Ambiguity, Low Risk-Free Rates, and Consumption Inequality*

Yulei Luo†
The University of Hong Kong

Jun Nie‡
Federal Reserve Bank of Kansas City

Eric R. Young§
Zhejiang University and University of Virginia

January 7, 2019

Abstract

The failure of macroeconomists to predict the Great Recession suggests possible misspecification of existing macroeconomic models. If agents bear in mind this misspecification, how are their optimal decisions changed and how large are the associated welfare costs? To shed light on these questions, we develop a tractable continuous-time recursive utility (RU) version of the Huggett (1993) model to study the effects of model uncertainty due to a preference for robustness (RB, or ambiguity aversion). We show that RB reduces the equilibrium interest rate and the relative dispersion of consumption to income, making them closer to the data, but our benchmark model cannot match the observed relative dispersion. An extension to a RU-RB model with a risky asset is successful at matching this dimension. Our analysis implies the welfare costs of model uncertainty are sizable: a typical consumer in equilibrium would be willing to sacrifice about 15 percent of his initial wealth to remove the model uncertainty he faces.

JEL Classification Numbers: *C61, D81, E21*.

Keywords: Ambiguity, Robustness, Precautionary Savings, Low Interest Rates, Consumption and Income Inequality, General Equilibrium.

---

*We are grateful to Evan Anderson, Martin Ellison, Ken Kasa, Lars Hansen, Tom Sargent, Yenen Sun, Neng Wang, Shenghao Zhu, and seminar and conference participants at Federal Reserve Bank of Kansas City, National University of Singapore, Seoul National University, Southwestern University of Finance and Economics, Wuhan University, the 6th Shanghai Macroeconomic Workshop, and the Asian Meeting of the Econometric Society for helpful comments and discussions related to this paper. The financial support from the General Research Fund (GRF No. HKU17500117 and HKU17500515) in Hong Kong is acknowledged. The views expressed here are the opinions of the authors only and do not necessarily represent those of the Federal Reserve Bank of Kansas City or the Federal Reserve System. All remaining errors are our responsibility.

†Faculty of Business and Economics, The University of Hong Kong, Hong Kong. Email address: yulei.luo@gmail.com.

‡Research Department, Federal Reserve Bank of Kansas City. E-mail: jun.nie@kc.frb.org.

§Department of Economics, University of Virginia, and Academy of Financial Research, Zhejiang University. E-mail: ey2d@virginia.edu.

1. Introduction

This paper is motivated by three important macroeconomic phenomena observed in the U.S. economy in the last few decades. The first is the declined real interest rate (Figure 1), which has been intensively discussed by macroeconomists and policymakers recently. Understanding changes in the real interest rate is important because it contains signals for future growth prospects and matters for financial stability (Fischer 2016). The second phenomenon is the decline in consumption inequality relative to income inequality, also shown in Figure 1. Consumption inequality, an important welfare measure, has received significantly less attention compared to the large literature studying income inequality. The third phenomenon is increased concerns about possible misspecification of existing macroeconomic models following the recent financial crisis – the failure of macroeconomists to predict the 2007-2009 Great Recession at least suggests existing macroeconomic models are missing critical ingredients. Some recent literature suggests that model uncertainty, the "unknown unknowns," played an important role in the recent economic and financial crises (see Caballero and Krishnamurthy 2008 and Boyarchenko 2012 as examples). The main purpose of this paper is to offer a unified framework to illustrate how model uncertainty, which measures the degree of model misspecification that will be defined more clearly below, can help explain the low real interest rates and the relative inequality/dispersion of consumption growth to income growth in the US economy.[1] We then use this framework to examine the welfare and policy implications of model uncertainty.

Explaining low real interest rates and consumption dispersion through the lens of model uncertainty is new in the literature. The determination of the risk-free rate is the subject of a large literature. Broadly speaking, there are three major factors determining the equilibrium real interest rate: (i) aggregate savings, (ii) aggregate investment, and (iii) the relative demand for safe versus risky assets. Huggett (1993) and Aiyagari (1994) study the importance of labor income risk in determining aggregate savings and equilibrium interest rates in a heterogenous-agent framework. They find that labor income risk itself cannot explain the observed low real interest rate. Summers (2014) and Blanchard et al. (2014) argue that the low equilibrium real interest rates in the U.S. and other advanced economies were caused by increases in global savings. Their explanations for higher savings rely on demographic trends (such as an aging population) or capital flows from emerging economies to advanced economies. On the other hand, there is relatively little work studying consumption dispersion. Blundell et al. (2008) show that consumption and income inequality diverged during the 1980-2004 period and explain it through the channels of the persistence of income shocks and the degree of consumption insurance. Krueger and Perri (2004) try to account for the observed difference between consumption and income inequality using a calibrated incomplete-markets model with limited commitment. All these papers have assumed

[1] In this paper we use "inequality" and "dispersion" interchangebly to describe the cross-sectional distributions of consumption and income.

the models that agents use to make decisions are correct. In this paper, we relax this assumption and show that a reasonable level of model uncertainty can help explain both the low real interest rate and consumption dispersion.

The basic idea of model uncertainty we pursue in this paper is based on Hansen and Sargent (1995) who first introduced a preference for robustness (RB) into economic models as a way to capture an agent's fear that their model is misspecified (a form of ambiguity aversion).[2] In RB models, agents have in mind a reference model that represents their best estimate of the model governing the dynamics of state variables. However, they are worried that this reference model is incorrect in some way, and they make their optimal decisions under the worst-case scenario (i.e., as if the subjective distribution over shocks is chosen by an evil agent whose aim is to minimize their expected lifetime utility).

Our paper follows this line of research and develops a tractable continuous-time general equilibrium model in which consumers have a recursive utility with an aversion to ambiguity and face uninsurable labor income.[3] We use this model to show that the key features of Figure 1 arise naturally from the interactions of ambiguity, the separation of risk aversion and intertemporal substitution, and fundamental risk. Our paper also contributes to the rapidly growing literature of using continuous-time heterogeneous-agent models to address inequality issues including Benhabib et al. (2011), Gabaix et al. (2016), and Kasa and Lei (2017). The key difference between their work and this paper is that the other papers focus on income or wealth distributions while we focus mainly on consumption inequality and equilibrium asset returns. Related to the discrete-time rational expectations (RE) heterogenous-agent models (such as Huggett (1993) and Aiyagari (1994)), we provide analytical solutions in a heterogenous-agent model to help illustrate the key mechanisms through which model uncertainty influences our results.

Our analysis generates three main findings. First, we derive analytic solutions for a model featuring robustness, recursive exponential utility, uninsurable labor income, and portfolio choice. We characterize how the equilibrium interest rate is affected by RB through interactions with the elasticity of intertemporal substitution (EIS) and constant absolute risk aversion (CARA). An increase in the EIS affects the equilibrium interest rate through two distinct channels: (i) high EIS increases the relative importance of the impatience-induced dissaving effect (*the direct channel*) and (ii) reduces the precautionary saving amount by reducing the effect of RB on the effective coefficient of absolute risk aversion (*the indirect channel*).[4] We then show that a general equilibrium

[2]See Hansen and Sargent (2007) for a textbook treatment on robustness. It is worth noting that we can use either robust decision-making or recursive multiple-prior utility (Chen and Epstein 2002) due to ambiguity aversion to capture the same idea that the decision maker is concerned that their model is misspecified. We follow Hansen and Sargent (2007) because it is technically easier.

[3]See Anderson, Hansen, and Sargent (2003); Maenhout (2004); Ju and Miao (2012); and Kasa and Lei (2017) for the applications of robustness in continuous-time models.

[4]The effective coefficient of absolute risk aversion ($\widetilde{\gamma}$) is determined by the interaction between the true coefficient of absolute risk aversion (γ), the EIS (ψ), and the degree of RB (ϑ) via the following formula: $\widetilde{\gamma} = \gamma + \vartheta/\psi$.

under RB can be constructed in the vein of Huggett (1993) and Wang (2003).[5] In general equilibrium, more model uncertainty leads to stronger precautionary savings effects and lower interest rates. In addition, we show that the relative dispersion of consumption to income is determined only by the equilibrium interest rate and the persistenceof the income process. In particular, the relative consumption dispersion decreases with RB if the income process is stationary.

Second, our model succeeds quantitatively in explaining the equilibrium interest rate and the relative dispersion of consumption to income in the data. In the US economy, the real risk-free interest rate averaged 1.87 percent between 1981 and 2010, and averaged 1.37 percent if the sample is extended to 2015.[6] To explain the observed real interest rate levels, an RE model without model uncertainty would require the coefficient of risk aversion parameter to be as high as 15 to 24 when the EIS takes reasonable values.[7] However, when consumers take into account model uncertainty, the model can generate a low equilibrium interest rate with much lower values for the coefficient of risk aversion.[8] Moreover, the reduction in the equilibrium interest rate following the 2007-2009 financial crisis can be rationalized by possible increases in concerns about model misspecification during this period.

