) + g(x) \\ &\geq \beta \int f\left(x'\right) F\left(dx', x\right) + g(x) \\ &= (Tf)(x) . \end{aligned}$$

Monotonicity is proven. For discounting on constants we have

$$\begin{aligned} (Tf + a)(x) &= \beta \int \left[f\left(x'\right) + a\right] F\left(dx', x\right) + g(x) \\ &= \beta \int f\left(x'\right) F\left(dx', x\right) + \beta a + g(x) \\ &= (Tf)(x) + \beta a. \end{aligned}$$

Thus T is a contraction on a complete metric space (the space of continuous and bounded functions endowed with the sup-norm); by the Contraction Mapping Theorem it possesses a unique fixed point.

2. Given a function $g(x)$ describe an algorithm to compute the unknown function $w(x)$ that solves the functional equation.

   From the Contraction Mapping Theorem we have

$$w(x) = \lim_{t \to \infty} \left(T^n w_0\right)(x)$$

from any initial $w_0(x)$. Choose $w_0(x) = 0$. Then we have

$$w_1(x) = (Tw_0)(x) = g(x).$$

Repeating this yields

$$\begin{aligned} w_2(x) &= (Tw_1)(x) \\ &= \beta \int g(x') F(dx', x) + g(x). \end{aligned}$$

Continuing this process we will converge to the unique fixed point $w(x)$.

2. Consider the following problem. A vintner has one unit of labor to use each day. He can allocate that labor between the making of bread and the pressing of grapes for grape juice. The bread he makes today he can consume today. The grape juice he makes today will become tomorrow's wine. The production technology is linear: it produces one unit of bread per unit of labor allocated to baking, one unit of juice per unit of labor allocated to grape pressing, and one unit of wine per unit of grape juice left to ferment. The transformation of juice into wine requires no labor input, only time. The vintner allocates his labor so as to maximize the utility of his own consumption. His utility function has the form

$$U = \sum_{t=0}^{\infty} \beta^t \sqrt{b_t w_t}$$

where b_t and w_t are the bread and wine consumption, respectively, in period t. The initial wine consumption w_0 is given. The discount factor is $\beta \in (0,1)$.

   1. Write down the sequence problem of the vintner, being careful to specify his resource constraints.

      The sequence problem for the vintner is

      $$\max_{\{w_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sqrt{(1-w_{t+1})\, w_t}$$

      subject to

      $$0 \le w_{t+1} \le 1$$

      for all t, w_0 given.

2. Write down this problem in recursive form. Prove that there exists a unique value function v^* that satisfies the functional equation and that this value function corresponds to the supremum of present discounted utilities in the sequence problem. Be careful to specify the complete metric space within which you look for solutions to the functional equation. Also, be careful to justify each step in your proof with reference to results discussed in class and to recall why these results are applicable.

   In recursive form we have

$$v(w) = \max_{w' \in [0,1]} \left\{ \sqrt{(1-w')\, w} + \beta v(w') \right\}.$$

   We will look for a solution in $C[0,1]$, the space of continuous functions that map the unit interval into $\mathcal{R}$. We first need to show that the operator

$$(Tf)(w) \equiv \max_{w' \in [0,1]} \left\{ \sqrt{(1-w')\, w} + \beta f(w') \right\}$$

   maps $C[0,1]$ into $C[0,1]$. Let $f \in C[0,1]$. Since

$$\sqrt{(1-w')\, w} \in [0,1]$$

   we must have $(Tf) \in C[0,1]$; it has domain only over $[0,1]$. We next need to show that T is a contraction. Setting $\phi = 1$ because our functions are bounded, we show monotonicity and discounting on constants. Let $f, g \in C[0,1]$ with $f \leq g$ and denote the policy function by $w' = \pi(w)$. Then

$$\begin{aligned} (Tf)(w) &= \sqrt{(1-\pi(w))\, w} + \beta f(\pi(w)) \\ &\leq \sqrt{(1-\pi(w))\, w} + \beta g(\pi(w)) \\ &\leq (Tg)(w); \end{aligned}$$

   the last is an inequality since $\pi(w)$ may not be the policy function given $g(w)$. We also have

$$\begin{aligned}(Tf+a)(w) &= \max_{w'\in[0,1]}\left\{\sqrt{(1-w')\,w}+\beta\left(f\left(w'\right)+a\right)\right\}\\ &= \max_{w'\in[0,1]}\left\{\sqrt{(1-w')\,w}+\beta f\left(w'\right)\right\}+\beta a\\ &= (Tf)(w)+\beta a.\end{aligned}$$

Thus we have shown that the operator is a contraction on a complete metric space. The Contraction Mapping Theorem ensures that there is a unique $v^*(w)\in C[0,1]$ such that $(Tv^*)(w)=v^*(w)$; since v^* is bounded we have

$$\lim_{t\to\infty}\beta^t v^*(w_t)=0$$

for all feasible sequences (this is transversality). Thus, we know from class that the solution to the recursive problem is the same as the solution to the sequence problem.

