

# Robust Control and Filtering under Rational Inattention in a Permanent Income Model\*

Yulei Luo<sup>†</sup>

University of Hong Kong

Eric R. Young<sup>‡</sup>

University of Virginia

August 16, 2010

## Abstract

In this paper we consider robust control and filtering under limited information-processing capacity (rational inattention or RI) in an otherwise standard permanent income model. We first solve the robustness (RB) versions of the permanent income model with RI explicitly and develop three main results. First, the observational equivalence between robustness (or risk-sensitivity) and the discount factor obtained by Hansen, Sargent, and Tallarini (1999) in the perfect-state-observations permanent income model holds exactly in the RI version of the model when the agent distrusts the Kalman filtering equation governing the evolution of the perceived state. Second, both a stronger preference for robustness in the Kalman filtering gain and higher channel capacity increase the Kalman gain. Third, concerns about the shocks to the perceived state and concerns about the Kalman gain have different effects on consumption dynamics, precautionary savings, and the welfare costs of uncertainty.

*JEL* Classification Numbers: *C61, D81, E21.*

Keywords: Robustness, Rational Inattention, Robust Kalman Gain, Observational Equivalence, Welfare Costs.

---

\*We would like to thank Richard Dennis, Tasos Karantounias, and Chris Sims for helpful discussions. Luo thanks the General Research Fund (GRF) in Hong Kong and the HKU seed funding program for basic research for financial support. All errors are the responsibility of the authors.

<sup>†</sup> School of Economics and Finance, University of Hong Kong, Hong Kong. Email address: [yloeo@econ.hku.hk](mailto:yloeo@econ.hku.hk).

<sup>‡</sup> Department of Economics, University of Virginia, Charlottesville, VA 22904. E-mail: [ey2d@virginia.edu](mailto:ey2d@virginia.edu).

## 1. Introduction

Sims (2003) first introduced rational inattention (RI) into economics and argued that it is a plausible method for introducing sluggishness, randomness, and delay into economic models. In his formulation agents have finite Shannon channel capacity, limiting their ability to process signals about the true state of the world. As a result, an impulse to the economy induces only gradual responses by individuals, as their limited capacity requires many periods to discover just how much the state has moved; one key change relative to the RE case is that consumption has a hump-shaped impulse response to income shocks. Luo (2008) used this model to explore anomalies in the consumption literature, particularly the well-known *excess sensitivity* and *excess smoothness* puzzles, employing a linear-quadratic (LQ) version of the standard permanent income model (as in Hall 1978 and Flavin 1981). In that model RI is equivalent to confronting the household with a signal extraction problem regarding the value of permanent income but permitting the agents to choose the distribution of the noise terms, subject to their limited capacity.<sup>1</sup>

Hansen and Sargent (1995, 2007) first introduce robustness (a concern for model misspecification) into economic models. In robust control problems, agents do not know the true model generating the data and are concerned about the possibility that their model (denoted the approximating model) is misspecified; consequently, they choose optimal decisions as if the subjective distribution over shocks was chosen by an evil nature in order to minimize their expected utility.<sup>2</sup> Robustness models produce precautionary savings but remain within the class of LQ Gaussian models, which leads to analytical simplicity. A second class of models that produces precautionary savings but remains within the class of LQ-Gaussian models is the risk-sensitive model of Hansen, Sargent, and Tallarini (1999) (henceforth, HST). As shown in Hansen and Sargent (2007), we can use the risk-sensitivity preference to characterize robustness as they lead to the same consumption, saving, and welfare. Since RI introduces additional uncertainty, the endogenous noise due to finite channel capacity, into economic models, RI by itself creates an additional demand for robustness. In this paper we consider several ways to robustify the RI version of an otherwise standard permanent income model, and show how different types of robustness affects optimal consumption, precautionary savings, and welfare costs of uncertainty via interacting with finite capacity.

---

<sup>1</sup>Luo and Young (2010b) develop the formal equivalence between RI and signal extraction.

<sup>2</sup>The solution to a robust decision-maker's problem is the equilibrium of a max-min game between the decision-maker and nature.

In Luo and Young (2010a), we study the properties of risk-sensitive (RS) and robust (RB) permanent income models with rational inattention (RI). Within the linear-quadratic setting, we show that risk-sensitivity (or robustness) and the discount factor are observationally equivalent in the sense that RB-RI (or RS-RI) and RI models possess combinations of parameters such that their implied consumption-savings rules are the same. In particular, when examining the RB-RI model, we incorporate RB into an RI version of the permanent income model proposed by Sims (2003) and assume that agents with finite capacity distrust their budget constraint, but still use an ordinary Kalman filter to estimate the true state; in this case, a distortion to the mean of permanent income is introduced to represent possible model misspecification. However, this case ignores the effect of the RI-induced noise on the demand for robustness. Specifically, both the fundamental shock and the RI-induced noise affect the Kalman filtering equation that governs the dynamics of the perceived state; that is, the endogenous RI noise could be another source of the demand for robustness. We therefore need to consider this demand for robustness in the RB-RI model. In this paper, we assume that the consumer not only has doubts about the process for the shock to permanent income, but also distrusts his regular Kalman filtering equation hitting the endogenous noise and updating the estimated state. After solving this RB-RI permanent income model explicitly, we show that the observational equivalence between robustness (or risk-sensitivity) and the discount factor obtained by HST (1999) in the perfect-state-observations permanent income model holds in exactly the same form in the RI model. For ease of presentation, we will refer to the first type of robustness as Type I and the second as Type II.

In this paper we construct an RB-RI permanent income model with two types of concerns about model misspecifications: (i) concerns about the disturbances to the perceived permanent income and (ii) concerns about the Kalman gain. We first show that given finite capacity, the two types of robustness have opposing impacts on the marginal propensity to consume out of perceived permanent income (MPC) and precautionary savings. Specifically, for Type I robustness, since agents with low capacity are very concerned about the confluence of low permanent income and high consumption (meaning they believe their permanent income is high so they consume a lot and then their new signal indicates that in fact their permanent income was low), they take actions which reduce the probability of this bad event – they save more. As for Type II robustness, an increase in the strength of this effect increases the Kalman gain, which leads to lower total uncertainty about the true level of permanent income and then low precautionary savings. In addition, the strength of the precautionary

effect is positively related to the amount of this uncertainty that always increases as finite capacity gets smaller. Using the explicit expression for consumption dynamics, we also show that increasing Type II robustness increases the robust Kalman filter gain and thus leads to greater relative volatility of consumption to income (less smooth consumption process). In contrast, Type I robustness increases the relative volatility of consumption by increasing the MPC out of changes in permanent income. This mechanism is similar to that examined in Luo and Young (2010a). Finally, after obtaining a series of observational equivalence results between robustness and the discount factor in this RB-RI model, we show that under the OE, robustness and the discount factor lead to different welfare costs of uncertainty. Specifically, we show that given finite capacity, the welfare losses for RB-RI agents are decreasing with both Type I robustness and Type II robustness.

The remainder of the paper is organized as follows. Section 2 presents several robustness version of the RI permanent income model, and discusses the different observational equivalence results between robustness and the discount factor. Section 3 studies how two types of robustness, concerns about the shocks to the perceived state and concerns about the Kalman filter gain, affect optimal consumption, precautionary savings, and consumption dynamics via interacting with finite capacity. Section 4 examines the welfare implications of the two types of robustness and RI. Section 5 concludes.

## 2. Robust Control and Filtering under Rational Inattention

### 2.1. A Rational Inattention Version of the Standard Permanent Income Model

In this section we consider a rational inattention (RI) version of the standard permanent income model. In the standard permanent income model (Hall 1978, Flavin 1981), households solve the dynamic consumption-savings problem

$$v(s_0) = \max_{\{c_t\}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right] \quad (2.1)$$

subject to

$$s_{t+1} = R s_t - c_t + \zeta_{t+1}, \quad (2.2)$$

where  $u(c_t) = -\frac{1}{2}(c_t - \bar{c})^2$  is the period utility function,  $\bar{c} > 0$  is the bliss point,  $c_t$  is consumption,

$$s_t = w_t + \frac{1}{R} \sum_{j=0}^{\infty} R^{-j} E_t [y_{t+j}] \quad (2.3)$$

is permanent income, i.e., the expected present value of lifetime resources, consisting of financial wealth plus human wealth,

$$\zeta_{t+1} = \frac{1}{R} \sum_{j=t+1}^{\infty} \left(\frac{1}{R}\right)^{j-(t+1)} (E_{t+1} - E_t) [y_j]; \quad (2.4)$$

is the time  $(t+1)$  innovation to permanent income,  $w_t$  is cash-on-hand (or market resources),  $y_t$  is a general income process with Gaussian white noise innovations,  $\beta$  is the discount factor, and  $R > 1$  is the constant gross interest rate at which the consumer can borrow and lend freely.<sup>3</sup> Note that when  $y$  follows the following AR(1) process with the persistence coefficient  $\rho \in [0, 1]$ ,  $y_{t+1} = \rho y_t + \varepsilon_{t+1}$ ,  $\zeta_{t+1} = \varepsilon_{t+1}/(R - \rho)$ . For the rest of the paper we will restrict attention to points where  $c_t < \bar{c}$ , so that utility is increasing and concave. This specification follows that in Hall (1978) and Flavin (1981) and implies that optimal consumption is determined by permanent income:

$$c_t = \left(R - \frac{1}{\beta R}\right) s_t - \frac{1}{R-1} \left(1 - \frac{1}{\beta R}\right) \bar{c}. \quad (2.5)$$

We assume for the remainder of this section that  $\beta R = 1$ , since this setting is the only one that implies zero drift in consumption. Under this assumption the model leads to the well-known random walk result of Hall (1978):

$$\Delta c_{t+1} = (R - 1) \zeta_{t+1}; \quad (2.6)$$

the change in consumption depends neither on the past history of labor income nor on anticipated changes in labor income. We also point out the well-known result that the standard PIH model with quadratic utility implies the certainty equivalence property holds; thus, uncertainty has no impact on optimal consumption, so that there is no precautionary saving.

Sims (2003) examines how RI affects consumption dynamics when the agent only has limited capacity when processing information. Luo (2008) shows that the RI permanent income model can be solved explicitly even if the income process is not iid, and then shows how RI can resolve the excess smoothness puzzle and the excess sensitivity puzzle. Following Sims (2003) and Luo

---

<sup>3</sup>We only require that  $y_t$  and  $R$  are such that permanent income is finite.

(2008), incorporating RI into the above otherwise standard permanent income model leads to the following consumption rule:

$$c_t = \left( R - \frac{1}{\beta R} \right) \widehat{s}_t - \frac{1}{R-1} \left( 1 - \frac{1}{\beta R} \right) \bar{c}, \quad (2.7)$$

where  $\widehat{s}_t = E_t[s_t]$  is the perceived state (the conditional mean of the state).  $\widehat{s}_t$  is governed by the following Kalman filtering equation

$$\widehat{s}_{t+1} = (1 - \theta) (R\widehat{s}_t - c_t) + \theta (s_{t+1} + \xi_{t+1}), \quad (2.8)$$

where  $\theta = \sigma^2/\lambda^2 = 1 - 1/\exp(2\kappa) \in [0, 1]$  is the constant optimal weight on any new observation,<sup>4</sup>  $\kappa$  is the consumer's channel capacity,  $\sigma^2 = \text{var}[s_t|\mathcal{I}_t] = \frac{\omega_\xi^2}{\exp(2\kappa) - R^2}$  is the conditional variance of  $s_t$ ,  $\xi_{t+1}$  is the iid endogenous noise with  $\lambda^2 = \text{var}[\xi_{t+1}] = \frac{[\omega_\xi^2 + R^2\sigma^2]\sigma^2}{\omega_\xi^2 + (R^2 - 1)\sigma^2}$ , and given  $s_0 \sim N(\widehat{s}_0, \sigma^2)$ .

In the next section, we will discuss alternative ways to robustify this RI-PIH model and their different implications for consumption, precautionary savings, and welfare costs of uncertainty. It is worth noting that the RB-RI model proposed here encompasses the hidden state (HS) model discussed in Hansen, Sargent, and Wang (2002) and Hansen and Sargent (2007b); the main difference is that agents in the RB-RI model cannot observe the entire state vector perfectly, whereas agents in the RB-HS model can observe some part of the state vector (in particular, the part they control).