The relative consumption dispersion turns out to depend only on the equilibrium interest rate in the benchmark model; as a result, the benchmark model cannot match both the real interest rate and the relative consumption dispersion with the same parametrization. However, we show that extending our model to include risky assets fixes this problem.[9] We show that the presence of the risky asset affects equilibrium precautionary saving through two channels: (i) the risky asset can be used to hedge labor income risk and (ii) the risky asset increases the amount of total risk faced by agents if the net supply of the risky asset is positive. We find that the relative dispersion of consumption to income is increasing in the supply of the risky asset and the risk-free rate is decreasing. For plausibly calibrated parameter values of RB, we find that the extended model can simultaneously generate the observed low risk free rate and high relative dispersion of consumption to income in the U.S. Moreover, if we divide our sample period into two periods, we find that allowing for time-varying degree of RB can help account for the observed decline in the relative

[5]Wang (2003) constructs a general equilibrium under RE in the same Huggett-type model economy with CARA expected utility.

[6]Here the numbers are computed using the Consumer Price Index (CPI) to measure inflation. Using Personal consumption expenditures (PCE) leads to similar results. See Table 1 for different measures of the risk-free rates.

[7]We normalize the mean consumption level to be 1, so the coefficient of relative risk aversion is equal to the coefficient of absolute risk aversion. While estimates of the EIS fluctuate wildly, in general risk aversion coefficients about 10 are not considered reasonable in the macroeconomic literature; finance is different, where very high risk aversion is seemingly accepted without comment.

[8]Barillas et al. (2009) show that most of the observed high market price of risk in the U.S. can be reinterpreted as a market price of model uncertainty and the risk-aversion parameter can thus be reinterpreted as measuring the representative agent's doubts about the model specification.

[9]A related literature has tried to understand how the supply of safe assets matters for low interest rates (Barro et al. 2017); our two-asset extension should be viewed as grappling with this question as well.

inequality of consumption to income.[10]

Finally, we evaluate the welfare costs associated with model uncertainty in two dimensions: the overall welfare costs and the marginal welfare costs. First, we assess the welfare gains associated with eliminating model uncertainty–what would an agent pay in order to find out exactly the true stochastic process governing his income? We find that these numbers are large–the cost can be as high as 15 percent of initial wealth. These costs are increasing in the degree of RB and decreasing in the EIS. Second, we provide formulas to evaluate, at the margin, the welfare costs/gains of changes in the degree of RB and labor income volatility. We find that, under our calibrated parameter values, a 10 percent increase in the degree of model uncertainty leads to a welfare cost equivalent to a 1.23 percent reduction in initial income, and this welfare cost is significantly larger in a more volatile environment. To highlight the policy implications of our analysis, we show that a macro policy that reduces the income variance by 10 percent would lead to a welfare gain equivalent to roughly a 16 percent increase in initial income. The gain is much smaller if there is no model uncertainty.

This paper is organized as follows. Section 2 presents a robustness version of the Caballero-Huggett type model with incomplete markets. Section 3 discusses the general equilibrium implications of RB for the interest rate and the joint dynamics of consumption and income. Section 4 presents our quantitative results after estimating the income process and calibrating the RB parameter. Section 5 considers the extension to the multiple-asset case. Section 6 examines the welfare implications. Section 7 concludes and briefly discusses future research.

2. A Continuous-Time Heterogeneous-Agent Economy with Robustness

In this section, we lay out our continuous-time consumption-portfolio choice model with recursive utility and uninsurable labor income. To help explain the key structure of the model, we will introduce each of the key elements one by one, starting with specifications of the recursive utility preference, followed by labor income and investment opportunity set, and finally model uncertainty due to robustness and ambiguity.

2.1. Recursive Utility

Although the expected power utility model has many attractive features, that model implies that the elasticity of intertemporal substitution is the reciprocal of the coefficient of relative risk aver-

[10] The result that the relative dispersion of consumption to income decreases with income uncertainty through the general equilibrium interest rate channel provides an explanation for the empirical evidence documented in Blundell et al. (2008) that income and consumption inequality diverged over the sampling period they study. It is worth noting that they use the variances of log consumption and log income to measure consumption and income inequality. Since our paper adopts the CARA-Gaussian setting and the consumption process is non-stationary, we use the standard deviations of changes in consumption and income to measure the cross-sectional dispersions/inequality of changes in consumption and income, respectively.

sion. Conceptually, risk aversion (attitudes towards atemporal risks) and intertemporal substitution (attitudes towards shifts in consumption over time) capture two distinct aspects of decision-making and need not be so tightly connected.[11] By contrast, the class of recursive utility functions (Epstein and Zin 1989; Duffie and Epstein 1992) enable one to disentangle risk aversion from intertemporal substitution. In this paper, we assume that agents in our model economy have the Kreps-Porteus type preference with *recursive exponential utility* (REU): for every stochastic consumption stream, $\{c_t\}_{t=0}^{\infty}$, the utility stream, $\{f(U_t)\}_{t=0}^{\infty}$, is recursively defined by[12]

$$f(U_t) = \left(1 - e^{-\delta\Delta t}\right) f(c_t) + e^{-\delta\Delta t} f\left(CE_t\left[U_{t+\Delta t}\right]\right). \tag{1}$$

where Δt is the time interval, $\delta > 0$ is the agent's subjective discount rate, $f(U_t) = -\psi \exp(-U_t/\psi)$, $f(c_t) = -\psi \exp(-c_t/\psi)$,

$$CE_t\left[U_{t+\Delta t}\right] = g^{-1}\left(E_t\left[g\left(U_{t+\Delta t}\right)\right]\right), \tag{2}$$

is the certainty equivalent of U_{t+1} conditional on the period t information, and $g(U_{t+\Delta t}) = -\exp(-\gamma U_{t+\Delta t})/\gamma$. In (1), $\psi > 0$ governs the elasticity of intertemporal substitution (EIS), while $\gamma > 0$ governs the coefficient of CARA.[13] A high value of ψ corresponds to a strong willingness to substitute consumption over time, and a high value of γ implies a low willingness to substitute consumption across states of nature. Note that if $\psi = 1/\gamma$, the functions f and g are the same and the recursive utility reduces to the standard time-separable expected utility function used in Caballero (1990) and Wang (2003).

2.2. Specifications of Investment Opportunity Set and Labor Income

We assume that there is only one risk-free asset in the model economy and there are a continuum of consumers who face uninsurable labor income. The evolution of the financial wealth (w_t) of a typical consumer is:

$$dw_t = (rw_t + y_t - c_t)\,dt; \tag{3}$$

[11] Risk aversion describes the agent's reluctance to substitute consumption across different states of the world and is meaningful even in a static setting. By contrast, intertemporal substitution describes the agent's willingness to substitute consumption over time and is meaningful even in a deterministic setting.

[12] Skiadas (Chapter 6, 2009) axiomatizes and systematically characterizes this type of recursive exponential utility (or transition-invariant recursive utility.) Skiadas (2009) also compares this type of recursive utility with the scale-invariant (SI) Kreps-Porteus recursive utility (e.g., the Epstein-Zin-Weil parametric utility form). See also Angeletos and Calvet (2006) for an application of REU in a business cycles model.

[13] It is well-known that the CARA utility specification is tractable for deriving optimal policies and constructing general equilibrium in different settings. See Caballero (1990), Wang (2003), and Angeletos and Calvet (2006).

r is the return to the risk-free asset and c_t and y_t are consumption and labor income at time t, respectively. Uninsurable labor income (y_t) follows an Ornstein-Uhlenbeck process:[14]

$$dy_t = \rho\left(\overline{y} - y_t\right)dt + \sigma_y dB_t, \tag{4}$$

where $\overline{y}$ is the unconditional mean of y_t, σ_y is the unconditional volatility of the income change over an incremental unit of time, $\sigma_y^2/\left(2\rho\right)$ is the unconditional variance of y_t, the persistence coefficient ρ governs the speed of convergence or divergence from the steady state, and B_t is a standard Brownian motion on the real line $\mathcal{R}$. To present the model more compactly, we define a new state variable, s_t:

$$s_t \equiv w_t + h_t,$$

where h_t is human wealth at time t and is defined as the expected present value of current and future labor income discounted at the risk-free interest rate r:

$$h_t \equiv E_t\left[\int_t^{\infty} \exp\left(-r\left(s-t\right)\right) y_s ds\right].$$

For the given the income process, (4), $h_t = y_t/\left(r+\rho\right) + \overline{y}/\left(r\left(r+\rho\right)\right)$.[15] Using s_t as the unique state variable, we can rewrite (3) as:

$$ds_t = \left(rs_t - c_t\right)dt + \sigma_s dB_t, \tag{5}$$

where $\sigma_s = \sigma_y/\left(r+\rho\right)$ is the unconditional variance of the innovation to s_t.[16]