3. Guess that the solution to the functional equation takes the form

$$v(w)=\alpha\sqrt{\gamma+w}$$

for some unknown parameters α and γ. Determine these parameters and compute the optimal policy function.

Using our guess in the Bellman equation yields

$$\alpha\sqrt{\gamma+w}=\max_{w'\in[0,1]}\left\{\sqrt{(1-w')\,w}+\beta\alpha\sqrt{\gamma+w'}\right\}.$$

The above problem is concave so the first-order condition will be necessary and sufficient for an optimum:

$$-\frac{1}{2}\left(1-w'\right)^{-\frac{1}{2}}w^{\frac{1}{2}}+\frac{\beta\alpha}{2}\left(\gamma+w'\right)^{-\frac{1}{2}}=0.$$

The solution to this equation is found by

$$\begin{aligned}\left(1-w'\right)^{-\frac{1}{2}}w^{\frac{1}{2}} &= \beta\alpha\left(\gamma+w'\right)^{\frac{-1}{2}}\\ \left(1-w'\right)w^{-1} &= (\beta\alpha)^{2}\left(\gamma+w'\right)\\ w'\left((\beta\alpha)^{2}+w^{-1}\right) &= w^{-1}-(\beta\alpha)^{-2}\gamma\\ w' &= \frac{w^{-1}-(\beta\alpha)^{-2}\gamma}{(\beta\alpha)^{-2}+w^{-1}}\\ &= \frac{1-w(\beta\alpha)^{-2}\gamma}{w(\beta\alpha)^{-2}+1}.\end{aligned}$$

Inserting this back into the Bellman equation yields

$$\begin{aligned}\alpha\sqrt{\gamma+w} &= \sqrt{\left(1-\frac{1-w(\beta\alpha)^{-2}\gamma}{w(\beta\alpha)^{-2}+1}\right)w}+\beta\alpha\sqrt{\frac{1-w(\beta\alpha)^{-2}\gamma}{w(\beta\alpha)^{-2}+1}}\\ &= \sqrt{\left(\frac{w(\beta\alpha)^{-2}+1-1+w(\beta\alpha)^{-2}\gamma}{w(\beta\alpha)^{-2}+1}\right)w}+\beta\alpha\sqrt{\frac{\gamma w(\beta\alpha)^{-2}+\gamma+1-w(\beta\alpha)^{-2}\gamma}{w(\beta\alpha)^{-2}+1}}\\ &= \sqrt{\left(\frac{w(\beta\alpha)^{-2}(1+\gamma)}{w(\beta\alpha)^{-2}+1}\right)w}+\beta\alpha\sqrt{\frac{1+\gamma}{w(\beta\alpha)^{-2}+1}}\\ &= \sqrt{\frac{1+\gamma}{w(\beta\alpha)^{-2}+1}}\left(\beta\alpha+\sqrt{\left(\frac{w}{\beta\alpha}\right)^{2}}\right)\\ &= \sqrt{\frac{(1+\gamma)(\beta\alpha)^{2}}{w(\beta\alpha)^{-2}+1}}\left(\left(\beta\alpha^{2}\right)+w\right)\frac{1}{\beta\alpha}\\ &= \sqrt{(1+\gamma)\left(w+(\beta\alpha)^{2}\right)}.\end{aligned}$$

Matching coefficients yields

$$\begin{aligned}\alpha &= \sqrt{1+\gamma}\\ \gamma &= (\beta\alpha)^{2}\end{aligned}$$

which implies

$$\begin{aligned}
\alpha &= \sqrt{\frac{1}{1-\beta^2}} \\
\gamma &= \frac{\beta^2}{1-\beta^2} \\
v(w) &= \sqrt{\frac{1}{1-\beta^2}}\sqrt{\frac{\beta^2}{1-\beta^2}+w}.
\end{aligned}$$

Inserting these into the policy function we obtain

$$\begin{aligned}
w' &= \frac{1-w\left(\beta\sqrt{\frac{1}{1-\beta^2}}\right)^{-2}\frac{\beta^2}{1-\beta^2}}{w\left(\beta\sqrt{\frac{1}{1-\beta^2}}\right)^{-2}+1} \\
&= \frac{1-w}{1+w\frac{1-\beta^2}{\beta^2}} \\
&= \frac{(1-w)\beta^2}{w+(1-w)\beta^2} \in (0,1).
\end{aligned}$$

3. The goal of this exercise is to study, in the context of a specific problem, two methods for solving dynamic programs: value function iteration and Howard's policy improvement. Consider McCall's model of intertemporal job search. An unemployed worker draws one offer from a c.d.f. F, with $F(0)=0$ and $F(B)=1$, $B<\infty$. If the worker rejects the offer, she receives unemployment compensation c and can draw a new wage offer next period. If she accepts the offer, she works forever at wage w. The objective of the worker is to maximize the expected discounted value of her earnings. Her discount factor is $0<\beta<1$.

    1. Write the Bellman equation. Show that the optimal policy is of the reservation wage form. Write an equation for the reservation wage w^*.