## 2.2. Concerns about the Income Shock

In this section we discuss one way to introduce robustness into the above RI model. Following Luo and Young (2010), we incorporate RB into the RI-PIH model by assuming that agents with finite capacity distrust their model generating the data (i.e., their income process), but still use an ordinary Kalman filter to estimate the true state. In this model, a distortion to the mean of income is introduced to represent possible model misspecification. Without the concern for model misspecification, the consumer has no doubts about the probability model used to form the conditional expectation of current and future income.

Specifically, the robust permanent income (PI) problem with inattentive consumers is for-

---

<sup>4</sup>That is,  $\theta$  measures how much new information is transmitted each period or how much uncertainty is removed upon the receipt of a new signal.

ulated as follows:

$$\widehat{v}(\widehat{s}_t) = \max_{c_t} \min_{\nu_t} \left\{ -\frac{1}{2} (c_t - \bar{c})^2 + \beta E_t [\vartheta \nu_t^2 + \widehat{v}(\widehat{s}_{t+1})] \right\} \quad (2.9)$$

subject to the budget constraint

$$s_{t+1} = R s_t - c_t + \omega_\zeta \nu_t + \zeta_{t+1}, \quad (2.10)$$

and the regular Kalman filter equation

$$\widehat{s}_{t+1} = (1 - \theta) (R \widehat{s}_t - c_t + \omega_\zeta \nu_t) + \theta (s_{t+1} + \xi_{t+1}), \quad (2.11)$$

where  $s_t$  is permanent income,  $c_t$  is consumption,  $\nu_t$  is the worst-possible case shock,  $\vartheta$  is the parameter measuring the preference for RB,  $\theta$  is the Kalman filter gain, and  $\xi_{t+1}$  is the iid endogenous noise due to RI. After substituting (2.10) into (2.11), the Kalman filter equation can be rewritten as

$$\widehat{s}_{t+1} = R \widehat{s}_t - c_t + \omega_\zeta \nu_t + \eta_{t+1}. \quad (2.12)$$

where

$$\eta_{t+1} = \theta R (s_t - \widehat{s}_t) + \theta (\zeta_{t+1} + \xi_{t+1}) \quad (2.13)$$

is the innovation to the mean of the distribution of perceived permanent income, and

$$s_t - \widehat{s}_t = \frac{(1 - \theta) \zeta_t}{1 - (1 - \theta) R \cdot L} - \frac{\theta \xi_t}{1 - (1 - \theta) R \cdot L}. \quad (2.14)$$

Note that  $E_t [\eta_{t+1}] = 0$  if the expectation is conditional on the perceived signals, as inattentive agents cannot perceive the lagged shocks perfectly. The following proposition summarizes the solution to the above RB-RI model.

**Proposition 1.** *Given  $\vartheta$  and  $\theta$ , the consumption function is*

$$c_t = \frac{R - 1}{1 - \Pi} \widehat{s}_t - \frac{\Pi}{1 - \Pi} \bar{c}, \quad (2.15)$$

where

$$\Pi = \frac{R \omega_\zeta^2}{2\vartheta} \in (0, 1). \quad (2.16)$$

**Proof.** See Appendix 6.1. ■

Risk-sensitivity (RS) was first introduced into the LQ-Gaussian framework by Jacobson (1973) and extended by Whittle (1981, 1990). Exploiting the recursive utility framework of Epstein and Zin (1989), Hansen and Sargent (1995) introduce discounting into the RS specification and show that the resulting decision rules are time-invariant. In the RS model agents effectively compute expectations through a distorted lens, increasing their effective risk aversion by overweighting negative outcomes. The resulting decision rules depend explicitly on the variance of the shocks, producing precautionary savings, but the value functions are still quadratic functions of the states.<sup>5</sup> In HST (1999) and Hansen and Sargent (2007), they interpret the RS preference in terms of a concern about model uncertainty (robustness or RB) and argue that RS introduces precautionary savings because RS consumers want to protect themselves against model specification errors.

Following Luo and Young (2010a), we formulate an RI version of risk-sensitive control based on recursive preferences with an exponential certainty equivalence function as follows:

$$\hat{v}(\hat{s}_t) = \max_{c_t} \left\{ -\frac{1}{2} (c_t - \bar{c})^2 + \beta \mathcal{R}_t [\hat{v}(\hat{s}_{t+1})] \right\} \quad (2.17)$$

subject to the budget constraint, (2.2), and the Kalman filter equation (2.8). Note that combining with yields the following constraint facing the risk-sensitive inattentive agent:

$$\hat{s}_{t+1} = R\hat{s}_t - c_t + \eta_{t+1}. \quad (2.18)$$

The distorted expectation operator is now given by

$$\mathcal{R}_t [\hat{v}(\hat{s}_{t+1})] = -\frac{1}{\alpha} \log E_t [\exp(-\alpha \hat{v}(\hat{s}_{t+1}))],$$

where  $s_0 | \mathcal{I}_0 \sim N(\hat{s}_0, \bar{\sigma}^2)$ ,  $\hat{s}_t = E_t[s_t]$  is the perceived state variable,  $\theta$  is the optimal weight on the new observation of the state, and  $\xi_{t+1}$  is the endogenous noise. The optimal choice of the weight  $\theta$  is given by  $\theta(\kappa) = \frac{\exp(2\kappa)-1}{\exp(2\kappa)} \in [0, 1]$ . The following proposition summarizes the solution to the RI-RS model when  $\beta R = 1$ :

**Proposition 2.** *Given finite channel capacity  $\kappa$  and the degree of risk-sensitivity  $\alpha$ , the con-*

---

<sup>5</sup>Formally, one can view risk-sensitive agents as ones who have non-state-separable preferences, as in Epstein and Zin (1989), but with a restricted value for the intertemporal elasticity of substitution (see Tallarini 2000).

sumption function of a risk-sensitive consumer under RI

$$c_t = \frac{R-1}{1-\Pi} \widehat{s}_t - \frac{\Pi \bar{c}}{1-\Pi}, \quad (2.19)$$

where

$$\Pi = R\alpha\omega_\eta^2 \in (0, 1), \quad (2.20)$$

$$\omega_\eta^2 = \text{var} [\eta_{t+1}] = \frac{\theta}{1 - (1-\theta)R^2} \omega_\zeta^2, \quad (2.21)$$

and  $\eta_{t+1} = \theta \left[ \frac{\zeta_{t+1}}{1-(1-\theta)RL} + \left( \xi_{t+1} - \frac{\theta R \xi_t}{1-(1-\theta)RL} \right) \right]$ .

**Proof.** See Appendix 6.2. ■

It is straightforward to show that when

$$\frac{\alpha\theta}{1 - (1-\theta)R^2} = \frac{1}{2\vartheta},$$

consumption and savings are identical in the RS-RI and RB-RI models. Note that in the RB model we assume that the agents distrust the income process hitting the budget constraint, but trust the RI-induced noise hitting the Kalman filtering equation.

HST (1999) show that as far as the *quantity* observations on consumption and savings are concerned, the robustness version ( $\vartheta > 0$  or  $\alpha > 0, \widetilde{\beta}$ ) of the PIH model is observationally equivalent to the standard version ( $\vartheta = \infty$  or  $\alpha = 0, \beta = 1/R$ ) of the PIH model for a unique pair of discount factors.<sup>6</sup> The intuition is that introducing a preference for risk-sensitivity (RS) or a concern about robustness (RB) increases savings in the same way as increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RS or RB on consumption and investment.<sup>7</sup> Alternatively, holding all parameters constant except the pair  $(\alpha, \beta)$ , the RI version of the PIH model with RB consumers ( $\vartheta > 0$  and  $\beta R = 1$ ) is observationally equivalent to the standard RI version of the model ( $\vartheta = \infty$  and  $\widetilde{\beta} > 1/R$ ). To do so, we compare (2.7) with (2.15) and obtain the following OE expression for the discount factor:

---

<sup>6</sup>HST (1999) derive the observational equivalence result by fixing all parameters, including  $R$ , except for the pair  $(\alpha, \beta)$ .

<sup>7</sup>As shown in HST (1999), the two models have different implications for asset prices because continuation valuations would alter as one alters  $(\alpha, \beta)$  within the observationally-equivalent set of parameters.

**Proposition 3.** *Let*

$$\tilde{\beta} = \frac{1}{R} \frac{1 - R\omega_\zeta^2 / (2\vartheta)}{1 - R^2\omega_\zeta^2 / (2\vartheta)} > \frac{1}{R}. \quad (2.22)$$

*Then consumption and savings are identical in the RI and RB-RI models.*

### 2.3. Concerns about Both Income Shock and Endogenous Noise

A key assumption in the RB-RI model proposed in Section 2.2 is that we assume that the consumer has doubts about the state equation hitting the exogenous shock to permanent income  $\zeta_{t+1}$ , but trusts his regular Kalman filter. However, the Kalman filter under RI is also affected by the endogenous noise ( $\xi_{t+1}$ ) induced by finite capacity that could be another source of the demand for robustness. We therefore need to consider this demand for robustness in the RB-RI model. By adding the additional concern for robustness developed here, we are able to strengthen the effects of robustness on decisions.<sup>8</sup>

First, substituting the budget constraint,

$$s_{t+1} = Rs_t - c_t + \zeta_{t+1},$$

into the regular Kalman filter equation,

$$\hat{s}_{t+1} = (1 - \theta) (R\hat{s}_t - c_t) + \theta (s_{t+1} + \xi_{t+1}),$$

we obtain the following equation governing the dynamics of the perceived state  $\hat{s}_t$  that matters in agents' decision problems:

$$\begin{aligned} \hat{s}_{t+1} &= R\hat{s}_t - c_t + \theta R (s_t - \hat{s}_t) + \theta (\zeta_{t+1} + \xi_{t+1}), \\ &= R\hat{s}_t - c_t + \eta_{t+1}, \end{aligned} \quad (2.23)$$

where  $\eta_{t+1} = \theta R (s_t - \hat{s}_t) + \theta (\zeta_{t+1} + \xi_{t+1})$  is the innovation to the mean of the distribution of perceived permanent income,

$$s_t - \hat{s}_t = \frac{(1 - \theta) \zeta_t}{1 - (1 - \theta)R \cdot L} - \frac{\theta \xi_t}{1 - (1 - \theta)R \cdot L}. \quad (2.24)$$

and  $E_t [\eta_{t+1}] = 0$ . To introduce robustness into the RI model, we assume that the agent thinks

---

<sup>8</sup>Luo, Nie, and Young (2010) use this approach to study the joint dynamics of consumption, income, and the current account, finding that the anomalies displayed by the basic model are partially resolved.

that (2.23) is the approximating model. Following Hansen and Sargent (2007), we surround (2.23) with a set of alternative models to represent his preference for robustness:

$$\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \omega_\eta \nu_t + \eta_{t+1}. \quad (2.25)$$

Under RI the innovation  $\eta_{t+1}$ , (2.13), that the agent distrusts is composed of two MA( $\infty$ ) processes and includes the entire history of the exogenous income shock and the endogenous noise,  $\{\zeta_{t+1}, \zeta_t, \dots, \zeta_0; \xi_{t+1}, \xi_t, \dots, \xi_0\}$ . The difference between 2.12 and 2.25 is the third term; in 2.12 the coefficient on  $\nu_t$  is  $\omega_\zeta$  while in 2.25 the coefficient is  $\omega_\eta$ ; note that with  $\theta < 1$  and  $R > 1$  it holds that  $\omega_\zeta < \omega_\eta$ .

The optimizing problem for this RB-RI model can be formulated as follows:

$$\widehat{v}(\widehat{s}_t) = \max_{c_t} \min_{\nu_t} \left\{ -\frac{1}{2} (c_t - \bar{c})^2 + \beta E_t [\vartheta \nu_t^2 + \widehat{v}(\widehat{s}_{t+1})] \right\} \quad (2.26)$$

subject to (2.25). (2.26) is a standard dynamic programming problem and can be easily solved using the standard procedure. The following proposition summarizes the solution to the RB-RI model.