The optimization problem under rational expectations (RE) can thus be written as:

$$f\left(J_t\right) = \max_{c_t}\left\{\left(1 - e^{-\delta\Delta t}\right) f\left(c_t\right) + e^{-\delta\Delta t} f\left(CE_t\left[J_{t+\Delta t}\right]\right)\right\}, \tag{6}$$

subject to (5). An educated guess is that $J_t = As_t + A_0$, where A and A_0 are undetermined coefficients. The J function at t time $t + \Delta t$ can thus be written as:

$$J\left(s_{t+\Delta t}\right) = As_{t+\Delta t} + A_0 \approx As_t + A\left(rs_t - c_t\right)\Delta t + A\sigma_s \Delta B_t + A_0,$$

where $\Delta s_t \equiv s_{t+\Delta t} - s_t$ and $\Delta s_t \approx \left(rs_t - c_t\right)\Delta t + \sigma_s \Delta B_t$ where $\Delta B_t = \sqrt{\Delta t}\epsilon$ and ϵ is a standard

[14]In this paper, we abstract from income growth. It is worth noting that higher income growth generates higher risk-free rates. However, within our REU-OU framework, assuming constant income growth leads to time-varying risk-free rates, which greatly complicates our model. The detailed proof is available from the corresponding author by request.

[15]If $\rho > 0$, the income process is stationary and deviations of income from the steady state are temporary; if $\rho \leq 0$, income is non-stationary. The larger ρ is, the less y tends to drift away from $\overline{y}$. As ρ goes to ∞, the variance of y goes to 0. We need to impose the restriction that $r > -\rho$ to guarantee the finiteness of human wealth.

[16]In the next section, we will introduce robustness directly into this "reduced" precautionary savings model. It is not difficult to show that the reduced univariate model and the original multivariate model are equivalent in the sense that they lead to the same consumption and saving functions, because the financial wealth part of total wealth is deterministic between periods. The detailed proof is available from the corresponding author by request.

normal innovation. The Hamilton-Jacobi-Bellman (HJB) equation is then:

$$\delta f\left(J_t\right) = \sup_{c_t}\left\{\delta f\left(c_t\right) + \mathcal{D}f\left(s_t\right)\right\} \tag{7}$$

where

$$\mathcal{D}f\left(s_t\right) = f'\left(J_t\right)\left(A\left(rs_t - c_t\right) - \frac{1}{2}\gamma A^2\sigma_s^2\right), \tag{8}$$

and the transversality condition, $\lim_{t\to\infty}\left\{E\left|\exp\left(-\delta t\right)f_t\right|\right\} = 0$, holds at the optimum.

2.3. Incorporating Fear of Model Uncertainty

To introduce aversion to model uncertainty into our model (and thus generate a demand for robust decision rules), we follow the continuous-time methodology proposed by Anderson, Hansen, and Sargent (2003) (henceforth, AHS) and adopted in Maenhout (2004). Households take Equation (5) as the approximating model. The corresponding set of distorting models can thus be obtained by adding endogenous distortions $b\left(s_t\right)$ to (5):

$$ds_t = \left(rs_t - c_t\right)dt + \sigma_s\left(\sigma_s b\left(s_t\right)dt + dB_t\right). \tag{9}$$

As shown in AHS (2003), the objective $\mathcal{D}J$ defined in (8) can be thought of as $E\left[dJ\right]/dt$ and plays a key role in generating robustness. Consumers accept (5) as the best approximating model, but are still concerned that the model is misspecified. They therefore want to consider a range of models (the distorted models (9)) surrounding the approximating model when computing the continuation payoff. A preference for robustness (ambiguity aversion) manifests by having the agent guard against the distorting model that is reasonably close to the approximating model. The drift adjustment $b\left(s_t\right)$ is chosen to minimize the sum of (i) the expected continuation payoff adjusted to reflect the additional drift component in (9) and (ii) an entropy penalty:

$$\inf_{v}\left[\mathcal{D}f + f'\left(J_t\right)Ab\left(s_t\right)\sigma_s^2 + \frac{1}{\theta_t}\frac{\left(b\left(s_t\right)\sigma_s\right)^2}{2}\right], \tag{10}$$

where the first two terms are the expected continuation payoff when the state variable follows (9), i.e., the alternative model based on drift distortion $b\left(s_t\right)$, $\mathcal{H} = \left(b\left(s_t\right)\sigma_s\right)^2/2$ is the relative entropy or the expected log likelihood ratio between the distorted model and the approximating model and measures the distance between the two models, and $1/\theta_t$ is the weight on the entropy penalty term.[17] θ_t is fixed and state independent in AHS (2003), whereas it is state-dependent in Maen-

[17]The last term in (10) is due to the consumer's preference for robustness. Note that the $\theta_t = 0$ case corresponds to the standard expected utility case. This robustness specification is called the *multiplier (or penalty)* robust control problem. It is worth noting that this multiplier preference of RB expresses *ambiguity* with a multiplier that penalizes deviations from the approximating model as measured by relative entropy, and they express *ambiguity aversion* with the minimization operator.

hout (2004). The role of the state-dependent counterpart to ϑ_t in Maenhout (2004) is to assure the homotheticity or scale invariance of the decision problem under a constant relative risk aversion (CRRA) utility function. Note that the evil agent's minimization problem, (10), is invariant to the scale of total resources s_t under the state-dependent specification for $\theta_t(s_t)$, which we use as well so that the demand for robustness does not disappear as the value of total wealth increases.

We can then obtain the HJB equation for the RB model:

$$\delta f(J_t) = \sup_{c_t \in \mathcal{C}} \inf_{v_t} \left\{ \delta f(c_t) + \mathcal{D} f(s_t) + b(s_t)\, \sigma_s^2 J_s + \frac{1}{\theta(s_t)} \mathcal{H} \right\}. \tag{11}$$

Note that here following Hansen and Sargent (2011) and Kasa and Lei (2017), we scale the robustness parameter (θ_t) by the sampling interval (Δt), effectively making the consumer have stronger preference for robustness as the sampling interval shrinks.[18] Solving first for the infimization part of (11) yields:

$$b^*(s_t) = -\theta(s_t)\, f_s,$$

where $\theta(s_t) = -\vartheta / f(s_t) > 0$ (see Online Appendix 8.1 for the derivation). Here we can also define "$1/f(s_t)$" in the $\theta(s_t)$ specification as a *normalization* factor that is introduced to convert relative entropy (the distance between the approximating model and a given distorted model) into units of utility so that it is consistent with the units of the expected future value function evaluated with the distorted model. It is worth noting that this state-dependent robustness parameter follows a geometric Brownian motion in general equilibrium. (See Section 3.2 for the details.) This resulting process is similar to the AR(1) ambiguity shocks proposed in Bhandari et al. (2017). They identified AR(1) ambiguity shocks using U.S. survey data, and found that in the data, the ambiguity shocks are an important source of variation in labor market variables.

Since $\theta(s_t) > 0$, the perturbation adds a negative drift term to the state transition equation because $J_s > 0$. Substituting for b^* in (11) gives:

$$\delta f(U_t) = \sup_{c_t \in \mathcal{C}} \left\{ \delta f(c_t) + f'(J_t)\, A \left(r s_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 + \frac{\vartheta}{f(U_t)} A f'(U_t)\, \sigma_s^2 \right) - \frac{\vartheta}{2 f(J_t)} A^2 \left(f'(J_t)\right)^2 \sigma_s^2 \right\}. \tag{12}$$

2.4. The Robust Consumption Function and Precautionary Saving

We can now solve (12) and obtain the consumption rule under robustness. The following proposition summarizes the solution.

[18] See also Hansen and Miao (2018) for a discussion on distinguishing ambiguity from risk in the continuous time limit.