       Let $v(w)$ be the value of an unemployed worker with offer w in hand and who behaves optimally. The Bellman equation is

       $$v(w) = \max_{\text{accept,reject}}\left\{\frac{w}{1-\beta}, c+\beta\int v(w')\,dF(w')\right\}.$$

       The right hand side takes the max of an increasing function and of a constant. Thus, the optimal policy is of the reservation wage form. There is a reservation wage w^* such that, for $w\le w^*$, the increasing function is less than the constant and the worker rejects the offer. For $w \geq w^*$, the increasing function is greater than the constant and the worker accepts the offer. The reservation wage w^* solves

$$\begin{aligned}
\frac{w^*}{1-\beta} &= c+\beta\int v\left(w'\right)dF\left(w'\right) \\
&= c+\beta\int_0^{w^*}\frac{w^*}{1-\beta}dF\left(w'\right)+\beta\int_{w^*}^{B}\frac{w'}{1-\beta}dF\left(w'\right) \\
&= c+\frac{\beta}{1-\beta}w^*F\left(w^*\right)+\frac{\beta}{1-\beta}w^*\left(1-F\left(w^*\right)\right)+\frac{\beta}{1-\beta}\int_{w^*}^{B}\left(1-F\left(w'\right)\right)dw' \\
&= c+\frac{\beta}{1-\beta}w^*+\frac{\beta}{1-\beta}\int_{w^*}^{B}\left(1-F\left(w'\right)\right)dw'.
\end{aligned}$$

Thus, the reservation wage is the unique solution to

$$w^* = c\left(1-\beta\right)+\beta w^*+\beta\int_{w^*}^{B}\left(1-F\left(w'\right)\right)dw'.$$

2. Consider the value function iteration method. Show that at each iteration, the optimal policy is of the reservation wage form. Let w_n be the reservation wage at iteration n. Derive a recursion for w_n. Show that w_n converges to w^* at rate β.

   The value function iteration algorithm iterates on the Bellman equation

$$v^{n+1}\left(w\right) = \max_{\text{accept,reject}}\left\{\frac{w}{1-\beta}, c+\beta\int v^n\left(w'\right)dF\left(w'\right)\right\}.$$

   Using the previous result it is straightforward that the optimal policy is of the reservation wage form. The reservation wage at iteration $n+1$ solves

$$\frac{w_{n+1}}{1-\beta} = c+\beta\int v^n\left(w'\right)dF\left(w'\right).$$

   Making the same manipulation as above one gets that

$$w_{n+1} = c\left(1-\beta\right)+\beta w_n+\beta\int_{w_n}^{B}\left(1-F\left(w'\right)\right)dw'.$$

   To show convergence, subtract the equation for w^* from the one for w_{n+1}:

$$w_{n+1}-w^* = \beta\left(w_n-w^*\right)+\beta\int_{w_n}^{w^*}\left(1-F\left(w'\right)\right)dw'.$$

Using the fact that

$$w_n - w^* = -\int_{w_n}^{w^*} dw'$$

one obtains

$$w_{n+1} - w^* = -\beta \int_{w_n}^{w^*} F(w')\, dw'.$$

Since $0 \leq F(w') \leq 1$, this equality implies

$$|w_{n+1} - w^*| \leq \beta |w_n - w^*| .$$

Thus, by the Contraction Mapping Theorem this recursion converges to w^*.

3. Consider Howard's policy improvement algorithm. Show that at each iteration, the optimal policy is of the reservation wage form. Let w_n be the reservation wage at iteration n. Derive a recursion for w_n. Show that the rate of convergence of w_n towards w^* is (locally) quadratic. Specfically use a Taylor expansion to show that, for w_n close to enough to w^*, there is a constant K such that $w_{n+1} - w^* \cong K(w_n - w^*)^2$.

   Assume that the optimal policy at iteration n is of the reservation wage form. Let w_n be this reservation wage. Let v^n be the value of a worker who uses forever the reservation wage policy w_n. For $w \geq w_n$, the worker accepts the offer and $v^n(w) = \frac{w}{1-\beta}$. For $w \leq w_n$, the worker rejects the offer and $v^n(w)$ =constant= Q_n; this constant solves

$$\begin{aligned} Q_n &= c + \beta \int_0^{w_n} Q_n dF(w') + \beta \int_{w_n}^{B} \frac{w'}{1-\beta} dF(w') \\ Q_n &= (1 - \beta F(w_n))^{-1} \left(c + \frac{\beta}{1-\beta} \int_{w_n}^{B} w' dF(w') \right). \end{aligned}$$

   Observe that the value function at iteration n is not continuous. There is a jump at $w = w_n$. The jump expresses that the reservation wage policy w_n is suboptimal. Namely, at $w = w_n$, the worker is not indifferent between accepting or rejecting the offer. Let's do iteration $n+1$. We need to solve

$$\begin{aligned} \widetilde{v}(w) &= \max_{\text{accept,reject}} \left\{ \frac{w}{1-\beta}, c + \beta \int v^n(w')\, dF(w') \right\} \\ &= \max_{\text{accept,reject}} \left\{ \frac{w}{1-\beta}, Q_n \right\}. \end{aligned}$$