**Proposition 4.** *Given  $\vartheta$  and  $\theta$ , the consumption function under RB and RI is*

$$c_t = \frac{R-1}{1-\Pi} \widehat{s}_t - \frac{\Pi \bar{c}}{1-\Pi}, \quad (2.27)$$

the mean of the worst-case shock is

$$\omega_\eta \nu_t = \frac{(R-1)\Pi}{1-\Pi} \widehat{s}_t - \frac{\Pi}{1-\Pi} \bar{c},$$

and  $\widehat{s}_t$  is governed by

$$\widehat{s}_{t+1} = \rho_s \widehat{s}_t + \eta_{t+1}. \quad (2.28)$$

where  $\rho_s = \frac{1-R\Pi}{1-\Pi} \in (0, 1)$ ,

$$\begin{aligned} \Pi &= \frac{R\omega_\eta^2}{2\vartheta} > 0, \\ \omega_\eta^2 &= \text{var}[\eta_{t+1}] = \frac{\theta}{1-(1-\theta)R^2} \omega_\zeta^2, \end{aligned} \quad (2.29)$$

**Proof.** See Appendix 6.3. ■

## 2.4. Observational Equivalence Between RS-RI, RB-RI, and RI with higher $\beta$

Given that the consumption function under RS and RI is

$$c_t = \frac{R-1}{1-\Pi} \widehat{s}_t - \frac{\Pi \bar{c}}{1-\Pi},$$

where  $\Pi = R\alpha\omega_\eta^2$ , and  $\alpha$  is the risk-sensitivity parameter and measures enhanced risk aversion, it is straightforward to show that it is impossible to distinguish between RB and RS under RI using only consumption-savings decisions.

**Proposition 5.** *Let the following expression hold:*

$$\alpha = \frac{1}{2\vartheta}. \tag{2.30}$$

*Then consumption and savings are identical in the RS-RI and RB-RI models.*

Note that (2.30) is *exactly the same as* the observational equivalence condition obtained in the full-information RE model (see Backus, Routledge, and Zin 2004). That is, under the assumption that the agent distrusts the Kalman filter equation, the OE result obtained under full-information RE still holds under RI.

As in , holding all parameters constant except the pair  $(\alpha, \beta)$ , the RI version of the PIH model with RB consumers ( $\vartheta > 0$  and  $\beta R = 1$ ) is observationally equivalent to the standard RI version of the model ( $\vartheta = \infty$  and  $\tilde{\beta} > 1/R$ ). To do so, we compare (2.7) with (2.15) and obtain the following OE expression for the discount factor:

**Proposition 6.** *Let*

$$\tilde{\beta} = \frac{1}{R} \frac{1 - R\omega_\eta^2/(2\vartheta)}{1 - R^2\omega_\eta^2/(2\vartheta)} > \frac{1}{R}.$$

*Then consumption and savings are identical in the RI and RB-RI models.*

## 3. Robust Kalman Filter Gain

### 3.1. Introducing Robustness in the Kalman Gain

Another source of robustness could arise from the Kalman filter gain. In Section 2, we assume that the agent distrusts the innovation to the perceived state but trusts the regular Kalman filter gain. Following Hansen and Sargent (Chapter 17, 2008), in this section we consider a

situation in which the agent pursues a robust Kalman gain. Specifically, assume that at  $t$  the agent observe the noisy signal

$$s_t^* = s_t + \xi_t, \quad (3.1)$$

where  $s_t$  is the true state and  $\xi_t$  is the iid endogenous noise and its variance,  $\varrho_t^2 = \text{var}[\xi_t]$ , is determined by

$$\varrho_t^2 = \varrho^2 = \text{var}[\xi_t] = \frac{(\omega_\zeta^2 + R^2\sigma^2)\sigma^2}{\omega_\zeta^2 + (R^2 - 1)\sigma^2},$$

and  $\sigma^2 = \frac{\omega_\zeta^2}{\exp(2\kappa) - R^2}$  is the steady state conditional variance. Given the budget constraint,

$$s_{t+1} = Rs_t - c_t + \zeta_{t+1}, \quad (3.2)$$

we consider the following time-invariant robust Kalman filter equation,

$$\hat{s}_{t+1} = (1 - \theta)(R\hat{s}_t - c_t) + \theta(s_{t+1} + \xi_{t+1}), \quad (3.3)$$

where  $\hat{s}_{t+1}$  is the estimate of the state using the history of the noisy signals,  $\{s_j^*\}_{j=0}^{t+1}$ . We want  $\theta$  to be robust to possible misspecification of Equations (3.1) and (3.2). To obtain robust Kalman filter gain, the agent considers the following distorted model:

$$s_{t+1} = Rs_t - c_t + \zeta_{t+1} + \omega\nu_{1,t+1}, \quad (3.4)$$

$$s_{t+1}^* = s_{t+1} + \xi_{t+1} + \varrho\nu_{2,t+1}, \quad (3.5)$$

where  $\nu_{t+1} = \begin{bmatrix} \nu_{1,t+1} & \nu_{2,t+1} \end{bmatrix}^T$  are distortions to the conditional means of the two shocks,  $\zeta_{t+1}$  and  $\xi_{t+1}$ .

Combining (3.4) with (3.5) gives the following dynamic equation for the estimation error:<sup>9</sup>

$$e_{t+1} = (1 - \theta)Re_t + (1 - \theta)\zeta_{t+1} - \theta\xi_{t+1} + (1 - \theta)\omega\nu_{1,t+1} - \theta\varrho\nu_{2,t+1}. \quad (3.6)$$

As shown in Hansen and Sargent (2008), we can solve for the robust Kalman filter gain corresponding to this problem by solving the following deterministic optimal linear regulator:

$$v(e_0) = e_0^T P e_0 = \max_{\{\nu_{t+1}\}} \sum_{t=0}^{\infty} (e_t^T e_t - \vartheta\nu_{t+1}^T \nu_{t+1}), \quad (3.7)$$

---

<sup>9</sup>Note that control variable,  $c$ , does not affect the estimation error equation.

subject to

$$e_{t+1} = (1 - \theta) R e_t + D \nu_{t+1}, \quad (3.8)$$

where  $D = \begin{bmatrix} (1 - \theta) \omega & -\theta \varrho \end{bmatrix}$ . We can compute the worst-case shock by solving the corresponding Bellman equation

$$\nu_{t+1}^* = Q e_t, \quad (3.9)$$

where

$$Q = (\vartheta I - D^T P D)^{-1} D^T P (1 - \theta) R. \quad (3.10)$$

Note that  $Q$  is a function of robustness ( $\vartheta$ ) and channel capacity ( $\kappa$ ).

For arbitrary Kalman filter gain  $\theta$  and (3.9), the error in reconstructing the state  $s$  can be written as

$$e_{t+1} = \{(1 - \theta) R + [(1 - \theta) \omega - \theta \varrho] Q\} e_t + (1 - \theta) \zeta_{t+1} - \theta \xi_{t+1}. \quad (3.11)$$

Taking unconditional mean on both sides of (3.11) gives

$$\Sigma_{t+1} = \{(1 - \theta) R + [(1 - \theta) \omega - \theta \varrho] Q\} \Sigma_t + (1 - \theta)^2 \omega_\zeta^2 + \theta^2 \omega_\xi^2, \quad (3.12)$$

where  $\Sigma_{t+1} = E [e_{t+1}^2]$ . From (3.12), it follows directly that in the steady state

$$\Sigma(\theta; Q) = \frac{(1 - \theta)^2 \omega_\zeta^2 + \theta^2 \omega_\xi^2}{1 - \chi^2},$$

where  $\chi = (1 - \theta) R + [(1 - \theta) \omega - \theta \varrho] Q$ . We use the program `rfilter.m` provided in Hansen and Sargent (2008) to compute the robust Kalman filter gain  $\theta(\vartheta, \kappa)$  that minimizes the variance of  $e_t$ ,  $\Sigma(\theta; Q)$ :

$$\theta(\vartheta, \kappa) = \arg \min (\Sigma(\theta; Q(\vartheta, \kappa))).$$

Figure 6.1 illustrates how robustness (measured by  $\vartheta$ ) and inattention (measured by  $\kappa$ ) affect the robust Kalman filter gain when  $R = 1.01$  and  $\omega_\zeta = 1$ . It clearly shows that holding the degree of attention (i.e., channel capacity  $\kappa$ ) fixed, increasing robustness (i.e., reducing  $\vartheta$ ) can increase the Kalman filter gain ( $\theta$ ). In addition, for given robustness ( $\vartheta$ ), the Kalman gain is increasing with capacity. For example, when  $\log(\vartheta) = 3$ , the robust Kalman gain will increase from 60.17% to 77.35% when capacity  $\kappa$  increases from 0.6 bits to 1 bit;<sup>10</sup> when  $\kappa = 0.6$  bits, the robust Kalman gain will increase from 58.31% to 60.17% when  $\vartheta$  reduces from  $\log(\vartheta) = 4$

---

<sup>10</sup> $\theta$  measures how much uncertainty can be removed upon receiving new signals on the state.

to 3.<sup>11</sup>

### 3.2. Optimal Consumption and Precautionary Savings

After obtaining the robust Kalman gain  $\theta(\vartheta, \kappa)$ , we can solve the Bellman equation proposed in Section 2.2 using the Kalman filtering equation with robust  $\theta$  and obtain a new consumption function under RB and RI:

$$c_t = \frac{R-1}{1-\Pi} \widehat{s}_t - \frac{\Pi \bar{c}}{1-\Pi}, \quad (3.13)$$

where

$$\begin{aligned} \Pi &= \frac{R\omega_\eta^2}{2\vartheta_0} > 0, \\ \omega_\eta^2 &= \text{var}[\eta_{t+1}] = \frac{\theta(\vartheta, \kappa)}{1 - (1 - \theta(\vartheta, \kappa))R^2} \omega_\zeta^2, \end{aligned} \quad (3.14)$$

and  $\widehat{s}_t$  is governed by

$$\widehat{s}_{t+1} = \rho_s \widehat{s}_t + \eta_{t+1}, \quad (3.15)$$

where  $\rho_s = \frac{1-R\Pi}{1-\Pi} \in (0, 1)$ .

Note that here  $\theta$  is a function of both  $\vartheta$  (concerns about Kalman gain) and  $\kappa$  (channel capacity), rather than  $1 - 1/\exp(2\kappa)$  obtain in Section 2.2. In this case the agent has two types of concerns about model misspecification: (i) concerns about the disturbances to the perceived permanent income ( $\vartheta_0$ ) and (ii) concerns about the Kalman gain ( $\vartheta$ ). It is clear from (3.13) and (3.14) that the two types of robustness have opposing effects on both the marginal propensity to consume out of permanent income, i.e., the responsiveness of  $c_t$  to  $\widehat{s}_t$  (MPC =  $\frac{R-1}{1-\Pi}$ ) and precautionary savings, i.e., the intercept of the consumption profile (PS =  $\frac{\Pi \bar{c}}{1-\Pi} = -\bar{c} + \frac{\bar{c}}{1-\Pi}$ ).<sup>12</sup> Specifically, the less the value of  $\vartheta_0$  (type i robustness) the larger the MPC and the larger the precautionary saving increment, since

$$\frac{\partial(\text{MPC})}{\partial \vartheta_0} < 0 \text{ and } \frac{\partial(\text{PS})}{\partial \vartheta_0} < 0.$$

For the effects of type ii robustness ( $\vartheta$ ), the less the value of  $\vartheta$  the less the MPC and the less

---

<sup>11</sup>This result is consistent with that obtained in a continuous-time setting discussed in Kasa (2003).

<sup>12</sup>Note that given the consumption function  $\Pi$  has the same impact on the marginal propensity to consume and precautionary savings.

the precautionary saving increment

$$\frac{\partial(\text{MPC})}{\partial\vartheta} > 0 \text{ and } \frac{\partial(\text{PS})}{\partial\vartheta} > 0$$

because  $\frac{\partial\omega_\eta^2}{\partial\theta} < 0$ ,  $\frac{\partial\theta}{\partial\vartheta} < 0$ .

From (3.13) and (3.14), it is clear that the precautionary savings increment in the RB-RI model is determined by the interaction of three factors: labor income uncertainty, preferences for robustness (RB), and finite information-processing capacity (RI). The intuition about the effects of type i robustness ( $\vartheta_0$ ) on precautionary savings is as follows. Since agents with low capacity are very concerned about the confluence of low permanent income and high consumption (meaning they believe their permanent income is high so they consume a lot and then their new signal indicates that in fact their permanent income was low), they take actions which reduce the probability of this bad event – they save more. The strength of the precautionary effect is positively related to the amount of uncertainty regarding the true level of permanent income, and this uncertainty increases as  $\theta$  gets smaller. The intuition about the effects of type ii robustness ( $\vartheta$ ) on precautionary savings is as follows. An increase in type ii robustness (a reduction in  $\vartheta$ ) will increase the Kalman gain  $\theta$ , which leads to lower  $\omega_\eta^2$  and then low precautionary savings. In addition, since

$$\frac{\partial\omega_\eta^2}{\partial\vartheta} > 0, \frac{\partial\omega_\eta^2}{\partial\kappa} < 0, \frac{\partial\omega_\eta^2}{\partial\vartheta_0} = 0$$

it is clear that under certain conditions a greater reaction to the shock can either be interpreted as an increased concern for robustness in the presence of model misspecification, or an increase in information-processing ability when agents only have finite channel capacity. Figure 6.3 illustrates  $\Pi$  as functions of  $\vartheta_0$  and  $\vartheta$ . It clearly shows that how increasing the robustness preference for the shock to the perceived state (i.e., decreasing  $\vartheta_0$ ) and reducing the preference for a robust gain (i.e., increasing  $\vartheta$ ) increases  $\Pi$  and then increase the impacts of the two types of robustness on consumption and precautionary savings.