Proposition 1. *Under robustness, the consumption function and the saving function are*

$$c_t^* = rs_t + \Psi(r) - \Gamma(\vartheta, r), \tag{13}$$

and

$$d_t^* = x_t + \Gamma - \Psi, \tag{14}$$

respectively, where $x_t \equiv \rho(y_t - \overline{y}) / (r + \rho)$ *is the demand for savings "for a rainy day",*

$$\Psi(r) \equiv \psi\left(\frac{\delta}{r} - 1\right) \tag{15}$$

captures the saving demand of relative patience,

$$\Gamma(\vartheta, r) \equiv \frac{1}{2} r \widetilde{\gamma} \sigma_s^2 \tag{16}$$

is the demand for precautionary savings due to the interaction of income uncertainty, intertemporal substitution, and risk and uncertainty aversion, and

$$\widetilde{\gamma} \equiv \gamma + \frac{\vartheta}{\psi} \tag{17}$$

is the effective coefficient of absolute risk aversion. The corresponding value function is

$$f(s_t) = -\frac{\delta\psi}{r} \exp\left(-\left(\frac{\delta}{r} - 1 - \frac{1}{2}\frac{r}{\psi}\widetilde{\gamma}\sigma_s^2\right) - \frac{r}{\psi}s_t\right). \tag{18}$$

Finally, the worst possible distortion is

$$b^* = -r\frac{\vartheta}{\psi}. \tag{19}$$

Proof. See Online Appendix A. ■

From (15), it is clear that if the consumer is impatient relative to the market ($\delta > r$), the higher the EIS, the stronger the demand for consumption. If $\delta > r$, households want consumption to fall over time, and a higher EIS implies that consumption will be allowed to fall faster for a given value of δ/r; as a result, consumption must initially be high. Following the literature on precautionary savings, we measure the demand for precautionary saving as the amount of saving induced by the combination of uninsurable labor income risk and risk aversion. Expression (16) shows that the precautionary savings demand now depends on the effective coefficient of risk aversion $\widetilde{\gamma}$, which is a function of the EIS (ψ), the CARA (γ), and the degree of robustness (ϑ). Specifically, it increases with γ and ϑ, whereas it decreases with ψ. From (16), one can also see that the precautionary saving demand is larger for a more volatile income innovation (higher σ_y) and a larger persistence coefficient (lower ρ).[19] Holding other parameters constant, we can see from (13) to (16) that intertemporal substitution and risk aversion have opposing effects on consumption and saving decisions if $\delta > r$ (which will be the case in general equilibrium).[20]

Another interesting question here is the relative importance of RB (ϑ) and CARA (γ) in determining the precautionary savings demand, holding other parameters constant. We can use the elasticities of precautionary saving as a measure of their importance.

Proposition 2. *The relative sensitivity of precautionary saving to risk aversion (γ), intertemporal substitution (ψ), and robustness (ϑ) can be measured by*

$$\mu_{\gamma\vartheta} \equiv \frac{e_\gamma}{e_\vartheta} = \frac{\gamma}{\vartheta/\psi}, \tag{20}$$

$$\mu_{\psi\vartheta} \equiv \frac{e_\psi}{e_\vartheta} = -1. \tag{21}$$

respectively, where $e_\gamma \equiv \frac{\partial\Gamma/\Gamma}{\partial\gamma/\gamma}$, $e_\psi \equiv \frac{\partial\Gamma/\Gamma}{\partial\psi/\psi}$, and $e_\vartheta \equiv \frac{\partial\Gamma/\Gamma}{\partial\vartheta/\vartheta}$ are the elasticities of the precautionary saving demand to CARA, EIS, and RB, respectively.

Proof. The proof is straightforward. ∎

The interpretation of (20) is that the precautionary savings demand is more sensitive to the actual coefficient of (absolute) risk aversion (γ) than it is to RB (ϑ) if the actual CARA is greater than RB amplified by the inverse of the EIS, i.e., $\gamma > \vartheta/\psi$. Of course, it is not exactly clear how to interpret a proportional change in either parameter since they do not have units, but we report this result to show that risk aversion does not clearly dominate the motives of the agents in the model.

Hansen, Sargent, and Tallarini (1999, henceforce HST) show that the discount factor and the concern about robustness are observationally equivalent in the sense that they lead to the same consumption and investment decisions in a discrete-time LQG representative-agent permanent income model. The reason for this result is that introducing a concern about robustness increases savings in the same way as increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RB on consumption and investment.[21] In contrast, in our continuous-time CARA-Gaussian model, we have a more general observational equivalence result between δ, γ, and ϑ.

[19] As argued in Caballero (1990) and Wang (2003), a more persistent income shock takes a longer time to wear off and thus induces a stronger precautionary saving demand by a prudent forward-looking consumer.

[20] As a side note, incomplete markets generally imply that aggregate dynamics depend on the wealth distribution, this "curse of dimensionality" is circumvented by our CARA-Gaussian specification since savings functions are linear.

[21] As shown in HST (1999), the two models have different implications for asset prices because continuation valuations would alter as one alters the values of the discount factor and the robustness parameter within the observational equivalence set.

Proposition 3. *Let*

$$\gamma^{fi} = \gamma + \frac{\vartheta}{\psi}, \tag{22}$$

where γ^{fi} is the coefficient of absolute risk aversion in the RE model. Then consumption and savings are identical in the RE and RB models, holding other parameter values constant. Furthermore, let $\delta = r$ in the RB model, and

$$\delta^{fi} = r - \frac{1}{2}\vartheta \left(\frac{r}{\psi}\right)^2 \sigma_s^2, \tag{23}$$

where δ^{fi} is the discount rate in the RE model. Then consumption and savings are identical in the RE and RB models, ceteris paribus.

Proof. Using (13) and (16), the proof is straightforward. ■

Expression (22) means that a consumer with a preference for robustness (ϑ) and recursive utility with EIS (ψ) and CARA (γ) is observationally equivalent to a consumer with full-information and recursive utility with EIS (ψ) and CARA ($\gamma + \vartheta/\psi$). In contrast, within a Merton model with recursive utility, Maenhout (2004) showed that an agent with a preference for robustness and Epstein-Zin recursive utility with EIS (ψ) and CRRA (γ) is observationally equivalent to an agent with full information and recursive utility with EIS (ψ) and CARA ($\gamma + \vartheta$). In other words, in Maenhout's model, the effective coefficient of relative risk aversion ($\gamma + \vartheta$) does not depend on the EIS (ψ).

3. General Equilibrium Implications of RB

3.1. Definition of the General Equilibrium

As in Huggett (1993) and Wang (2003), we assume that the economy is populated by a continuum of *ex ante* identical, but *ex post* heterogeneous agents, with each agent having the saving function, (16). In addition, we also assume that the risk-free asset in our model economy is a pure-consumption loan and is in zero net supply.[22] The key insights can be also obtained in a CARA-Gaussian production economy with incomplete markets (as in Angeletos and Calvet 2006) using a neoclassical production function with capital and bonds as saving instruments. We consider the simpler Huggett-type endowment economy for two reasons. First, in the endowment economy, we can directly compare the model's predictions on the dynamics of individual consumption and income with its empirical counterpart, and do not need to infer the idiosyncratic productivity shock process. Second, the endowment economy allows us to solve the models explicitly, and thus helps us identify distinct channels via which RB interacts with risk aversion, discounting, and intertemporal substitution and affects the consumption-saving behavior.

[22]We can easily generalize to fixed positive net savings, as in a Lucas-style tree model. Nothing would change.

In the model economy, the initial cross-sectional distribution of income is assumed to be its stationary distribution $\Phi(\cdot)$. By the law of large numbers (LLN), provided that the spaces of agents and the probability space are constructed appropriately, aggregate income and the cross-sectional distribution of permanent income $\Phi(\cdot)$ will be constant over time.

Proposition 4. *The total savings demand "for a rainy day" in the precautionary savings model with RB equals zero for any positive interest rate. That is,* $F_t(r) = \int_{y_t} x_t(r)\, d\Phi(y_t) = 0$, *for* $r > 0$.

Proof. Given that labor income is a stationary process, the LLN can be directly applied. The proof is the same as that in Wang (2003). ∎

This proposition states that the total savings "for a rainy day" is zero, at any positive interest rate; with a constant income distribution and linear decision rules, agents in the stationary wealth distribution follow the "American dream and American nightmare" path, where any rise in income today is eventually offset by a future decline. Therefore, from (14), for $r > 0$, the expression for total savings under RB in the economy at time t can be written as:

$$D(\vartheta, r) \equiv \Gamma(\vartheta, r) - \Psi(r). \tag{24}$$

where $\Psi(r)$ and $\Gamma(\vartheta, r)$ are defined in (15) and (16), respectively. We can now define a general equilibrium.