   It is apparent that the optimal policy is of the reservation wage form. The reservation wage at iteration $n+1$ solves

$$w_{n+1} = (1-\beta F(w_n))^{-1}\left(c(1-\beta)+\beta\int_{w_N}^{B} w' dF(w')\right) \equiv G(w_n).$$

The optimal reservation wage is the fixed point of G. For w_n close enough to w^*, we have

$$w_{n+1}-w^* \cong G'(w^*)(w_n-w^*)+\frac{1}{2}G''(w^*)(w_n-w^*)^2.$$

Since $w^*=G(w^*)$ we have $G'(w^*)=0$. Thus, for w_n close enough to w^* we have

$$w_{n+1}-w^* \cong \frac{1}{2}G''(w^*)(w_n-w^*)^2.$$

Thus, convergence is locally quadratic.

4. Consider an economy with a continuum of identical households with preferences given by

$$\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\sigma}}{1-\sigma}$$

and a budget constraint of the form

$$c_t+k_{t+1}\le(1+r_t-\delta)k_t+w_t.$$

Factor prices r_t and w_t are competitively determined as a function of the aggregate capital stock K_t. In equilibrium, $k_t=K_t$.

1. Display the Bellman equation for this household.

   The states for the household are the individual capital stock k and the aggregate stock K, so the Bellman equation is

$$v(k,K)=\max_{k'\le[1+r(K)-\delta]k+w(K)}\left\{\frac{1}{1-\sigma}\left([1+r(K)-\delta]k+w(K)-k'\right)^{1-\sigma}+\beta v(k',K')\right\}.$$

2. Guess that the value function takes the form

$$v(k)=\frac{1}{1-\beta}a(K)(k+b(K))^{1-\sigma}.$$

   Find two functional equations that determine the unknown functions $a(K)$ and $b(K)$.

The first-order condition is

$$\begin{aligned}
\left(\left[1+r\left(K\right)-\delta\right]k+w\left(K\right)-k'\right)^{-\sigma} &= \frac{\beta}{1-\beta}a\left(K'\right)\left(k'+b\left(K'\right)\right)^{-\sigma} \\
\left[1+r\left(K\right)-\delta\right]k+w\left(K\right)-k' &= \left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1}{\sigma}}\left(k'+b\left(K'\right)\right) \\
\left[1+\left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1}{\sigma}}\right]k' &= \left[1+r\left(K\right)-\delta\right]k+w\left(K\right)-\left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1}{\sigma}}b\left(K'\right) \\
k' &= \frac{\left[1+r\left(K\right)-\delta\right]k+w\left(K\right)}{\left[1+\left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1}{\sigma}}\right]}-\frac{\left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1}{\sigma}}b\left(K'\right)}{\left[1+\left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1}{\sigma}}\right]}.
\end{aligned}$$

Inserting k' into Bellman equation yields

$$\begin{aligned}
\frac{1}{1-\beta}a\left(K\right)\left(k+b\left(K\right)\right)^{1-\sigma} &= \frac{1}{1-\sigma}\frac{\left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1-\sigma}{\sigma}}}{\left[1+\left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1}{\sigma}}\right]^{1-\sigma}}\times \\
&\quad \frac{1}{\left[1+r\left(K\right)-\delta\right]^{1-\sigma}}\left(k+\frac{w\left(K\right)+b\left(K'\right)}{\left[1+r\left(K\right)-\delta\right]}\right)^{1-\sigma}+ \\
&\quad \beta\frac{1}{1-\beta}a\left(K'\right)\frac{1}{\left[1+\left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1}{\sigma}}\right]^{1-\sigma}}\times \\
&\quad \frac{1}{\left[1+r\left(K\right)-\delta\right]^{1-\sigma}}\left(k+\frac{w\left(K\right)+b\left(K'\right)}{\left[1+r\left(K\right)-\delta\right]}\right)^{1-\sigma}.
\end{aligned}$$

Matching coefficients then yields

$$\begin{aligned}
\frac{1}{1-\beta}a\left(K\right) &= \frac{1}{1-\sigma}\frac{\left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1-\sigma}{\sigma}}}{\left[1+\left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1}{\sigma}}\right]^{1-\sigma}}\frac{1}{\left[1+r\left(K\right)-\delta\right]^{1-\sigma}}+ \\
&\quad \beta\frac{1}{1-\beta}a\left(K'\right)\frac{1}{\left[1+\left[\frac{\beta}{1-\beta}a\left(K'\right)\right]^{-\frac{1}{\sigma}}\right]^{1-\sigma}}\frac{1}{\left[1+r\left(K\right)-\delta\right]^{1-\sigma}} \\
b\left(K\right) &= \frac{w\left(K\right)+b\left(K'\right)}{\left[1+r\left(K\right)-\delta\right]}.
\end{aligned}$$

3. Find an equation that implicitly determines K' as a function of K.

Imposing equilibrium on the function that determines k' yields

$$K' = \frac{\left[1 + r\left(K\right) - \delta\right] K + w\left(K\right)}{\left[1 + \left[\frac{\beta}{1-\beta} a\left(K'\right)\right]^{-\frac{1}{\sigma}}\right]} - \frac{\left[\frac{\beta}{1-\beta} a\left(K'\right)\right]^{-\frac{1}{\sigma}} b\left(K'\right)}{\left[1 + \left[\frac{\beta}{1-\beta} a\left(K'\right)\right]^{-\frac{1}{\sigma}}\right]},$$

which in principle could be solved for K'.