### 3.3. Sensitivity and Smoothness of Consumption Process

Combining (3.13) with (3.15) yields the change in individual consumption in the RI-RB economy:

$$\Delta c_t = \frac{(1-R)\Pi}{1-\Pi} (c_{t-1} - \bar{c}) + \frac{R-1}{1-\Pi} \left[ \frac{\theta\zeta_t}{1-(1-\theta)R \cdot L} + \theta \left( \xi_t - \frac{\theta R \xi_{t-1}}{1-(1-\theta)R \cdot L} \right) \right], \quad (3.16)$$

where  $L$  is the lag operator and we assume that  $(1 - \theta)R < 1$ .<sup>13</sup> This expression shows that consumption growth is a weighted average of all past permanent income and noise shocks. In addition, it is also clear from (3.16) that the propagation mechanism of the model is determined by the robust Kalman filter gain,  $\theta(\vartheta, \kappa)$ . Figure 6.4 illustrates that consumption in the RB-RI model reacts gradually to income shocks, with monotone adjustments to the corresponding RB asymptote. Note that when  $\vartheta_0 = 1$  and  $\log \vartheta = 3$ , the robust Kalman gain  $\theta = 42.56\%$ . This case is illustrated by the dash-dotted line in Figure 6.4. Similarly, the dotted line corresponds to the case in which  $\vartheta_0 = 2$  and  $\log \vartheta = 5$  ( $\theta = 0.3541$ ). With a stronger preference for robustness, the precautionary savings increment is larger and thus an income shock that is initially undetected would have larger impacts on consumption during the adjustment process.

Using (3.16), the relative volatility of consumption growth relative to income growth can be written as

$$\mu = \frac{\text{sd}[\Delta c_t]}{\text{sd}[\Delta y_t]} = \frac{\theta}{1 - \Pi} \sqrt{\sum_{j=0}^{\infty} \Gamma_j^2 + \frac{1 - \theta}{\theta(1 - (1 - \theta)R^2)} \sum_{j=0}^{\infty} (\Gamma_j - R\Gamma_{j-1})^2}, \quad (3.17)$$

where we use the fact that  $\omega_\xi^2 = \text{var}[\xi_t] = \frac{1 - \theta}{\theta(1 - (1 - \theta)R^2)} \omega_\varepsilon^2$ ,  $\rho_1 = \frac{1 - R\Pi}{1 - \Pi} \in (0, 1)$ ,  $\rho_2 = (1 - \theta)R \in (0, 1)$ , and

$$\Gamma_j = \sum_{k=0}^j \left( \rho_1^{j-k} \rho_2^k \right) - \sum_{k=0}^{j-1} \left( \rho_1^{j-1-k} \rho_2^k \right), \text{ for } j \geq 1,$$

and  $\Gamma_0 = 1$ . Figure 6.5 illustrates how the combination of the two types of robustness,  $\vartheta_0$  and  $\vartheta$ , affects the relative volatility of consumption growth to income growth when  $\kappa = 0.3$  bits. It clearly shows that given  $\vartheta_0$ , the relative volatility  $\mu$  is increasing with  $\vartheta$ . The intuition is that reducing  $\vartheta$  (i.e., increasing type i robustness) increases the robust Kalman gain  $\theta$  and reduces  $\omega_\eta^2$  and  $\Pi$ , which leads to more smooth consumption process. Note that  $\theta$  is independent of  $\vartheta_0$ . Hence, given  $\vartheta$ ,  $\mu$  is decreasing with  $\vartheta_0$  because  $\Pi$  is increasing with  $\vartheta_0$ . To explore the intuition behind this result, we consider the perfect-state-observation case in which  $\kappa = \infty$ . In this case, the relative volatility of consumption growth to income growth reduces to

$$\mu = \frac{1}{1 - \Pi} \sqrt{\frac{2}{1 + \rho_1}}, \quad (3.18)$$

which clearly shows that  $\vartheta_0$  increases the relative volatility via two channels. First, a higher

---

<sup>13</sup>This assumption requires  $\kappa > \frac{1}{2} \log(R) \approx \frac{R-1}{2}$ , which is weaker than the condition needed for convergence of the filter.

$\vartheta_0$  increases the marginal propensity to consume out of permanent income  $\left(\frac{R-1}{1-\Pi}\right)$ , and second, it increases consumption volatility by reducing the persistence of permanent income measured by  $\rho_1$ :  $\frac{\partial \rho_1}{\partial \Pi} < 0$ .

In the presence of robustness, rational inattention measured by  $\kappa$  affects consumption volatility via two channels: (i) the gradual and smooth responses to income shocks (i.e., the  $1 - (1 - \theta)R \cdot L$  term in (3.16)) and (ii) the RI-induced noises ( $\xi_t$ ). Specifically, a reduction in capacity  $\kappa$  decreases the Kalman gain  $\theta$ , which strengthens the smooth responses to income shock and increases the volatility of the RI-induced noise. Luo (2008) shows that the noise effect dominates the smooth response effect, and the volatility of consumption growth decreases with  $\kappa$ . Figure 6.6 illustrates how the combination of  $\vartheta_0$  and  $\vartheta$  affects the relative volatility of consumption growth to income growth when  $\kappa = 0.3$  bits and there is no noise term. In this case,

$$\Delta c_t = \frac{(1-R)\Pi}{1-\Pi} (c_{t-1} - \bar{c}) + \frac{R-1}{1-\Pi} \frac{\theta \zeta_t}{1 - (1-\theta)R \cdot L}.$$

Figure 6.6 shows that given  $\vartheta_0$ , the relative volatility  $\mu$  is decreasing with  $\vartheta$ . The intuition is that reducing  $\vartheta$  (i.e., increasing type i robustness) increases the robust Kalman gain  $\theta$ , which leads to more volatile consumption process because the smooth response effect completely dominates the noise effect.

#### 4. Welfare Effects of Robustness and Rational Inattention

In this section, we examine how the interaction of the two types of robustness and inattention affect consumer welfare, and compare different welfare implications in the RB-RI and RI settings. To make the comparisons meaningful, we restrict our attention to combinations of preferences that imply observational equivalence – that is, how agents whose consumption/savings decisions are identical (but for different reasons) suffer different welfare losses from information-processing capacity limitations.

Specifically, in the RB-RI model, the value function can be written as:

$$\begin{aligned} \widehat{v}^{RB-RI}(\widehat{s}_t) = & -\frac{\beta R^2 - 1}{2(\beta - \omega_\eta^2/(2\vartheta))} \widehat{s}_t^2 + \frac{(\beta R^2 - 1)\bar{c}}{(R-1)(\beta - \omega_\eta^2/(2\vartheta))} s \\ & - \left[ \frac{1}{2} \frac{\beta R^2 - 1}{(R-1)^2(\beta - \omega_\eta^2/(2\vartheta))} \bar{c}^2 - \frac{\beta}{1-\beta} \vartheta \log \left( 1 - \frac{\beta R^2 - 1}{\beta R - R\omega_\eta^2/(2\vartheta)} \frac{R\omega_\eta^2}{2\vartheta} \right) \right]. \end{aligned} \quad (4.1)$$

and the corresponding value function for the standard RI model is

$$\widehat{v}(\widehat{s}_t) = -\frac{\beta R^2 - 1}{2\beta} \widehat{s}_t^2 + \frac{(\beta R^2 - 1)\bar{c}}{\beta(R-1)} \widehat{s}_t - \frac{1}{2} \frac{\beta R^2 - 1}{\beta(R-1)^2} \bar{c}^2 - \frac{\beta R^2 - 1}{2(1-\beta)} \omega_\eta^2. \quad (4.2)$$

In the RI case, when the standard restriction on the discount factor  $\beta$ ,  $\beta R = 1$ , holds, the value function can be rewritten as

$$\widehat{v}(\widehat{s}_t) = -\frac{(R-1)R}{2} \widehat{s}_t^2 + R\bar{c}\widehat{s}_t - \frac{R}{2(R-1)} \bar{c}^2 - \frac{R}{2} \omega_\eta^2. \quad (4.3)$$

It is straightforward to show that when the discount factor  $\beta$  in Expression (4.1) satisfies

$$\beta = \beta^{RB-RI} = \frac{1}{R} - (R-1) \frac{\omega_\eta^2(\kappa, \vartheta)}{2\vartheta_0}, \quad (4.4)$$

the observational equivalence between the RB-RI model and the RI model with  $\beta R = 1$  holds in the sense that they lead to the same consumption and savings rules. Given (4.4), (4.1) can be rewritten as

$$\widehat{v}^{RB-RI}(\widehat{s}_t) = -\frac{(R-1)R}{2} \widehat{s}_t^2 + R\bar{c}\widehat{s}_t - \frac{R}{2(R-1)} \bar{c}^2 - \frac{\beta^{RB-RI}}{1-\beta^{RB-RI}} \vartheta_0 \log \left( 1 - (R-1) \frac{R\omega_\eta^2}{2\vartheta_0} \right) \quad (4.5)$$

The only difference between the two functions, (4.3) and (4.5), is the constant term involving the volatility  $\omega_\eta^2$ ; this equivalence arises because the slope and curvature parameters are pinned down by the equalization of the consumption decisions. We note that

$$\lim_{\vartheta \rightarrow \infty} \left\{ -\vartheta \log \left( 1 - (R-1) R \frac{\omega_\eta^2}{2\vartheta} \right) \right\} = \frac{(R-1)R}{2} \omega_\eta^2,$$

so that the two expressions are equal if  $\vartheta = \infty$  and  $\beta^{RB-RI} = 1/R$ . We also note that

$$\frac{\partial}{\partial \vartheta} \left( -\vartheta \log \left( 1 - (R-1) R \frac{\omega_\eta^2}{2\vartheta} \right) \right) < 0.$$

Thus, we have

$$-\vartheta \log \left( 1 - (R-1) R \frac{\omega_\eta^2}{2\vartheta} \right) > \frac{(R-1)R}{2} \omega_\eta^2,$$

for finite  $\vartheta > 0$ . However, this result does not mean that RB-RI households have lower lifetime utility than RI agents do, conditional on being observationally equivalent, because  $\beta^{RB-RI} =$

$1/R - (R-1)\omega_\eta^2(\kappa, \vartheta)/(2\vartheta_0) < 1/R$  and  $\frac{\beta^{RB-RI}}{1-\beta^{RB-RI}}$  in (4.5) is increasing with  $\beta^{RB-RI}$ . In other words, under the OE, i.e., given the same levels of consumption and precautionary savings, the welfare costs of uncertainty of RB agents are affected by two channels: the log term,  $-\vartheta_0 \log\left(1 - (R-1)R\frac{\omega_\eta^2}{2\vartheta_0}\right)$ , and the discount factor term,  $\frac{\beta^{RB-RI}}{1-\beta^{RB-RI}}$ . To analyze the effects of finite capacity and the two types of robustness on the welfare costs of uncertainty, denote

$$\Gamma = -\frac{\beta^{RB-RI}}{1-\beta^{RB-RI}}\vartheta_0 \log(\beta^{RB-RI}R), \quad (4.6)$$

since  $\log\left(1 - (R-1)R\frac{\omega_\eta^2}{2\vartheta_0}\right) = \log(\beta^{RB-RI}R)$ . It is straightforward to show that

$$\begin{aligned} \frac{\partial \Gamma}{\partial \kappa} &= \left[ -\vartheta_0 \frac{1}{1-\beta^{RB-RI}} - \vartheta_0 \log(\beta^{RB-RI}R) \frac{1}{(1-\beta^{RB-RI})^2} \right] \frac{\partial \beta^{RB-RI}}{\partial \kappa} \\ &= -\vartheta_0 \frac{1-\beta^{RB-RI} + \log(\beta^{RB-RI}R)}{(1-\beta^{RB-RI})^2} \underbrace{\frac{\partial \beta^{RB-RI}}{\partial \omega_\eta^2}}_{<0} \underbrace{\frac{\partial \omega_\eta^2}{\partial \kappa}}_{<0} < 0 \end{aligned}$$

if  $1 - \beta^{RB-RI} + \log(\beta^{RB-RI}R) > 0$ . Note that given that  $R = 1.02$  and  $\beta^{RB-RI} \in [0.9, 0.99]$ ,  $1 - \beta^{RB-RI} + \log(\beta^{RB-RI}R) > 0$  always holds. That is, for reasonable discount factors, the welfare loss is decreasing with finite capacity,  $\kappa$ . This result is similar to that obtained in Luo (2008) and Luo and Young (2009). Similarly, we have

$$\frac{\partial \Gamma}{\partial \vartheta} = -\vartheta_0 \frac{1-\beta^{RB-RI} + \log(\beta^{RB-RI}R)}{(1-\beta^{RB-RI})^2} \underbrace{\frac{\partial \beta^{RB-RI}}{\partial \omega_\eta^2}}_{<0} \underbrace{\frac{\partial \omega_\eta^2}{\partial \vartheta}}_{>0} > 0$$

if  $1 - \beta^{RB-RI} + \log(\beta^{RB-RI}R) > 0$ . That is, for reasonable discount factors, the welfare loss is decreasing with type ii robustness (i.e., a reduction in  $\vartheta$ ), the concern about misspecifying the Kalman gain. The intuition is simple: the smaller the value of  $\vartheta$  (i.e., the stronger the preference for robustness in the Kalman gain), the larger the value of  $\theta$  and the smaller the amount of the total uncertainty facing the consumer,  $\omega_\eta^2$ . Figure 6.7 illustrates how  $\vartheta$  and  $\kappa$  affect the welfare loss measured by  $\Gamma$  given  $\vartheta_0 = 1$  and  $R = 1.02$ . It clearly shows that  $\Gamma$  is increasing with  $\vartheta$  and decreasing with  $\kappa$ .