Definition 1. *Given (24), a general equilibrium under RB is defined by an interest rate* r^* *satisfying*

$$D(\vartheta, r^*) = 0. \tag{25}$$

3.2. Theoretical Results

The following proposition summarizes the equilibrium dynamics of consumption and wealth:

Proposition 5. *Suppose the true economy is governed by the approximating model, where* $\theta_t = -\vartheta / f(s_t)$ *and* $f(s_t)$ *is provided in (18). In equilibrium, each consumer's optimal consumption is described by the PIH:*

$$c_t^* = r^* s_t. \tag{26}$$

Furthermore, the evolution equations of wealth and consumption are:

$$dw_t^* = x_t dt, \tag{27}$$

$$dc_t^* = \frac{r^*}{r^* + \rho} \sigma_y dB_t, \tag{28}$$

respectively, where $x_t = \rho (y_t - \overline{y}) / (r^* + \rho)$. *Finally, in equilibrium,* θ_t *follows a geometric Brownian motion*

$$\frac{d\theta_t}{\theta_t} = \frac{1}{2} (r^* \sigma_s)^2 dt + (r^* \sigma_s) dB_t, \tag{29}$$

If $\rho > \delta$*, the equilibrium is unique.*

Proof. See Appendix 8.1. ■

The intuition behind this proposition is similar to that in Wang (2003). With an individual's constant total precautionary savings demand $\Gamma(\vartheta, r)$, for any $r > 0$, the equilibrium interest rate r^* must be at a level with the property that individual's dissavings demand due to impatience is exactly balanced by their total precautionary-savings demand, $\Gamma(\vartheta, r^*) = \Psi(r^*)$. We can see from (25) that EIS affects the equilibrium interest rate via two channels: (i) the precautionary saving channel and (ii) the impatience-induced dissaving channel. As EIS decreases, it increases the precautionary saving demand via increasing the effective coefficient of risk aversion and also reduces the impatience-induced dissaving effect; both channels drive down the equilibrium interest rate. It is also clear from (25) that a high value of ψ would amplify the relative importance of the dissaving effect $\Psi(r)$ for the equilibrium interest rate. The intuition behind this result is simple. When ψ is higher, consumption growth responds less to changes in the interest rate. In order to clear the market, the consumer must be offered a higher equilibrium risk free rate in order to be induced to save more and make his consumption tomorrow even more in excess of what it is today (less smoothing).

From the equilibrium condition:

$$\frac{1}{2} r^* \left(\gamma + \frac{\vartheta}{\psi}\right) \frac{\sigma^2}{(r^* + \rho)^2} - \psi \left(\frac{\delta}{r^*} - 1\right) = 0, \tag{30}$$

it is straightforward to show that

$$\frac{dr^*}{d\vartheta} = -\frac{r^* \sigma_s^2}{\psi} \left(\widetilde{\gamma} \sigma_s^2 \frac{\rho - r^*}{\rho + r^*} + \frac{2\delta\psi}{r^{*2}}\right)^{-1}; \tag{31}$$

if $\rho > \delta > r^*$, then this derivative is negative, so that r^* is decreasing in the degree of RB, ϑ. In addition, it is straightforward to see that:

$$\frac{dr^*}{d\gamma} < 0 \text{ and } \frac{dr^*}{d\psi} > 0.$$

That is, the equilibrium interest rate decreases with the degree of risk aversion and increases with the degree of intertemporal substitution. From (27) and (28), we can conclude that although both the CARA model and the linear-quadratic (LQ) model lead to the PIH in general equilibrium, both risk aversion and intertemporal substitution play roles in affecting the dynamics of consumption and wealth in the CARA model via the equilibrium interest rate channel. Figure 2 shows that the aggregate saving function $D(\vartheta, r)$ is increasing with the interest rate for different values of ϑ when $\gamma = 3$, $\psi = 0.5$, $\delta = 0.036$, $\sigma_y = 0.182$, and $\rho = 0.083$, and there exists a unique interest rate r^* for every given ϑ such that $D(\vartheta, r^*) = 0$.[23]

Note that mathematically, the cross-sectional dispersion of consumption (relative to income) can be measured by the relative volatility of consumption to income, as our model satisfies a mixing condition in the steady state. The following result is then immediate.

Proposition 6. *The relative dispersion of consumption growth to income growth is:*

$$\mu \equiv \frac{\mathrm{sd}\left(dc_t^*\right)}{\mathrm{sd}\left(dy_t\right)} = \frac{r^*}{r^* + \rho}. \tag{32}$$

Figure 2 also shows how RB (ϑ) affects the equilibrium interest rate (r^*). It is clear from the figure that the stronger the preference for robustness, the lower the equilibrium interest rate. From (32), we can see that RB will affect the dispersion of consumption by reducing the equilibrium interest rate. The following proposition summarizes the results about how the persistence coefficient of income changes the effect of RB on μ.

Proposition 7. *Using (32), we have:*

$$\frac{\partial \mu}{\partial \vartheta} = \frac{\rho}{\left(r^* + \rho\right)^2} \frac{\partial r^*}{\partial \vartheta} < 0$$

because $\rho > 0$ *and* $\partial r^* / \partial \vartheta < 0$.

Proof. The proof is straightforward. ∎

4. Quantitative Analysis

In this section, we first describe how we estimate the income process and calibrate the robustness parameter. We then present quantitative results on how RB affects the equilibrium interest rate and relative dispersion of consumption to income.

4.1. Estimation of the Income Process

To implement the quantitative analysis, we need to first estimate ρ and σ_y in the income process specification (4). We use micro data from the the Panel Study of Income Dynamics (PSID). Following Blundell et al. (2008), we define the household income as total household income (including

[23] In Section 4.1, we will discuss the choice of these preference parameters and provide more details about how to estimate the income process using the U.S. panel data. The main result here is robust to the choices of these parameter values.

wage, financial, and transfer income of head, wife, and all others in household) minus financial income (defined as the sum of annual dividend income, interest income, rental income, trust fund income, and income from royalties for the head of the household only) minus the tax liability of non-financial income. This tax liability is defined as the total tax liability multiplied by the non-financial share of total income. Tax liabilities after 1992 are not reported in the PSID, and so we estimate them using the TAXSIM program from the NBER. We report details on sample selection in Online Appendix B.

To exclude extreme outliers, following Floden and Lindé (2001), we normalize both income and consumption measures as ratios of the mean of each year, and exclude households in the bottom and top 1 percent of the distribution of those ratios. To eliminate possible heteroskedasticity in the income measures, we regress each on a series of demographic variables to remove variation caused by differences in age and education. We next subtract these fitted values from each measure to create a panel of income residuals. We then use this panel to estimate the household income process as a stationary AR(1) process with Gaussian innovations:

$$y_t = \phi_0 + \phi_1 y_{t-1} + \sigma\varepsilon_t,\ t \geq 1,\ \ |\phi_1| < 1, \tag{33}$$

where $\varepsilon_t \sim N(0,1)$, $\phi_0 = (1-\phi_1)\,\overline{y}$, $\overline{y}$ is the mean of y_t, and the initial level of labor income y_0 are given. Once we have estimates of ϕ_1 and σ, we can recover the drift and diffusion coefficients in the Ornstein-Uhlenbeck process specified in (4) by rewriting (33) in the time interval $[t, t+\Delta t]$ as

$$y_{t+\Delta t} = \phi_0 + \phi_1 y_t + \sigma\sqrt{\Delta t}\varepsilon_{t+\Delta t}, \tag{34}$$

where $\phi_0 = \kappa\,(1-\exp(-\rho\Delta t))\,/\,(\rho\Delta t)$, $\phi_1 = \exp(-\rho\Delta t)$, $\sigma = \sigma_y\sqrt{(1-\exp(-2\rho\Delta t))\,/\,(2\rho\Delta t)}$, and $\varepsilon_{t+\Delta t}$ is the time-$(t+\Delta t)$ standard normal distributed innovation to income.[24] As the time interval, Δt, converges to 0, (34) reduces to the Ornstein-Uhlenbeck process, (4). The estimation results and the recovered persistence and volatility coefficients in (4) are reported in Table 2.

4.2. Calibration of the Robustness Parameter

We adopt the calibration procedure outlined in AHS (2003) to calibrate the value of the RB parameter (ϑ) that governs the degree of robustness. Specifically, we calibrate ϑ by using the method of detection error probabilities (DEP) that is based on a statistical theory of model selection. We can then infer what values of ϑ imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by $p(\vartheta)$ is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard many models,

[24] Note that here we use the fact that $\Delta B_t = \varepsilon_t\sqrt{\Delta t}$, where ΔB_t represents the increment of a Wiener process.

implying that the cloud of models surrounding the approximating model is large. Let model P denote the approximating model, (5), and model Q be the distorted model, (9). See Appendix 8.2 for the detailed description of the calibration procedure.

In Appendix 8.2, we show that the DEP, $p(\vartheta)$, can be written as:

$$p(\vartheta) = \Pr\left(x < \frac{\overline{b}}{2}\sqrt{N}\right), \tag{35}$$

where $\overline{b} \equiv b^* \sigma_s = -\frac{\vartheta}{\psi} r^* \sigma_s$, and x follows a standard normal distribution. From the expressions of $\overline{b}$ and $p(\vartheta)$, it is clear that the value of p is decreasing with the value of ϑ. Under the observational equivalence condition between the multiplier and constraint robustness formulations, (35) can be rewritten as $p(\vartheta) = \Pr\left(x < -\sqrt{2\eta}\sqrt{N}\right)$, where η is the upper bound on the distance between the two models and measures the consumer's tolerance for model misspecification.