4. Show that, if $\sigma = 1$, the value function takes the form

$$v\left(k, K\right) = a\left(K\right) + \frac{1}{1-\beta} \log\left(k + b\left(K\right)\right).$$

With $\sigma = 1$ the first-order condition becomes

$$\begin{aligned}
\left(\left[1 + r\left(K\right) - \delta\right] k + w\left(K\right) - k'\right)^{-1} &= \frac{\beta}{1-\beta}\left(k' + b\left(K'\right)\right)^{-1} \\
\left[1 + r\left(K\right) - \delta\right] k + w\left(K\right) - k' &= \frac{1-\beta}{\beta}\left(k' + b\left(K'\right)\right) \\
k' &= \beta\left[\left[1 + r\left(K\right) - \delta\right] k + w\left(K\right) - \frac{1-\beta}{\beta} b\left(K'\right)\right] \\
&= \beta\left[\left[1 + r\left(K\right) - \delta\right] k + w\left(K\right)\right] - \left(1-\beta\right) b\left(K'\right)
\end{aligned}$$

so that

$$\begin{aligned}
c &= \left[1 + r\left(K\right) - \delta\right] k + w\left(K\right) - \beta\left[\left[1 + r\left(K\right) - \delta\right] k + w\left(K\right)\right] + \left(1-\beta\right) b\left(K'\right) \\
&= \left(1-\beta\right)\left[\left[1 + r\left(K\right) - \delta\right] k + w\left(K\right)\right] + \left(1-\beta\right) b\left(K'\right).
\end{aligned}$$

Inserting this into the Bellman equation yields

$$\begin{aligned}
a\left(K\right) + \frac{1}{1-\beta}\log\left(k + b\left(K\right)\right) &= \log\left(\left(1-\beta\right)\left[\left[1 + r\left(K\right) - \delta\right] k + w\left(K\right)\right] + \left(1-\beta\right) b\left(K'\right)\right) + \\
&\quad \beta a\left(K'\right) + \\
&\quad \frac{\beta}{1-\beta}\log\left(\beta\left[\left[1 + r\left(K\right) - \delta\right] k + w\left(K\right)\right] + \beta b\left(K'\right)\right) \\
a\left(K\right) + \frac{1}{1-\beta}\log\left(k + b\left(K\right)\right) &= \log\left(\left(1-\beta\right)\left[1 + r\left(K\right) - \delta\right]\right) + \beta a\left(K'\right) + \\
&\quad \frac{\beta}{1-\beta}\log\left(\beta\left[1 + r\left(K\right) - \delta\right]\right) + \\
&\quad \frac{1}{1-\beta}\log\left(k + \frac{w\left(K\right) + b\left(K'\right)}{\left[1 + r\left(K\right) - \delta\right]}\right)
\end{aligned}$$

Matching coefficients yields

$$\begin{aligned} a(K) &= \log(1-\beta) + \frac{\beta}{1-\beta}\log(\beta) + \\ &\quad \frac{1}{1-\beta}\log([1+r(K)-\delta]) + \beta a(K') \\ b(K) &= \frac{w(K)+b(K')}{[1+r(K)-\delta]}. \end{aligned}$$

We can find K' by solving the equation

$$K' = \beta\left[[1+r(K)-\delta]K + w(K)\right] - (1-\beta)\,b(K').$$

5. Consider an economy in which both human and physical capital can be accumulated. Let preferences be given by

$$\sum_{t=0}^{\infty}\beta^t \log(c_t).$$

The resource constraint is

$$c_t + k_{t+1} = Ak_t^{\alpha}(h_t n_t)^{1-\alpha}\overline{h}_t^{\eta},$$

where n_t is the current fraction of time spent on market work, and $\overline{h}_t$ is the aggregate stock of human capital which exerts an externality on market production. Human capital evolves according to

$$h_{t+1} = B(1-n_t)h_t.$$

Both A and B are positive scalars.