Given the complexity of the expression for  $\frac{\partial \Gamma}{\partial \vartheta_0}$  that measures the effects of type i robustness

for the welfare costs,

$$\frac{\partial \Gamma}{\partial \vartheta_0} = -\frac{1 - \beta^{RB-RI} + \log(\beta^{RB-RI} R)}{(1 - \beta^{RB-RI})^2} \left( \frac{1}{R} - \beta^{RB-RI} \right) - \frac{\beta^{RB-RI}}{1 - \beta^{RB-RI}} \log(\beta^{RB-RI} R),$$

we cannot obtain an explicit result about how  $\vartheta_0$  affects  $\Gamma$  via interacting with  $\kappa$  and  $\vartheta$ . We thus use a figure to illustrate how  $\vartheta_0$  affects the welfare costs  $\Gamma$ . Figure 6.8 illustrates the effects of  $\vartheta_0$  and  $\vartheta$  on  $\Gamma$  when  $R = 1.02$  and  $\kappa = 0.3$  bits.<sup>14</sup> It is clear from the figure that given  $\kappa$ , the welfare loss is decreasing with type i robustness (i.e., a reduction in  $\vartheta_0$ ), the concern about misspecifying the Kalman filtering equation hitting the income shock and the endogenous noise. The intuition for this result is that under the OE, i.e., given the same levels of consumption and precautionary savings,  $\vartheta_0$  affects the welfare costs of uncertainty via two channels: the log term,  $-\vartheta_0 \log\left(1 - (R-1)R\frac{\omega_\eta^2}{2\vartheta_0}\right)$ , and the discount factor term,  $\frac{\beta^{RB-RI}}{1 - \beta^{RB-RI}}$ , in (4.5).<sup>15</sup> The log term is increasing with the degree of type i robustness (i.e., a reduction in  $\vartheta_0$ ), while the discount factor term is decreasing with type i robustness because  $\beta^{RB-RI}$  is decreasing with type i robustness; and the effect of  $\vartheta_0$  on the discount factor term dominates that of  $\vartheta_0$  on the log term. This result implies that the costs of uncertainty could be less for RB-RI agents with stronger preference for type i robustness than RI agents whenever the parameters are such that their observable behavior is identical.

Furthermore, we can use the two value functions, (4.3) and (4.5), to compute the *marginal* welfare losses due to RI at different levels of (i) channel capacities ( $\kappa$ ), (ii) type i robustness ( $\vartheta_0$ ), and (iii) type ii robustness ( $\vartheta$ ). Specifically, following Barro (2007), the marginal welfare costs (mwc) due to finite capacity in the RB-RI model and the RI model can be written as

$$\begin{aligned} \text{mwc}^{RB-RI} &= \frac{\partial \widehat{v}^{RB-RI} / \partial \kappa}{(\partial \widehat{v}^{RB-RI} / \partial \widehat{s}_0) \widehat{s}_0} \\ &= \frac{-\frac{1}{2} \left[ \frac{\beta^{RB-RI}}{1 - \beta^{RB-RI}} \frac{R-1}{1 - (R-1)\omega_\eta^2 / (2\vartheta)} + \frac{R-1}{(1 - \beta^{RB-RI})^2} \log\left(1 - (R-1)\frac{\omega_\eta^2}{2\vartheta_0}\right) \right] \frac{\partial \omega_\eta^2}{\partial \theta} \frac{\partial \theta}{\partial \kappa}}{-(R-1)R\widehat{s}_0^2 + R\bar{c}\widehat{s}_0} \\ \text{mwc}^{RI} &= \frac{\partial \widehat{v} / \partial \kappa}{(\partial \widehat{v} / \partial \widehat{s}_0) \widehat{s}_0} = \frac{-\frac{R}{2} \frac{\partial \omega_\eta^2}{\partial \theta} \frac{\partial \theta}{\partial \kappa}}{-(R-1)R\widehat{s}_0^2 + R\bar{c}\widehat{s}_0} \end{aligned}$$

respectively. Note that here  $\partial \widehat{v}^{RB-RI} / \partial \kappa$  and  $\partial \widehat{v}^{RB-RI} / \partial \theta$  are evaluated for given  $\widehat{s}_0$ , and the welfare costs due to RI are compared with that from a small proportional change in the

<sup>14</sup>The result is robust to other values of channel capacity  $\kappa$ .

<sup>15</sup>Note that  $\vartheta_0$  has no effect on the total uncertainty  $\omega_\eta^2$ .

initial level of the perceived state  $\widehat{s}_0$ . Therefore, the following ratio,  $\pi$ , measures the *relative* marginal welfare losses due to RI in the two economies at various capacities ( $\theta$ ):

$$\pi = \frac{\text{mwc}^{RB-RI}}{\text{mwc}^{RI}} = \frac{R-1}{1-\beta^{RB-RI}} \left( \frac{1}{R} + \frac{\log(\beta^{RB-RI} R)}{1-\beta^{RB-RI}} \right). \quad (4.7)$$

Expression (4.7) clearly shows that this ratio is different at various capacities and is determined by  $\beta^{RB-RI}$  that is a function of  $\kappa$ ,  $\vartheta$ ,  $\vartheta_0$ , and labor income uncertainty given  $R$ . Figures 6.9 and 6.10 clearly show that the relative marginal welfare losses of RI-RB agents to RI agents are decreasing with type i robustness (i.e., decreasing with  $\vartheta_0$ ), increasing with type ii robustness (i.e., decreasing with  $\vartheta$ ), and increasing with  $\kappa$ .

## 5. Conclusion

This paper has provided a characterization of the consumption-savings behavior of a single agent who has a preference for robustness (worries about model misspecification) and limited information-processing ability. Specifically, we discuss how two concerns about model misspecification – (i) concerns about the disturbances to the perceived permanent income and (ii) concerns about the Kalman gain – interact with finite information-processing capacity to affect the optimal choices of consumption and savings in an otherwise standard permanent income model. After obtaining the explicit solution, we showed that there exists an observational equivalence between the RB-RI model and the RI model with an adjusted discount factor determined by the levels of the two types of robustness, finite capacity, and income uncertainty, extending the results of HST (1999) to a broader class of models. We finally examined the welfare effects of the preference for robustness via interacting with inattention and income uncertainty. Within the observationally-equivalent class of models the welfare costs of uncertainty are not constant – agents in the RI model with the discount factor  $\beta = 1/R$  suffer more from income uncertainty than those in the RB-RI model with lower adjusted discount factors.

## 6. Appendix (Not for Publication)

### 6.1. Solving the Robust-RI Model with Concerns about the Income Shock

To solve the Bellman equation (2.9) subject to (2.12), we conjecture that

$$v(\hat{s}_t) = -C - B\hat{s}_t - A\hat{s}_t^2, \quad (6.1)$$

where  $A$ ,  $B$ , and  $C$  are constants to be determined. Substituting this guessed value function into the Bellman equation (2.9) gives

$$-C - B\hat{s}_t - A\hat{s}_t^2 = \max_{c_t} \min_{\nu_t} \left\{ -\frac{1}{2} (c_t - \bar{c})^2 + \beta E_t [\vartheta \nu_t^2 - C - B\hat{s}_{t+1} - A\hat{s}_{t+1}^2] \right\}, \quad (6.2)$$

subject to (2.12):

$$\hat{s}_{t+1} = R\hat{s}_t - c_t + \omega_\zeta \nu_t + \eta_{t+1}.$$

We can do the min and max operations in any order, so we choose to do the minimization first. The first-order condition for  $\nu_t$  is

$$2\vartheta \nu_t - 2AE_t [\omega_\zeta \nu_t + R\hat{s}_t - c_t] \omega_\zeta - B\omega_\zeta = 0,$$

which means that

$$\nu_t = \frac{B + 2A(R\hat{s}_t - c_t)}{2(\vartheta - A\omega_\zeta^2)} \omega_\zeta. \quad (6.3)$$

Substituting (6.3) back into (6.2) gives

$$-A\hat{s}_t^2 - B\hat{s}_t - C = \max_{c_t} \left\{ -\frac{1}{2} (\bar{c} - c_t)^2 + \beta E_t \left[ \vartheta \left[ \frac{B + 2A(R\hat{s}_t - c_t)}{2(\vartheta - A\omega_\zeta^2)} \omega_\zeta \right]^2 - A\hat{s}_{t+1}^2 - B\hat{s}_{t+1} - C \right] \right\},$$

where  $\hat{s}_{t+1} = R\hat{s}_t - c_t + \omega_\zeta \nu_t + \eta_{t+1}$ . The first-order condition for  $c_t$  is

$$(\bar{c} - c_t) - 2\beta\vartheta \frac{A\omega_\zeta}{\vartheta - A\omega_\zeta^2} \nu_t + 2\beta A \left( 1 + \frac{A\omega_\zeta^2}{\vartheta - A\omega_\zeta^2} \right) (R\hat{s}_t - c_t + \omega_\zeta \nu_t) + \beta B \left( 1 + \frac{A\omega_\zeta^2}{\vartheta - A\omega_\zeta^2} \right) = 0.$$

Using the solution for  $\nu_t$  the solution for consumption is

$$c_t = \frac{2A\beta R}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \hat{s}_t + \frac{\bar{c} \left( 1 - A\omega_\zeta^2/\vartheta \right) + \beta B}{1 - A\omega_\zeta^2/\vartheta + 2\beta A}. \quad (6.4)$$

Substituting the above expressions into the Bellman equation gives

$$\begin{aligned}
& -A\widehat{s}_t^2 - B\widehat{s}_t - C \\
& = -\frac{1}{2} \left( \frac{2A\beta R}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \widehat{s}_t + \frac{-2\beta A\bar{c} + \beta B}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \right)^2 \\
& + \frac{\beta\vartheta\omega_\zeta^2}{\left(2(\vartheta - A\omega_\zeta^2)\right)^2} \left[ \frac{2AR(1 - A\omega_\zeta^2/\vartheta)}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \widehat{s}_t + B - \frac{2\bar{c}(1 - A\omega_\zeta^2/\vartheta)A + 2\beta AB}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \right]^2 \\
& - \beta A \left\{ \left[ \frac{R}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \widehat{s}_t - \frac{-B\omega_\zeta^2/\vartheta + 2c + 2B\beta}{2(1 - A\omega_\zeta^2/\vartheta + 2\beta A)} \right]^2 + \omega_\zeta^2 \right\} \\
& - \beta B \left[ \frac{R}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \widehat{s}_t - \frac{-B\omega_\zeta^2/\vartheta + 2c + 2B\beta}{2(1 - A\omega_\zeta^2/\vartheta + 2\beta A)} \right] - \beta C.
\end{aligned}$$

Collecting and matching terms, the constant coefficients turn out to be

$$\begin{aligned}
A &= \frac{\beta R^2 - 1}{2\beta - \omega_\zeta^2/\vartheta}, \\
B &= \frac{(\beta R^2 - 1)\bar{c}}{(R - 1)(\omega_\zeta^2/(2\vartheta) - \beta)}, \\
C &= \frac{R(\beta R^2 - 1)}{2(\beta R - R\omega_\zeta^2/2\vartheta)(R - 1)^2} ((R - 1)\omega_\zeta^2 + \bar{c}^2).
\end{aligned}$$

When  $\beta R = 1$ , they reduce to

$$A = \frac{R(R - 1)}{2 - R\omega_\zeta^2/\vartheta}, \quad (6.5)$$

$$B = -\frac{R\bar{c}}{1 - R\omega_\zeta^2/(2\vartheta)}, \quad (6.6)$$

$$C = \frac{R}{2(1 - R\omega_\zeta^2/(2\vartheta))} \omega_\zeta^2 + \frac{R}{2(1 - R\omega_\zeta^2/2\vartheta)(R - 1)} \bar{c}^2. \quad (6.7)$$

Substituting (6.5) and (6.6) into (6.4) yields the consumption function (2.15) in the text.