We first explore the relationship between the DEP (p) and the value of the RB parameter, ϑ. A general finding is a negative relationship between these two variables. The upper panels of Figure 3 illustrates how p varies with the value of ϑ for different values of EIS (ψ) and CARA (γ).[25] We can see from the figures that the stronger the preference for robustness (higher ϑ), the less the DEP (p) is. Following the literature on precautionary savings (e.g., Caballero 1990 and Wang 2003), we set $\gamma = 3$. The magnitude of the EIS (ψ) is an open and unresolved question, as the literature has found a very wide range of values. Vissing-Jorgensen and Attanasio (2003) estimate the EIS to be well in excess of one, while Campbell (2003) estimates a value well below one (and possibly zero). Guvenen (2006) finds that stockholders have a higher EIS (around 1.0) than non-stockholders (around 0.1). Havránek (2015) surveys the vast literature and suggests that a range around 0.3-0.4 is appropriate after correcting for selective reporting bias, while Crump et al. (2015) find that the EIS is precisely and robustly estimated to be around 0.8 in the general population using the newly released FRBNY Survey of Consumer Expectations (SCE). For example, let $\psi = 0.5$, then $p = 0.403$ and $r^* = 2.83$ percent when $\vartheta = 1$, while $p = 0.163$ and $r^* = 1.99$ percent when $\vartheta = 5$. Both values of p are reasonable as argued in AHS (2002), Maenhout (2004), and Hansen and Sargent (Chapter 9, 2007). In other words, a value of ϑ below 5 is reasonable in this case in which $\gamma = 3$ and $\psi = 0.5$. Furthermore, from the two upper panels of Figure 3, we can also see that the DEP increases with both ψ and γ for given values of ϑ, and the impact of a change in ψ is much larger than that of a change in γ. The intuition is that a change in the EIS has two channels to affect the values of $\overline{b}$ and p: (i) the direct channel and (ii) the indirect channel via affecting the general interest rate channel (r^*), and the direct channel dominates the indirect channel. In contrast, a change in

[25] Based on the estimation results, we set $\overline{y} = 1$, $\sigma_y = 0.182$, and $\rho = 0.083$. The implied CRRA in our CARA utility specification can be written as either γc or γy. Given that the value of the CRRA is very stable and $\overline{v}$ can be expressed as $r(\vartheta/\psi)\sigma_y/(r+\rho)$, proportional changes in the mean and standard deviation of y do not change our calibration results because their effects on γ and σ_y cancel each other out.

CARA only affects the values of $\bar{b}$ and p via the indirect equilibrium interest rate channel, which is relatively weak. Furthermore, using (20), in this case, we have $\mu_{\gamma\vartheta} = 1.5$ and 0.3 when we set $\vartheta = 1$ and 5, respectively. That is, the relative importance of risk aversion to RB in determining the precautionary savings demand decreases with the value of ϑ, holding other parameters constant.

The two lower panels of Figure 3 illustrate how DEP (p) varies with ϑ for different values of σ_y and ρ if $\psi = 0.5$ and $\gamma = 3$.[26] It also shows that the higher the value of ϑ, the less the DEP (p). In addition, to calibrate the same value of p, smaller values of σ_y (less volatile labor income processes) or higher values of ρ (less persistent income processes) lead to higher values of ϑ. The intuition behind this result is that σ_s and ϑ have opposite effects on $\bar{b}$ and then p (see (66)).

As emphasized in Hansen and Sargent (2007), in the robustness model, p is a measure of the amount of model uncertainty, whereas ϑ is a measure of the agent's aversion to model uncertainty. If we keep p constant when recalibrating ϑ for different values of γ, ρ, or σ_y, the amount of model uncertainty is held constant–that is, the set of distorted models with which we surround the approximating model does not change. In contrast, if we keep ϑ constant, p will change accordingly if the values of γ, ρ, or σ_y change; in this case, the amount of model uncertainty is "elastic" and will change accordingly as the agent's aversion to uncertainty changes.

4.3. Effects of RB on the Equilibrium Interest Rate and Consumption Dispersion

The equilibrium interest rate and relative dispersion of consumption to income are jointly determined by the degree of robustness, risk aversion, intertemporal substitution, and the income process. To better see how RB affects the equilibrium interest rate and the relative dispersion, we present two quantitative exercises here. The first exercise fixes the parameters of the income process at the estimated values and allows the risk aversion and intertemporal substitution parameters to change, while the second exercise fixes the risk aversion and intertemporal substitution parameters and allows the key income process parameter to vary.

Figure 4 shows that the equilibrium interest rate and the equilibrium relative consumption dispersion decrease with the calibrated value of ϑ for different values of ψ and γ when $\sigma_y = 0.182$ and $\rho = 0.083$. For example, if ϑ is increased from 1 to 5 (p decreases from 0.403 to 0.163), r^* falls from 2.63 percent to 1.98 percent and μ falls from 0.241 to 0.193, given $\psi = 0.5$ and $\gamma = 3$.[27] In addition, the figure also shows that the interest rate and the relative dispersion decrease with γ and increase with ψ for different values of ϑ.

Our model has the potential to explain the observed low real interest rate in the U.S. economy; see Laubach and Williams (2015) or Hall (2017) for evidence on low real rates. One of our the-

[26] Since $\sigma_s = \sigma_y / (r + \rho)$, both changes in the persistence coefficient (ρ) and changes in volatility coefficient (σ_y) will change the value of σ_s.

[27] In the RE case, $r^* = 2.92$ percent and $\mu = 0.26$.

oretical results shows that a stronger aversion to model uncertainty lowers the equilibrium real interest rate. In the US, the average real risk-free interest rate has been about 1.87 percent between 1981 and 2010 if we use CPI to measure inflation, and about 1.96 percent if we use PCE to measure inflation.[28] Therefore, depending on what inflation index is used, the risk-free rate is between 1.87 and 1.96 percent. In our following discussion, we set the risk-free rate to be 1.91 percent, which is the average of the two real interest rates under CPI and PCE. Using the equilibrium condition, we find that the RE model without RB requires the coefficient of risk aversion parameter to be 23 to match this rate if $\psi = 0.8$, and requires the coefficient to be 14.5 if $\psi = 0.5$.[29]

In contrast, when consumers take into account model uncertainty, the model can generate an equilibrium interest rate of 1.91 percent with much lower values of the coefficient of risk aversion.[30] Figure 5 shows the relationship between γ and ϑ for interest rates equal to 1.91 percent for different values of ψ. For example, if $\psi = 0.5$, the RB model with $\gamma = 4.5$ and $\vartheta = 5$ leads to the same interest rate as in the RE model with $\gamma = 14.5$. Note that $\gamma = 4.5$ is much lower than the risk aversion levels used in most macro-asset pricing models. Using the same calibration procedure discussed in Section 4.1, we find that the corresponding DEP is $p = 0.171$. In other words, agents tolerate a 17.1 percent probability that they cannot distinguish the distorted model from the approximating model. We have summarized these results in Table 3. As argued in Hansen and Sargent (2007) and in Section 4.2, this value is viewed as reasonable in the literature.

The explanation that agents have become more concerned about model misspecification after the 2007-2009 financial crisis does not seem unreasonable given the long and deep recession that generated skepticism (at least in the popular press) about whether the standard macro models fully capture the key features of the economy. To provide a numerical example, under our calibrated parameter values, $\psi = 0.5$, and $\gamma = 4.5$, an increase in model uncertainty reflected by a reduction in the DEP from $p = 0.297$ to $p = 0.171$ (an increase in ϑ from 2.5 to 5) leads to a reduction in the equilibrium interest rate from 2.21 percent to 1.91 percent.

To examine how RB affects the relative dispersion of consumption to income ($\mu = \text{sd}\,(dc_t^*)\ /\ \text{sd}\,(dy_t)$), we follow Luo et al. (2017) and construct a panel data set that contains both consumption and income at the household level. Figure 1 shows the relative dispersion of consumption to income

[28] Following Campbell (2003), we calculate the average of the real 3-month Treasury yields. Here we choose the $1981-2010$ period because it is more consistent with our sample period of the panel data in estimating the joint consumption and income process. When we consider an extended period from 1981 to 2015, the real interest rate is 1.37 percent when using CPI and is 1.75 percent when using PCE. Hall (2017) finds that real rates (computed using TIPS) have been consistently falling for several decades, so we are overstating the current rate.

[29] Note that since we set the mean income level to be 1, the coefficient of CRRA evaluated at this level is equal to the coefficient of CARA.