1. Write out the Bellman equation for this economy.

   The states are $(h_t, k_t, \overline{h}_t)$ and the controls are (c_t, n_t) and the multiplier on the human capital constraint μ_t. The Bellman equation is then

$$v\left(k,h,\overline{h}\right) = \max_{c,n}\min_{\mu}\left\{\log\left(Ak(hn)^{1-\alpha}\overline{h}^{\eta} - k'\right) + \beta v\left(k',h',\overline{h}'\right) + \mu\left(h' - h - B(1+n)h\right)\right\}$$

   subject to the equilibrium law of motion for $\overline{h}$:

$$\overline{h}' = B(1-\overline{n})\overline{h}.$$

2. Find the equations which describe a competitive equilibrium for this economy.

   The first-order conditions can be reduced to

$$\begin{aligned}
h_{t+1} &= B\left(1-n_t\right)h_t \\
c_{t+1} &= c_t\alpha\beta Ak_{t+1}^{\alpha-1}\left(h_{t+1}n_{t+1}\right)^{1-\alpha}\overline{h}_{t+1}^{\eta} \\
\left(1-\alpha\right)c_t^{-1}Ak_t^{\alpha}h_t^{1-\alpha}n_t^{-\alpha}\overline{h}_t^{\eta} &= \mu_tBh_t \\
\mu_t &= \beta\mu_{t+1}\left(B-1+n_t\right)+\left(1-\alpha\right)\beta c_{t+1}^{-1}Ak_{t+1}^{\alpha}n_{t+1}^{1-\alpha}h_{t+1}^{-\alpha}\overline{h}_{t+1}^{\eta} \\
c_t+k_{t+1} &= Ak_t^{\alpha}\left(n_th_t\right)^{1-\alpha}\overline{h}_t^{\eta}
\end{aligned}$$

   to determine the time paths of $\{c_t, k_t, h_t, n_t, \mu_t\}$. In equilibrium we must have $h_t = \overline{h}_t$ so that these equations can be written as

$$\begin{aligned}
h_{t+1} &= B\left(1-n_t\right)h_t \\
c_{t+1} &= c_t\alpha\beta Ak_{t+1}^{\alpha-1}n_{t+1}^{1-\alpha}h_{t+1}^{1-\alpha+\eta} \\
\left(1-\alpha\right)c_t^{-1}Ak_t^{\alpha}h_t^{1-\alpha+\eta}n_t^{-\alpha} &= \mu_tBh_t \\
\mu_t &= \beta\mu_{t+1}\left(B-1+n_t\right)+\left(1-\alpha\right)\beta c_{t+1}^{-1}Ak_{t+1}^{\alpha}n_{t+1}^{1-\alpha}h_{t+1}^{-\alpha+\eta} \\
c_t+k_{t+1} &= Ak_t^{\alpha}n_t^{1-\alpha}h_t^{1-\alpha+\eta}.
\end{aligned}$$

3. Guess that

$$\begin{aligned}
c_t &= \phi_{10}k_t^{\phi_{11}}h_t^{\phi_{12}} \\
k_{t+1} &= \phi_{20}k_t^{\phi_{21}}h_t^{\phi_{22}} \\
h_{t+1} &= \phi_{30}k_t^{\phi_{31}}h_t^{\phi_{32}} \\
n_t &= \phi_{40}k_t^{\phi_{41}}h_t^{\phi_{42}} \\
\mu_t &= \phi_{50}k_t^{\phi_{51}}h_t^{\phi_{52}}.
\end{aligned}$$

   Solve for the unknown coefficients ϕ_{ij}.

   Substitution into the first equilibrium equation yields

$$\phi_{30}k_t^{\phi_{31}}h_t^{\phi_{32}} = Bh_t - \phi_{40}k_t^{\phi_{41}}h_t^{\phi_{42}}h_t.$$

Matching coefficients then requires that

$$\begin{aligned}\phi_{31} &= \phi_{41} = \phi_{42} = 0 \\ \phi_{32} &= 1.\end{aligned}$$

This implies

$$n_t = \phi_{40},$$

a constant. Substitution into the last equation yields

$$\phi_{10} k_t^{\phi_{11}} h_t^{\phi_{12}} + \phi_{20} k_t^{\phi_{21}} h_t^{\phi_{22}} = A k_t^{\alpha} \phi_{40}^{1-\alpha} h_t^{1-\alpha+\eta}.$$

Clearly,

$$\begin{aligned}\phi_{11} &= \phi_{21} = \alpha \\ \phi_{12} &= \phi_{22} = 1 - \alpha + \eta.\end{aligned}$$

That is,

$$\begin{aligned}c_t &= \phi_{10} k_t^{\alpha} h_t^{1-\alpha+\eta} \\ k_{t+1} &= \phi_{20} k_t^{\alpha} h_t^{1-\alpha+\eta}.\end{aligned}$$

Using the third equation implies

$$\frac{(1-\alpha)}{\phi_{10} k_t^{\alpha} h_t^{1-\alpha+\eta}} A k_t^{\alpha} h_t^{1-\alpha+\eta} \phi_{40}^{-\alpha} = \mu_t B h_t$$

which simplifies to

$$(1-\alpha) A \phi_{40}^{-\alpha} = \phi_{50} k_t^{\phi_{51}} h_t^{\phi_{52}} B h_t$$

which requires

$$\begin{aligned}\phi_{51} &= 0 \\ \phi_{52} &= -1.\end{aligned}$$

We have now obtained the following solutions:

$$\begin{aligned}
c_t &= \phi_{10}k_t^{\alpha}h_t^{1-\alpha+\eta} \\
k_{t+1} &= \phi_{20}k_t^{\alpha}h_t^{1-\alpha+\eta} \\
h_{t+1} &= \phi_{30}h_t \\
n_t &= \phi_{40} \\
\mu_t &= \phi_{50}h_t^{-1}.
\end{aligned}$$