The proof that  $\Pi \in (0, 1)$  follows from an investigation of the second-order condition for the minimizing player.

## 6.2. Solving the Risk-sensitive RI Model

To solve the Bellman equation (2.17) subject to (2.18), we conjecture that

$$v(\hat{s}_t) = -C - B\hat{s}_t - A\hat{s}_t^2, \quad (6.8)$$

where  $A$ ,  $B$ , and  $C$  are constants to be determined. We can then evaluate  $E_t[\exp(-\alpha v(\hat{s}_{t+1}))]$  to obtain

$$\begin{aligned} & E_t[\exp(-\alpha v(\hat{s}_{t+1}))] \\ &= E_t[\exp(\alpha A\hat{s}_{t+1}^2 + \alpha B\hat{s}_{t+1} + \alpha C)] \\ &= E_t\left[\exp\left(\alpha A(R\hat{s}_t - c_t)^2 + \alpha B(R\hat{s}_t - c_t) + [2\alpha A(R\hat{s}_t - c_t) + \alpha B]\eta_{t+1} + \alpha A\eta_{t+1}^2 + \alpha C\right)\right] \\ &= (1 - 2c)^{-1/2} \exp\left(a + \frac{b^2}{2(1 - 2c)}\right), \end{aligned}$$

where

$$\begin{aligned} a &= \alpha A(R\hat{s}_t - c_t)^2 + \alpha B(R\hat{s}_t - c_t) + \alpha C, \\ b &= [2\alpha A(R\hat{s}_t - c_t) + \alpha B]\omega_\eta, \\ c &= \alpha A\omega_\eta^2. \end{aligned}$$

Thus, the distorted expectations operator can be written as

$$\begin{aligned} \mathcal{R}_t[v(\hat{s}_{t+1})] &= -\frac{1}{\alpha} \left\{ -\frac{1}{2} \log(1 - 2c) + a + \frac{b^2}{2(1 - 2c)} \right\} \\ &= \frac{1}{2\alpha} \log(1 - 2\alpha A\omega_\eta^2) - \frac{A}{1 - 2\alpha A\omega_\eta^2} (R\hat{s}_t - c_t)^2 - \frac{B}{1 - 2\alpha A\omega_\eta^2} (R\hat{s}_t - c_t) - \left[ C + \frac{\alpha B^2\omega_\eta^2}{2(1 - 2\alpha A\omega_\eta^2)} \right]. \end{aligned} \quad (6.9)$$

Maximizing the RHS of (6.9) with respect to  $c_t$  yields the first-order condition

$$-(c_t - \bar{c}) + \frac{2\beta A}{1 - 2\alpha A\omega_\eta^2} (R\hat{s}_t - c_t) + \frac{B\beta}{1 - 2\alpha A\omega_\eta^2} = 0,$$

which means that

$$c_t = \frac{2A\beta R}{1 - 2\alpha A\omega_\eta^2 + 2A\beta} \hat{s}_t + \frac{\bar{c}(1 - 2\alpha A\omega_\eta^2) + B\beta}{1 - 2\alpha A\omega_\eta^2 + 2A\beta}. \quad (6.10)$$

Substituting (6.10) and (6.8) into (2.17), and collecting and matching terms, the constant coefficients turn out to be

$$A = \frac{\beta R^2 - 1}{2\beta - 2\alpha\omega_\eta^2}, \quad (6.11)$$

$$B = \frac{(\beta R^2 - 1)\bar{c}}{(R-1)(\alpha\omega_\eta^2 - \beta)}, \quad (6.12)$$

$$C = \frac{R(\beta R^2 - 1)}{2(\beta R - R\alpha\omega_\eta^2)(R-1)^2} ((R-1)\omega_\eta^2 + \bar{c}^2). \quad (6.13)$$

Substituting (6.11) and (6.12) into (6.10) yields the consumption function (2.19) in the text.

### 6.3. Solving the Robust-RI Model with Concerns about the Income Shock and the Noise

To solve the Bellman equation (2.26) subject to (2.25),<sup>16</sup> we simply need to replace  $\omega_\zeta^2$  with  $\omega_\eta^2$  (we omit the steps) in the constant terms obtained in Appendix 6.1:

$$\begin{aligned} A &= \frac{\beta R^2 - 1}{2\beta - \omega_\eta^2/\vartheta}, \\ B &= \frac{(\beta R^2 - 1)\bar{c}}{(R-1)(\omega_\eta^2/(2\vartheta) - \beta)}, \\ C &= \frac{R(\beta R^2 - 1)}{2(\beta R - R\omega_\eta^2/2\vartheta)(R-1)^2} ((R-1)\omega_\eta^2 + \bar{c}^2), \end{aligned}$$

which reduces to

$$\begin{aligned} A &= \frac{R(R-1)}{2 - R\omega_\eta^2/\vartheta}, \\ B &= -\frac{R\bar{c}}{1 - R\omega_\eta^2/(2\vartheta)}, \\ C &= \frac{R}{2(1 - R\omega_\eta^2/2\vartheta)}\omega_\eta^2 + \frac{R}{2(1 - R\omega_\eta^2/2\vartheta)(R-1)}\bar{c}^2. \end{aligned}$$

when  $\beta R = 1$ . Using (6.5) and (6.6), we can obtain the consumption function (2.27) in the text.

---

<sup>16</sup>Note that the only difference between (2.12) and (2.25) is that the distortion term in (2.12) is  $\omega_\zeta\nu_t$ , whereas it is  $\omega_\eta\nu_t$  in (2.25).

## References

- [1] Backus, David K., Bryan R. Routledge, and Stanley E. Zin (2004), “Exotic Preferences for Macroeconomists,” *NBER Macroeconomics Annual 2004*, 319-414.
- [2] Barro, Robert J. (2007), “On the Welfare Costs of Consumption Uncertainty,” manuscript.
- [3] Epstein, Larry G. and Stanley E. Zin (1989), “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework,” *Econometrica* 57(4), 937-969.
- [4] Flavin, Marjorie A. (1981), “The Adjustment of Consumption to Changing Expectations About Future Income,” *Journal of Political Economy* 89(5), 974-1009.
- [5] Hall, Robert E (1978), “Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence,” *Journal of Political Economy* 86(6), 971-987.
- [6] Hansen, Lars Peter and Thomas J. Sargent (1995), “Discounted Linear Exponential Quadratic Gaussian Control,” *IEEE Transactions on Automatic Control* 40, 968-971.
- [7] Hansen, Lars Peter and Thomas J. Sargent (2005), “Robust Estimation and Control under Commitment,” *Journal of Economic Theory* 124(9), 258-301.
- [8] Hansen, Lars Peter and Thomas J. Sargent (2007), *Robustness*, Princeton University Press.
- [9] Hansen, Lars Peter, Thomas J. Sargent, and Neng Wang (2002) “Robust Permanent Income and Pricing with Filtering,” *Macroeconomic Dynamics* 6(1), 40-84.
- [10] Jacobson, David H. (1973), “Optimal Stochastic Linear Systems with Exponential Performance Criteria and Their Relation to Deterministic Differential Games,” *IEEE Transactions on Automatic Control* 18, 124-131.
- [11] Kasa, Kenneth (2006), “Robustness and Information Processing,” *Review of Economic Dynamics* 9(1), 1-33.
- [12] Luo, Yulei (2008), “Consumption Dynamics under Information Processing Constraints,” *Review of Economic Dynamics* 11(2), 366-385.
- [13] Luo, Yulei, Jun Nie, and Eric R. Young (2010), “Robustness, Information-Processing Constraints, and the Current Account in Small Open Economies,” manuscript.

- [14] Luo, Yulei and Eric R. Young (2010a), “Risk-sensitive Consumption and Savings under Rational Inattention,” forthcoming, *American Economic Journal: Macroeconomics*.
- [15] Luo, Yulei and Eric R. Young (2010b), “Signal Extraction and Rational Inattention,” manuscript.
- [16] Muth, John F. (1960), “Optimal Properties of Exponentially Weighted Forecasts,” *Journal of the American Statistical Association* 55, 299-306.
- [17] Sims, Christopher A. (2003), “Implications of Rational Inattention,” *Journal of Monetary Economics* 50(3), 665-690.
- [18] van der Ploeg, Frederick (1993), “A Closed-Form Solution for a Model of Precautionary Saving,” *Review of Economic Studies* 60(2), 385-95.
- [19] Wang, Neng (2004), “Precautionary Saving and Partially Observed Income,” *Journal of Monetary Economics* 51(8), 1645-1681.
- [20] Whittle, Peter (1981), “Risk-Sensitive Linear/Quadratic/Gaussian Control,” *Advances in Applied Probability* 13, 764-777.
- [21] Whittle, Peter (1990), *Risk-Sensitive Optimal Control*, John Wiley and Sons.