[30] This result is comparable to that obtained in Barillas et al. (2009). They find that most of the observed high market price of uncertainty in the U.S. can be reinterpreted as a market price of model uncertainty rather than the traditional market price of risk.

between 1980 and 2000.[31] From the figure, the average empirical value of the relative dispersion (μ) is 0.377 for the 1980-1996 period and 0.326 for the 1980-2010 period. The minimum and maximum values of the empirical relative dispersion from 1980 to 2010 are 0.195 (year 2006) and 0.55 (year 1982), respectively. From the expression for the equilibrium relative dispersion (32), we can see that when the real interest rate is low, it is impossible for the model to generate sufficiently high relative dispersion of consumption to income without using an implausible value for ρ. For example, when $r^* = 1.91$ percent, we obtain $\mu = 0.19$, which is well below the average value $\mu = 0.326$. To get $\mu = 0.326$, we would need $\rho = 0.039$, a value that can be rejected given our estimated value of $\rho = 0.082$. Because this moment matters for the welfare calculations that are the focus of the paper, in the next section, we will resolve the disparity between data and model using a risky asset in positive net supply.

5. Extension to an RU-RB Model with Multiple Financial Assets

5.1. Model Specification

In this section, we follow Viceira (2001) and Maenhout (2004), and assume that consumers can assess two financial assets: one risk-free asset and one risky asset. Our aim here is to resolve the anomaly from the benchmark model regarding the relative dispersion of consumption to income at low interest rates. Specifically, the consumer can purchase both a risk-free asset with a constant interest rate r and a risky asset (the market portfolio) with a risky return r_t^e. The instantaneous return dr_t^e of the risky market portfolio over dt is given by:

$$dr_t^e = (r + \pi)\, dt + \sigma_e dB_{e,t}, \tag{36}$$

where π is the market risk premium, σ_e is the standard deviation of the market return, and $B_{e,t}$ is a standard Brownian motion. Let ρ_{ye} be the contemporaneous correlation between the labor income process and the return of the risky asset. If $\rho_{ye} = 0$, the labor income risk is purely idiosyncratic, so the risky asset does not provide a hedge against labor income declines. The agent's financial wealth evolution is then given by:

$$dw_t = (rw_t + y_t - c_t)\, dt + \alpha_t\, (\pi dt + \sigma_e dB_{e,t})\,, \tag{37}$$

where α_t denotes the amount of wealth that the investor allocates to the market portfolio at time t.

As in the benchmark model, we define a new state variable, s_t: $s_t \equiv w_t + h_t$, where h_t is human wealth at time t and is defined as the expected present value of current and future labor income discounted at the risk-free interest rate r: $h_t \equiv E_t \left[\int_t^\infty \exp\left(-r\,(s-t)\right) y_s ds\right]$. Following the same state-space-reduction approach used in the benchmark model, the budget constraint can be written

[31] See Online Appendix B for more details on how we constructed the panel.

as:

$$ds_t = (rs_t - c_t + \pi\alpha_t)\,dt + \sigma dB_t, \tag{38}$$

where $\sigma dB_t = \sigma_e \alpha_t dB_{e,t} + \sigma_s dB_{y,t}$, $\sigma_s = \sigma_y / (r+\rho)$, and

$$\sigma = \sqrt{\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t} \tag{39}$$

is the unconditional variance of the innovation to s_t.

5.2. Consumption and Portfolio Rules Under RB

To introduce robustness into the above recursive utility model, we follow the same procedure as in Section 2.3 and write the distorting model by adding an endogenous distortion $b(s_t)$ to the law of motion of the state variable s_t, (38):

$$ds_t = (rs_t - c_t + \pi\alpha_t)\,dt + \sigma\left(\sigma b(s_t)\,dt + dB_t\right). \tag{40}$$

Following the same procedure we used in solving the benchmark model, we can also solve the multiple-asset case explicitly. The following proposition summarizes the solution to the dynamic problem.

Proposition 8. *Under robustness, the consumption function, the portfolio rule, and the saving function are*

$$c_t^* = r\left(s_t - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{r\sigma_e^2}\right) + \Psi(r) - \Gamma(\vartheta, r) + \Pi(\vartheta, r), \tag{41}$$

$$\alpha^* = \frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{42}$$

and

$$d_t^* = x_t + \Gamma - \Psi + \Pi, \tag{43}$$

respectively, where $x_t \equiv \rho(y_t - \overline{y}) / (r+\rho)$ is the demand for savings "for a rainy day,"

$$\Gamma(\vartheta, r) \equiv \frac{1}{2} r\widetilde{\gamma}\left(1 - \rho_{ye}^2\right)\sigma_s^2 \tag{44}$$

is the demand for precautionary savings due to the interactions of income uncertainty, intertemporal substitution, and risk and uncertainty aversion;

$$\Psi(r) \equiv \psi\left(\frac{\delta}{r} - 1\right) \tag{45}$$

captures the saving demand of relative patience;

$$\Pi\left(\vartheta, r\right) \equiv \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2} \tag{46}$$

is the additional saving demand due to the higher expected return of the risky asset; and $\widetilde{\gamma} \equiv \gamma + \vartheta/\psi$ *is the effective coefficient of absolute risk aversion. Finally, the worst possible distortion is* $b^* = -r\left(\vartheta/\psi\right)$.

Proof. See Online Appendix C. ■

Expression (41) shows that the presence of the risky asset in the agent's investment opportunity has two effects on current consumption. First, it reduces the risk-adjusted certainty equivalent human wealth by $\pi\rho_{ye}\sigma_s\sigma_e/\left(r\sigma_e^2\right)$ because the agent faces more risk when holding the risky asset. Second, it increases current consumption because it offers a higher expected return. In general equilibrium, the second effect dominates the first effect. Furthermore, from the definition of individual saving, we can see that the presence of $\pi\alpha^*$ term has the potential to increase saving because it offers a higher expected return. Combining these two effects, it is clear from (43) that the net effect of the risky asset on current saving is governed by $\Pi > 0$ defined in (46). In addition, since the risky asset can be used to hedge labor income risk (provided the correlation is not zero), it will reduce the precautionary saving demand arising from income uncertainty by a factor $1 - \rho_{ye}^2 \in (0, 1)$. It is thus clear from (43) that there are four saving motives in the model with a risky asset. The first three saving motives–x_t, Γ, and Ψ–are the same as that mentioned in our benchmark model. The fourth term captures the additional saving demand due to the higher expected return of the risky asset and obviously does not appear in the benchmark model.

Since the effective coefficient of absolute risk aversion depends on both the EIS and the degree of RB, it is clear from (42) that even if the consumer only has a constant investment opportunity set, the optimal share invested in the risky asset not only depends on risk aversion, but also depends on intertemporal substitution if $\vartheta > 0$.[32]

5.3. General Equilibrium Implications

We first consider the equilibrium in the market for the risky asset. Assuming that the net supply of the risky asset is $\overline{\alpha} \geq 0$, the equilibrium condition in the market for the risky asset is:

$$\overline{\alpha} = \frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} \tag{47}$$

for a given risk free rate, r.

[32] A constant investment opportunity set means a constant interest rate, a constant expected return on risky assets, and a constant volatility of the returns on risky assets.

Using the individual saving function (43) and following the same aggregation procedure used in the previous section, we have the following result on savings:

Proposition 9. *The total demand of savings "for a rainy day" equals zero for any positive interest rate. That is,* $F_t(r) = \int_{y_t} x_t(r)\, d\Phi(y_t) = 0$*, for* $r > 0$*.*

Proof. The proof uses the LLN and is the same as that in Wang (2003). ∎

Using this result, from (43), after aggregating across all consumers, the expression for total savings can be written as:

$$D^{total}(\vartheta, r) \equiv \Gamma(\vartheta, r) - \Psi(r) + \Pi(\vartheta, r), \tag{48}$$

where $\Gamma(\vartheta, r)$, $\Psi(r)$, and $\Pi(\vartheta, r)$ are given in (44), (45), and (46), respectively. To compare $D(\vartheta, r)$ with the aggregate saving function obtained in the benchmark model, we rewrite $D(\vartheta, r)$ as follows:

$$D^{total}(\vartheta, r) = \widetilde{\Gamma}(\vartheta, r) - \Psi(r) + \widetilde{\Pi}(\vartheta, r), \tag{49}$$

where $\widetilde{\Gamma}(\vartheta, r) = r\widetilde{\gamma}\sigma_s^2/2$, $\widetilde{\Pi}(\vartheta, r) = \overline{\alpha} r \widetilde{\gamma}\rho_{ye}\sigma_s\sigma_e + \overline{\alpha}^2 r\widetilde{\gamma}\sigma_e^2/2$, and $\overline{\alpha}$ is determined by (47). Comparing the two aggregate saving functions, $\widetilde{\Pi}(\vartheta, r)$ is an additional term due to the positive net supply of the risky asset in this model. As in the benchmark model, we still assume that the net supply of the risk-free asset is zero in equilibrium, i.e., an equilibrium interest rate r^* satisfies:

$$D(\vartheta, r^*) \equiv \Gamma(\vartheta, r^*) - \Psi(r^*) = 0, \tag{50}$$

where $D(\vartheta, r^*)$ denotes the amount of saving in the risk-free asset. The following proposition proves that an equilibrium exists and that the PIH is satisfied.