Inserting these into the five equations yields the system

$$\phi_{30}h_t = B\left(1-\phi_{40}\right)h_t$$

$$\phi_{10}\left(\phi_{20}k_t^{\alpha}h_t^{1-\alpha+\eta}\right)^{\alpha}\left(\phi_{30}h_t\right)^{1-\alpha+\eta} = \phi_{10}k_t^{\alpha}h_t^{1-\alpha+\eta}\alpha\beta A\left(\phi_{20}k_t^{\alpha}h_t^{1-\alpha+\eta}\right)^{\alpha-1}\left(\phi_{40}\right)^{1-\alpha}\left(\phi_{30}h_t\right)^{1-\alpha+\eta}$$

$$\left(1-\alpha\right)Ak_t^{\alpha}h_t^{1-\alpha+\eta}\phi_{40}^{-\alpha} = \phi_{50}h_t^{-1}Bh_t\phi_{10}k_t^{\alpha}h_t^{1-\alpha+\eta}$$

$$\phi_{10}\left(\phi_{20}k_t^{\alpha}h_t^{1-\alpha+\eta}\right)^{\alpha}\left(\phi_{30}h_t\right)^{1-\alpha+\eta}\phi_{50}h_t^{-1} = \beta\phi_{50}\left(\phi_{30}h_t\right)^{-1}\left(B-1+\phi_{40}\right)+\left(1-\alpha\right)\beta^{-1}A\left(\phi_{20}k_t^{\alpha}h_t^{1-\alpha+}\right.$$

$$\phi_{10}k_t^{\alpha}h_t^{1-\alpha+\eta}+\phi_{20}k_t^{\alpha}h_t^{1-\alpha+\eta} = Ak_t^{\alpha}\phi_{40}^{1-\alpha}h_t^{1-\alpha+\eta}.$$

Making some simplifications yields

$$\begin{aligned}
\phi_{30} &= B\left(1-\phi_{40}\right) \\
\phi_{20} &= \alpha\beta A\phi_{40}^{1-\alpha} \\
\left(1-\alpha\right)A\phi_{40}^{-\alpha} &= \phi_{50}B\phi_{10} \\
\phi_{10}\phi_{20}^{\alpha}\phi_{30}^{1-\alpha+\eta}\phi_{50} &= \beta\phi_{50}\phi_{30}^{-1}\left(B-1+\phi_{40}\right)+ \\
&\quad \left(1-\alpha\right)\beta^{-1}A\phi_{20}^{\alpha}\phi_{40}^{1-\alpha}\phi_{30}^{-\alpha+\eta} \\
\phi_{10}+\phi_{20} &= A\phi_{40}^{1-\alpha}.
\end{aligned}$$

The solutions are

$$\begin{aligned}
\phi_{10} &= (1-\alpha\beta)(1-\beta)^{1-\alpha} A \\
\phi_{20} &= \alpha\beta(1-\beta)^{1-\alpha} A \\
\phi_{30} &= B\beta \\
\phi_{40} &= 1-\beta \\
\phi_{50} &= \frac{1-\alpha}{B(1-\alpha\beta)(1-\beta)}.
\end{aligned}$$

The decision rules are then

$$\begin{aligned}
c_t &= (1-\alpha\beta)(1-\beta)^{1-\alpha} A k_t^{\alpha} h_t^{1-\alpha+\eta} \\
k_{t+1} &= \alpha\beta(1-\beta)^{1-\alpha} A k_t^{\alpha} h_t^{1-\alpha+\eta} \\
h_{t+1} &= B\beta h_t \\
n_t &= 1-\beta \\
\mu_t &= \frac{1-\alpha}{B(1-\alpha\beta)(1-\beta)} h_t^{-1}.
\end{aligned}$$

4. Obtain the growth rate of output for this economy.

   The growth rate of output is

$$\begin{aligned}
\frac{y_{t+1}}{y_t} &= \frac{A k_{t+1}^{\alpha} h_{t+1}^{1-\alpha+\eta} n_{t+1}^{1-\alpha}}{A k_t^{\alpha} h_t^{1-\alpha+\eta} n_t^{1-\alpha}} \\
&= \left(\frac{k_{t+1}}{k_t}\right)^{\alpha} \left(\frac{h_{t+1}}{h_t}\right)^{1-\alpha+\eta} \\
&= \left(\alpha\beta(1-\beta)^{1-\alpha} A k_t^{\alpha-1} h_t^{1-\alpha+\eta}\right)^{\alpha} (B\beta)^{1-\alpha+\eta} \\
&= \left(\alpha\beta(1-\beta)^{1-\alpha} A\right)^{\alpha} (B\beta)^{1-\alpha+\eta} \left(k_t^{\alpha-1} h_t^{1-\alpha+\eta}\right)^{\alpha}.
\end{aligned}$$

6. Consider a household savings problem with uncertain return. Wealth evolves as

$$a_{t+1} = R_{t+1}(a_t - c_t)$$

where c_t is consumption and R_{t+1} is the return on savings. R_{t+1} has a density $f(R_{t+1})$ which does not depend on R_t (that is, R is independently and identically distributed over time).