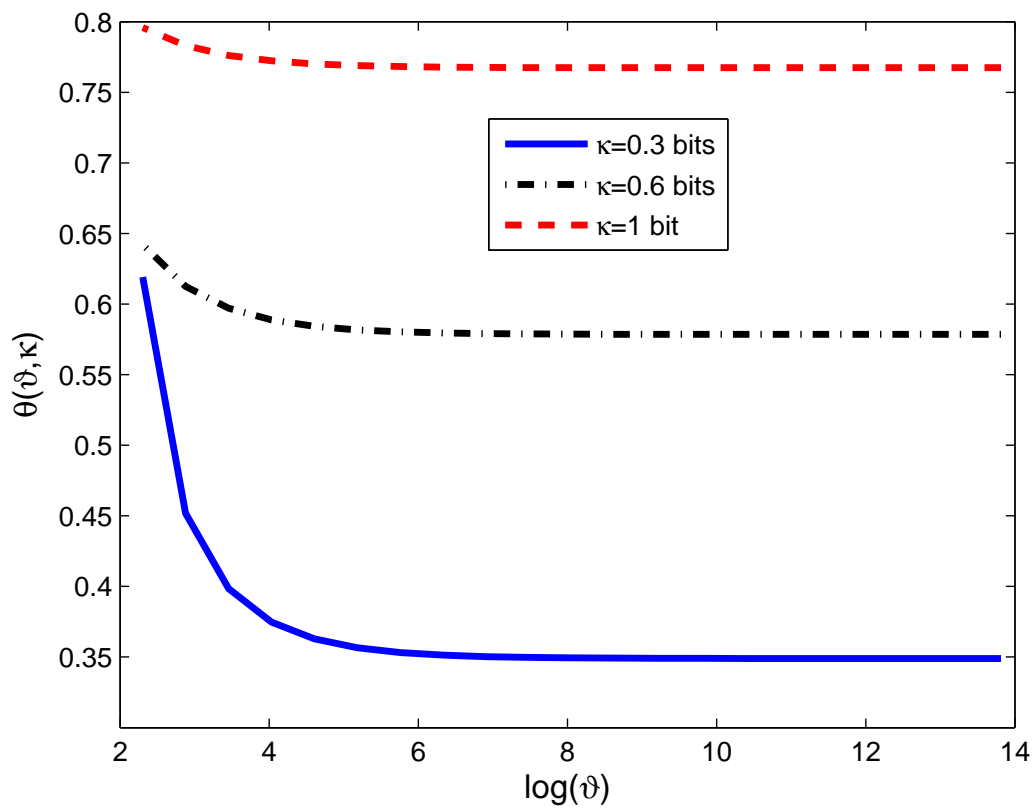


Figure 6.1: Effects of RB and RI on Robust Kalman Gain  $\theta$

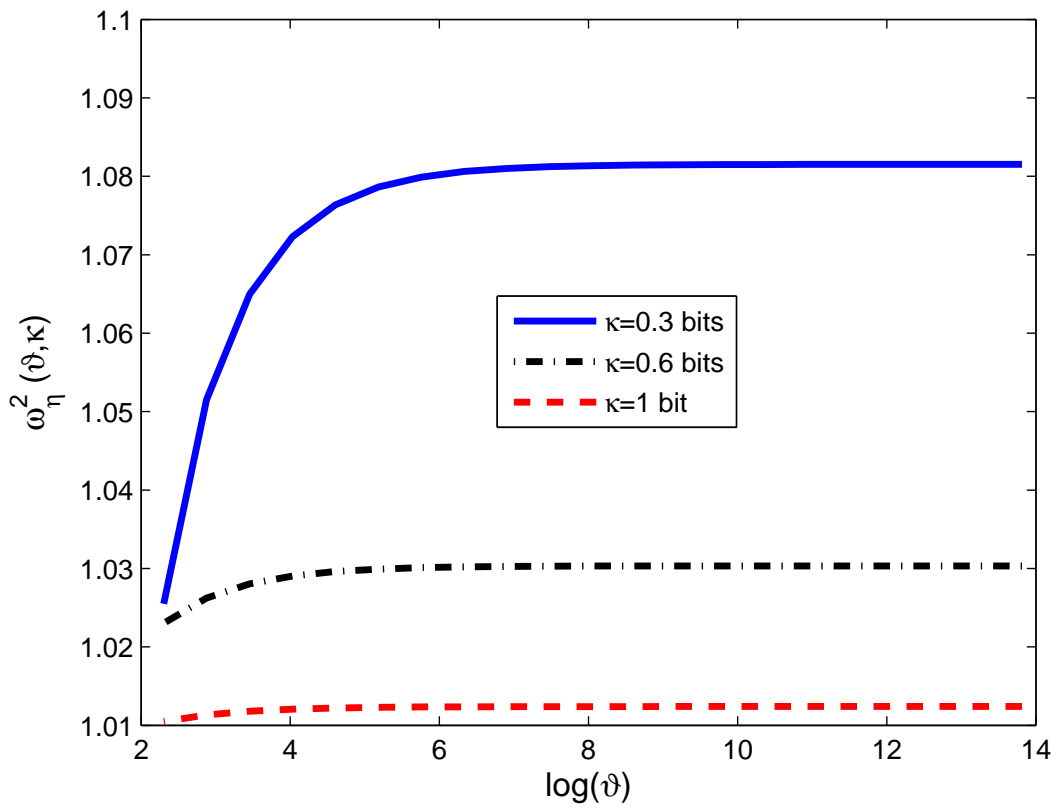


Figure 6.2: Effects of Robustness and Capacity,  $\vartheta$  and  $\kappa$ , on  $\omega_\eta^2$

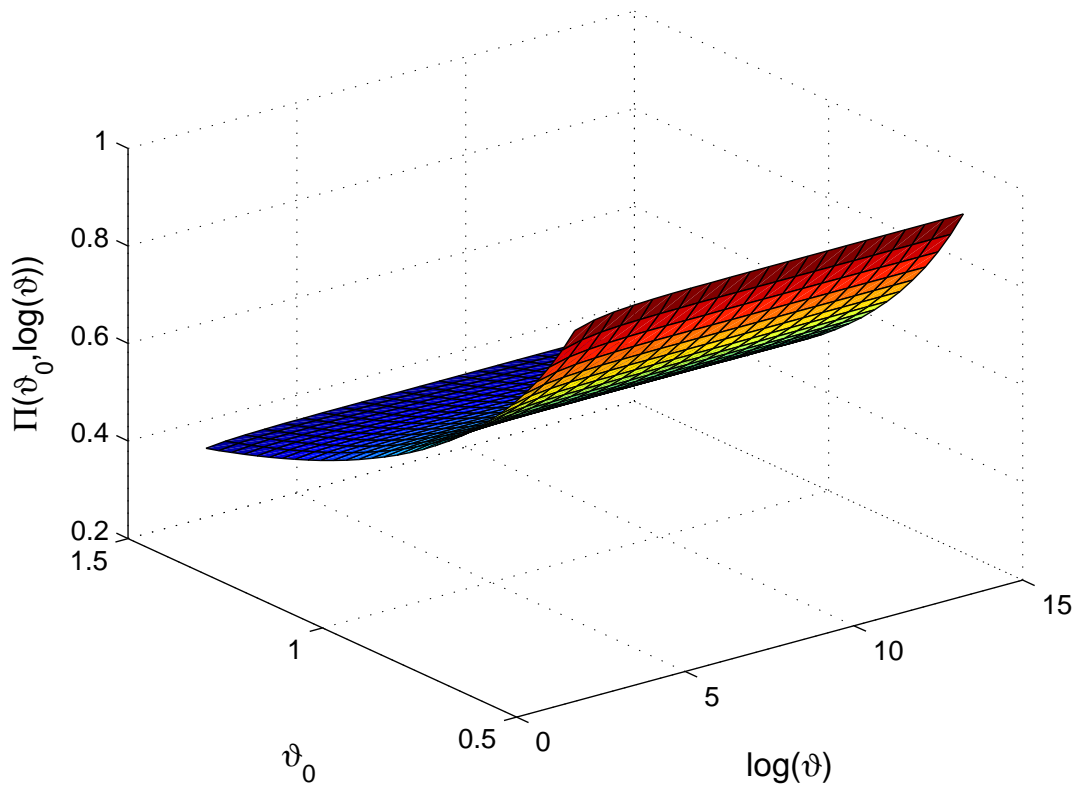


Figure 6.3: Effects of Two Types of Robustness,  $\vartheta_0$  and  $\vartheta$ , on  $\Pi$

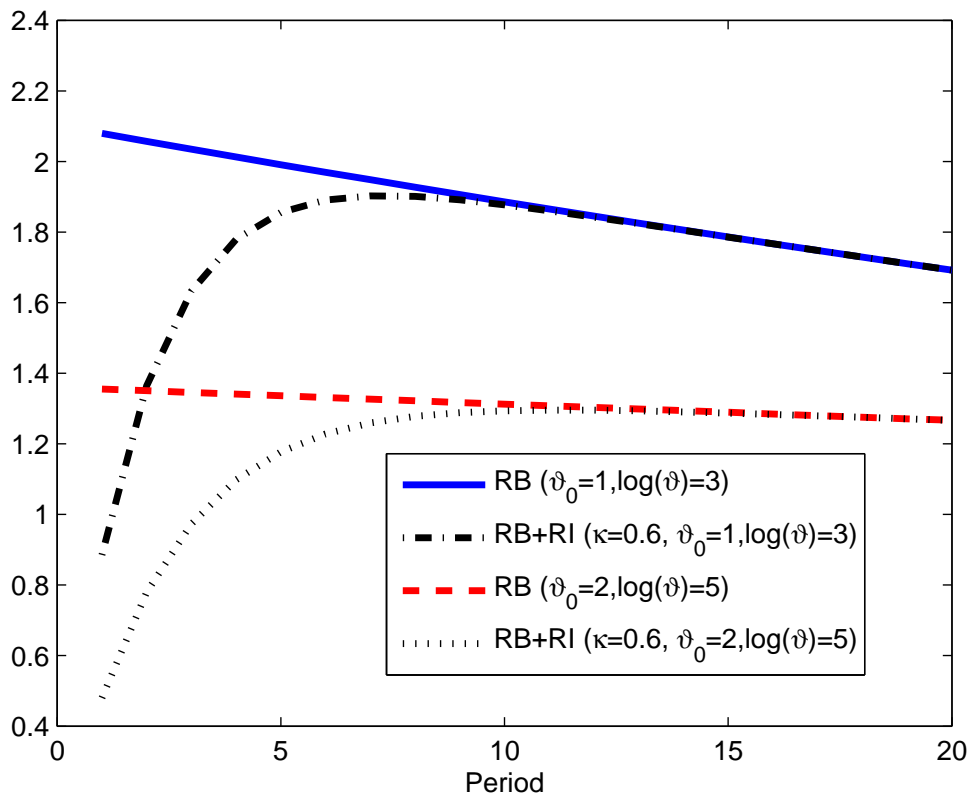


Figure 6.4: Impulse Responses of Consumption to Income Shock

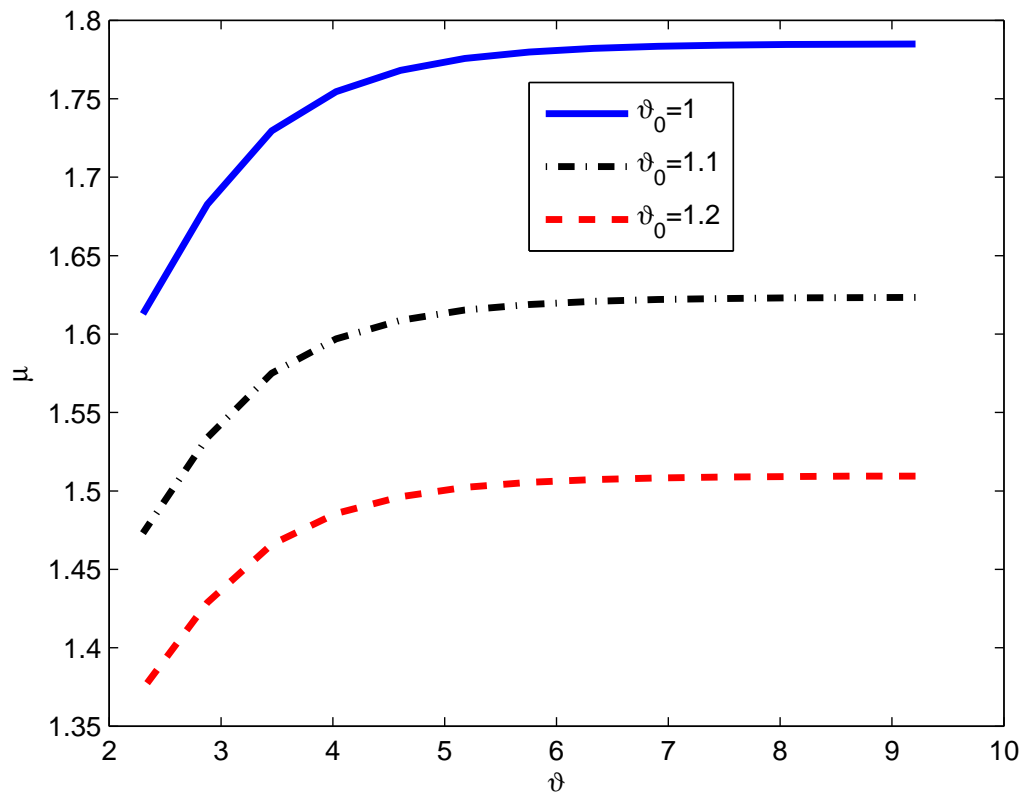


Figure 6.5: The Relative Volatility of Consumption to Income under RB and RI

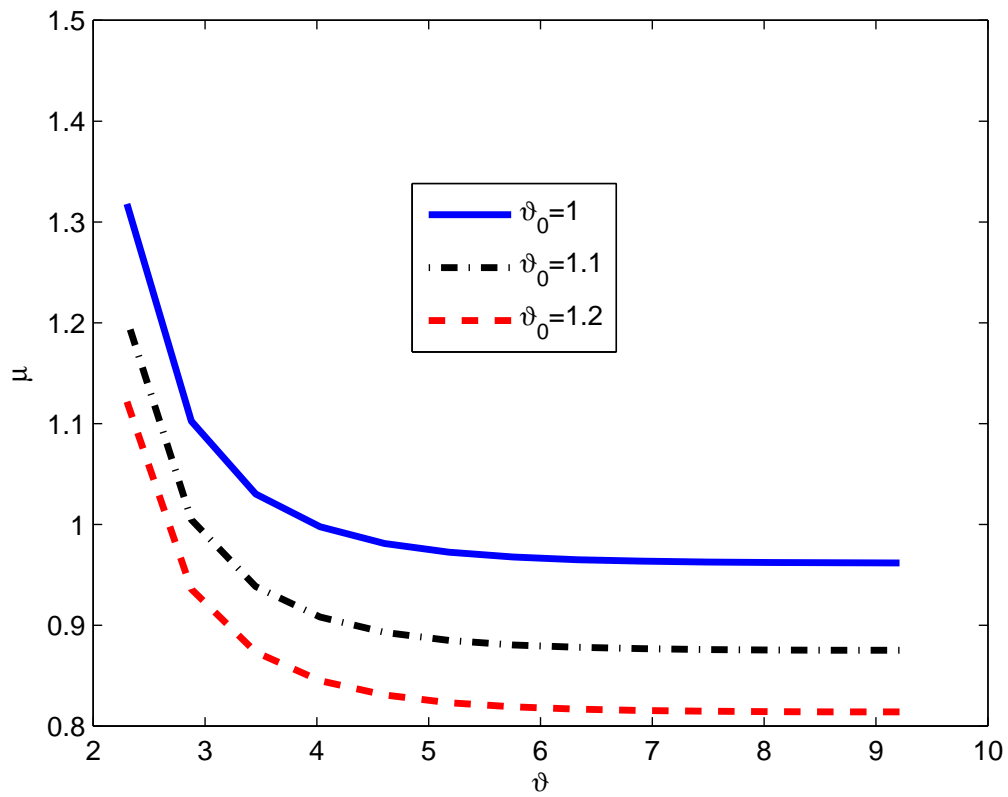


Figure 6.6: The Relative Volatility of Consumption to Income under RB-RI without Noise