1. Show that a_{t+1} has a density given by

   $$p\left(a_{t+1}; a_t, c_t\right) = f\left(\frac{a_{t+1}}{a_t - c_t}\right)\frac{1}{a_t - c_t},$$

   which is the form needed to apply the existence theorems.

   This is just the standard change of variables formula after noting that

   $$R_{t+1} = \frac{a_{t+1}}{a_t - c_t};$$

   the Jacobian is $\left|\frac{1}{a_t - c_t}\right| = \frac{1}{a_t - c_t}$ since $a_t \geq c_t$.

2. Write down the Bellman equation.

   The Bellman equation is

   $$v\left(a\right) = \max_{a'}\left\{u\left(c\right) + \beta\int v\left(R'\left(a - c\right)\right) f\left(R'\right) dR'\right\}.$$

3. Suppose that

   $$u\left(c\right) = \frac{c^{1-\sigma}}{1-\sigma}.$$

   Prove that the value function takes the form

   $$v\left(a\right) = B\frac{a^{1-\sigma}}{1-\sigma}$$

   for some constant B.

   Using this guess we obtain

   $$\begin{aligned}\frac{B}{1-\sigma}a^{1-\sigma} &= \max_c\left\{\frac{c^{1-\sigma}}{1-\sigma} + \beta\int\frac{B}{1-\sigma}\left(R'\left(a - c\right)\right)^{1-\sigma} f\left(R'\right) dR'\right\}\\ &= \max_c\left\{\frac{c^{1-\sigma}}{1-\sigma} + \frac{\beta B}{1-\sigma}\left(a - c\right)^{1-\sigma}\int\left(R'\right)^{1-\sigma} f\left(R'\right) dR'\right\}\\ &= \max_c\left\{\frac{c^{1-\sigma}}{1-\sigma} + \frac{\beta B}{1-\sigma}\left(a - c\right)^{1-\sigma} E\left[\left(R'\right)^{1-\sigma}\right]\right\}.\end{aligned}$$

   Taking the first-order condition yields

   $$c^{-\sigma} = \beta B\left(a - c\right)^{-\sigma} E\left[\left(R'\right)^{1-\sigma}\right]$$

which can be solved to obtain

$$c = \frac{1}{1 + \left(\beta B E\left[(R')^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}} a.$$

Substituting back into the Bellman equation yields

$$\begin{aligned}\frac{B}{1-\sigma} a^{1-\sigma} &= \frac{1}{1-\sigma}\left(\frac{1}{1 + \left(\beta B E\left[(R')^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}}\right)^{1-\sigma} a^{1-\sigma} + \\ &\quad \frac{\beta B E\left[(R')^{1-\sigma}\right]}{1-\sigma}\left(\frac{\left(\beta B E\left[(R')^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}}{1 + \left(\beta B E\left[(R')^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}}\right)^{1-\sigma} a^{1-\sigma}.\end{aligned}$$

Matching the constants then yields

$$B = \left(\frac{1}{1 + \left(\beta B E\left[(R')^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}}\right)^{1-\sigma} + \left(\beta B E\left[(R')^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}\left(\frac{1}{1 + \left(\beta B E\left[(R')^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}}\right)^{1-\sigma}.$$

A nontrivial amount of algebra gets the solution as

$$B = \left(\frac{1}{1 + \left(\beta E\left[(R')^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}}\right)^{\sigma}.$$

4. Obtain consumption as a function of wealth.

   Using the constant B we get

$$\begin{aligned}c &= \frac{1}{1 + \left(\beta B E\left[(R')^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}} a \\ &= \left(1 - \left(\beta E\left[(R')^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}\right) a.\end{aligned}$$

5. Consider a mean-preserving spread in the distribution of R' – that is, an increase in the variance holding the mean fixed. If $\sigma > 1$ prove that the fraction of wealth devoted to consumption decreases if the distribution undergoes a mean-preserving spread. What happens if $\sigma < 1$ or $\sigma = 1$? Hint: an MPS is a second-degree stochastically-dominated

change in the distribution of a random variable and therefore has different effects on concave and convex functions of that random variable.

If $\sigma > 1$, $(R')^{1-\sigma}$ is a decreasing convex function of R'. Therefore, a mean-preserving spread will increase $E\left[(R')^{1-\sigma}\right]$, leading to a rise in the expression $\left(\beta E\left[(R')^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}$ and therefore a decline in the fraction of wealth devoted to consumption. If $\sigma > 1$ the opposite occurs – a rise in the fraction of wealth devoted to consumption. If $\sigma = 1$ we get the expression

$$c = (1-\beta)\,a$$

so that the fraction of wealth devoted to consumption is independent of the distribution of R'.