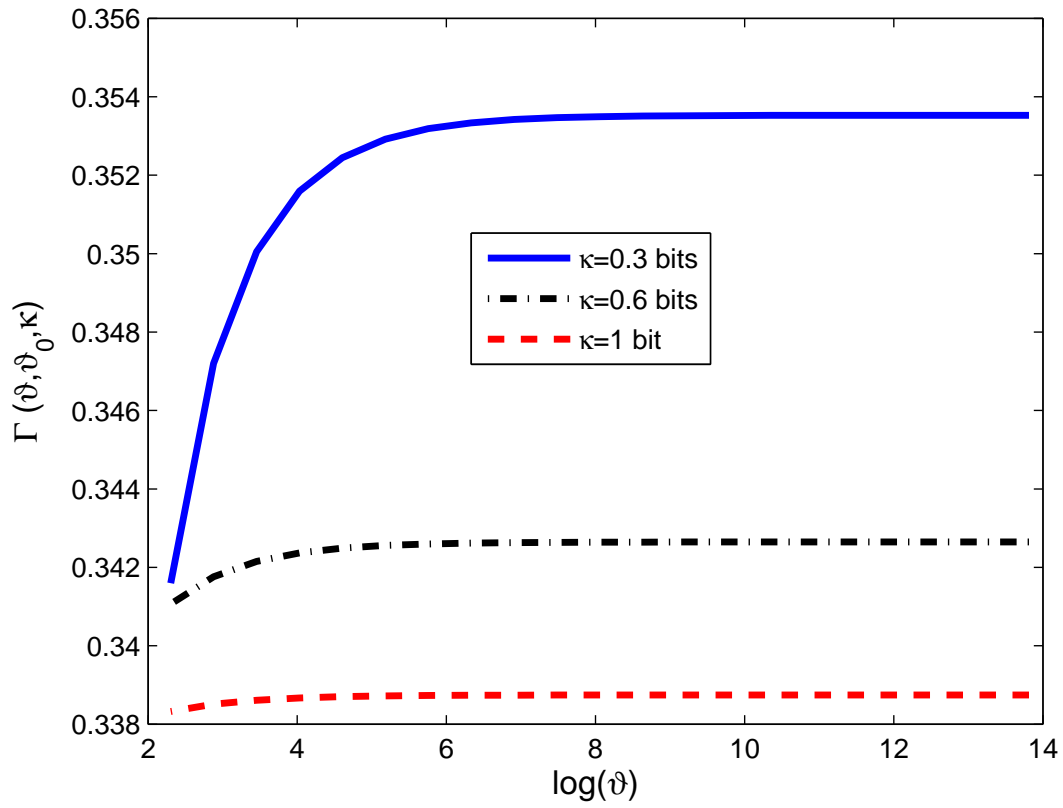


Figure 6.7: Effects of  $\vartheta$  and  $\kappa$  on Welfare Costs

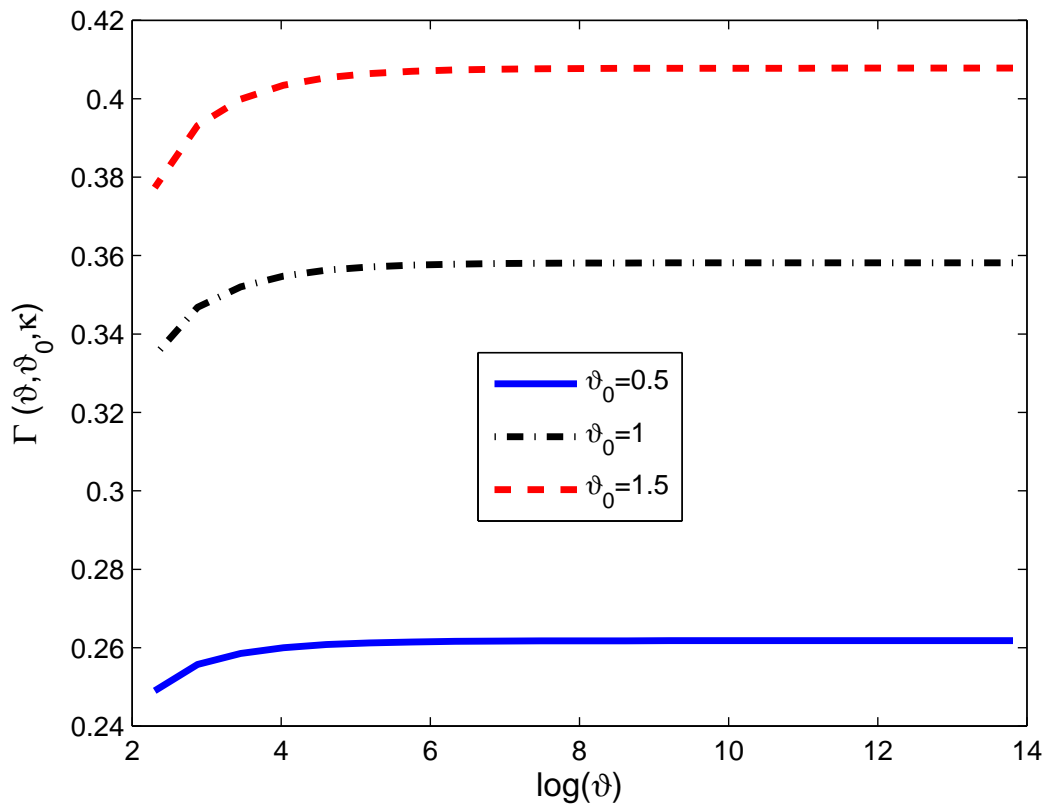


Figure 6.8: Effects of  $\vartheta_0$  and  $\kappa\vartheta$  on Welfare Costs

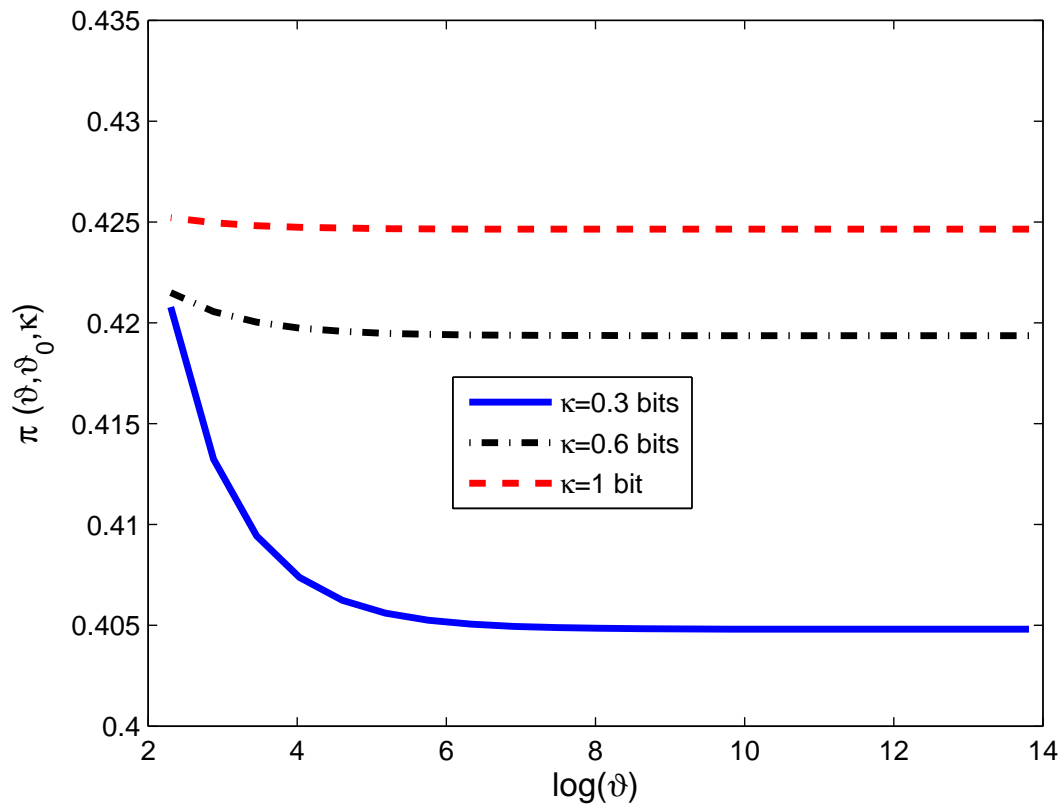


Figure 6.9: Effects of  $\vartheta$  and  $\kappa$  on Relative Marginal Welfare Losses

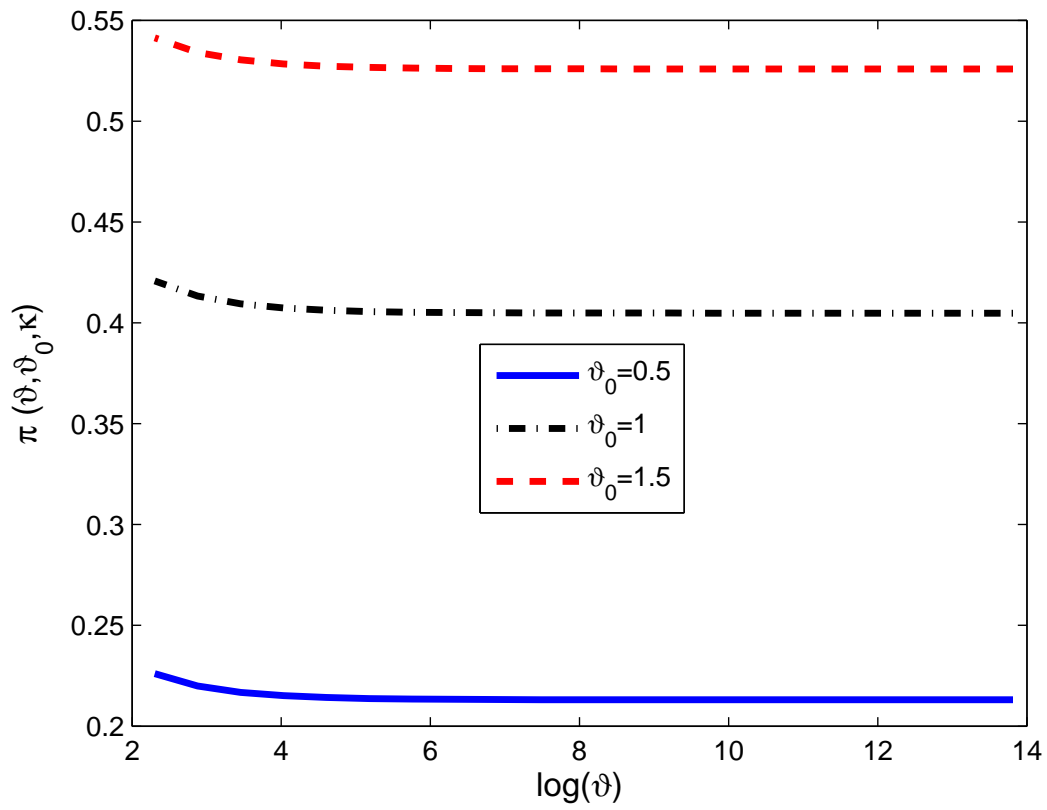


Figure 6.10: Effects of  $\vartheta$  and  $\vartheta_0$  on Relative Marginal Welfare Losses