Proposition 10. *Suppose the true economy is governed by the approximating model. In equilibrium, each consumer's optimal consumption-portfolio rules are described by:*

$$c_t^* = r^* s_t, \tag{51}$$

and

$$\alpha^* = \overline{\alpha}, \tag{52}$$

respectively. Furthermore, in equilibrium, the evolution equation of s_t *is:*

$$ds_t = \left(\frac{\pi^{*2}}{r^*\widetilde{\gamma}\sigma_e^2}\right) dt + \sigma dB_t, \tag{53}$$

and

$$\pi^* = r^*\widetilde{\gamma}\sigma_e\left(\rho_{ye}\sigma_s + \overline{\alpha}\sigma_e\right). \tag{54}$$

If $\rho > \delta$ and $\rho_{ye} \geq 0$, the equilibrium is unique.

Proof. See Online Appendix C. ■

Figure 6 shows that the aggregate saving function $D(\vartheta, r)$ is increasing with the interest rate for different values of ρ_{ye}.[33] It also clearly shows that there exists a unique interest rate r^* for every given ρ_{ye} such that $D(\vartheta, r^*) = 0$, and a higher correlation between the equity return and labor income leads to a higher equilibrium interest rate given ϑ. The intuition behind this result is that the presence of the risky asset helps hedge labor income risk, leading to less precautionary savings. However, we can see from the figure that ρ_{ye} does not have significant effects on the equilibrium interest rate. The following result is an immediate implication on how the presence of the risky asset affects the relative dispersion of consumption growth to income growth under RB.

Proposition 11. *The relative dispersion of consumption growth to income growth is*

$$\mu \equiv \frac{\mathrm{sd}\left(dc_t^*\right)}{\mathrm{sd}\left(dy_t\right)} = r^* \sqrt{\left(\frac{1}{r^* + \rho}\right)^2 + \left(\frac{\sigma_e}{\sigma_y}\right)^2 \overline{\alpha}^2 + 2\frac{\rho_{ye}}{r^* + \rho}\frac{\sigma_e}{\sigma_y}\overline{\alpha}}. \tag{55}$$

Comparing (32) with (55), it is clear that the positive net supply of the risky asset will be helpful at increasing the relative dispersion of consumption to income while keeping the real interest rate at a low level. To quantitatively examine the effects of RB on the relative dispersion of consumption growth to income growth, we first use the observed risk premium of 7.2 percent to calibrate the value of $\overline{\alpha}$ using (47). Estimating the correlation between individual labor income and the equity return is complicated by the lack of panel data on household portfolio choice, and we find several estimates in the literature: Viceira (2001) adopts $\rho_{ye} = 0.35$ when simulating his life-cycle consumption-portfolio choice model. Davis and Willen (2000) estimate that the correlation is between 0.1 and 0.3 for college-educated men, and is 0.25 or more for college-educated women. Here, we follow Viceira (2001) and set $\rho_{ye} = 0.35$. In our model, if $\gamma = 4.5$, $\psi = 0.5$, $\vartheta = 2.9$, $\delta = 0.03$, and $\rho_{ye} = 0.35$, the corresponding DEP (p) is 0.19, the equilibrium interest rate (r^*) is 1.91 percent, and the relative dispersion (μ) is 0.31, which equals the empirical counterpart for the sample from 1980 to 1996. We summarize these results in Table 3.

5.4. Explaining the Decline in the Relative Consumption Dispersion

To test the model's predictions on the effects of robustness on the dynamic relative consumption dispersion, we quantitatively examine how well a calibrated version of our extended model can explain the decline in the relative consumption dispersion from 1980 to 2010 (see Figure 1). To do this, we divide our sample into two periods, 1980-1995 and 1996-2010. We calibrate our model to

[33] Here, we set $\gamma = 4.5$, $\psi = 0.5$, $\vartheta = 2.9$, and $\delta = 0.03$. The parameters in the income process are the same as before.

the first period by choosing the robustness parameter (ϑ) and the aggregate asset supply parameter ($\overline{\alpha}$) to match the observed real interest rate (r) and the relative consumption dispersion (μ_{cy}). To focus on the effects of robustness, we fix the other parameters at the same values used in the previous section. Then, we let the robustness parameter vary to match the real interest rate in the second period. In other words, this is the amount of model uncertainty the model needs to explain the decline in the real interest rate from the first period to the second period. Finally, we check how much this change in the amount of model uncertainty can explain the decline in the relative consumption dispersion from the first period to the second period.

The results in Table 4 show the model does a good job of explaining the decline in the relative consumption dispersion. The first row shows that the model with $\overline{\alpha} = 8.8$ and $\vartheta = 0.57$ matches the average real interest rate of 3.1% and the average relative consumption dispersion of 0.39 in the first period, 1980-1995. To generate a real interest rate of 1.5% in the second period, 1996-2010, we need ϑ to increase from 0.57 to 19, which corresponds to a decrease in the DEP from 0.46 to 0.06 (third column in the table). Remember a larger DEP means that either it is more difficult to distinguish the approximating model and the distorted model, or there is less model uncertainty. The calibrated results therefore suggest that both the degree of robustness and the amount of model uncertainty increased significantly from the first period to the second period. With this increase in the degree of robustness, the model predicts a decrease in the relative consumption dispersion from 0.39 to 0.21, which nearly matches the observed decrease in the data. This accurate out-of-sample prediction provides additional evidence that incorporating model uncertainty due to ambiguity and robustness can help the model to better explain the data.

6. The Welfare Cost of Model Uncertainty

The uncertainty about model specifications due to a preference for robustness or ambiguity aversion generates welfare losses. We measure the welfare cost of model uncertainty in a standard and intuitive way–the amount of wealth or income an average consumer is willing to pay to remove or reduce such uncertainty. In particular, we provide two approaches to calculate the welfare cost of model uncertainty from different angles. The first approach is based on Lucas' (1987) elimination-of-risk method that tells us how much the consumers would pay to fully resolve all model uncertainty. The second approach is based on Barro's (2009) local welfare analysis which allows us to answer questions such as "how much consumers would pay to reduce partial uncertainty, such as 10 percent of model uncertainty, in order to keep the level of lifetime utility unchanged." We evaluate welfare costs based on our extended model that better accounts for data on the joint behavior of consumption, income, and asset returns than the benchmark model, consistent with Barro's (2009) argument that welfare and policy analyses of the impacts of consumption uncertainty should be carried out within models that can at least roughly capture the stylized facts in the asset markets. However, our qualitative points remain valid in the benchmark model without the risky asset.

It is worth noting that although our welfare analysis under robustness is related to that discussed in the endowment economies proposed in Barillas, et al. (2009) and Ellison and Sargent (2015), our paper is the first to examine the welfare costs of model uncertainty if consumers are allowed to choose optimal consumption-portfolio rules.

6.1. Total Welfare Gains from Eliminating Model Uncertainty

In this section, we follow Lucas's elimination-of-risk method (see Lucas (1987), Obstfeld (1994), and Tallarini (2000)) to quantify the welfare cost of RB in the general equilibrium. It is worth noting that in Lucas (1987), the welfare cost of aggregate fluctuations is expressed in terms of percentage of consumption the representative agent in the *endowment* economy is willing to give up at all dates to switch to the deterministic world. However, such a welfare measure may not be very informative when the consumers can choose optimal consumption–saving-portfolio plans because the marginal propensity of consumption out of total wealth is now endogenous and affects consumption growth. We thus follow Obstfeld (1994) and compute the welfare gains of removing uncertainty as an equivalent variation: by what percentage of initial wealth a typical consumer is willing to give up to be as well off in the RE economy as he is in the RB economy.[34]

Specifically, we define:

$$\widetilde{f}\left(s_0\left(1-\Delta\right)\right) = f\left(s_0\right), \tag{56}$$

where

$$\widetilde{f}\left(s_0\left(1-\Delta\right)\right) = -\frac{\delta}{\widetilde{\alpha}_1}\exp\left(-\widetilde{\alpha}_0 - \widetilde{\alpha}_1 s_0\left(1-\Delta\right)\right) \text{ and } f\left(s_0\right) = -\frac{\delta}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1 s_0\right) \tag{57}$$

are the value functions under RE and RB, respectively, Δ is the compensating amount measured as a percentage of s_0, the initial wealth,

$$\alpha_1 = \frac{r^*}{\psi}, \widetilde{\alpha}_1 = \frac{\widetilde{r}^*}{\psi},$$
$$\alpha_0 = \frac{\delta}{r^*} - 1 - \frac{1}{2}\frac{r^*}{\psi}\left(\gamma + \frac{\vartheta}{\psi}\right)\left(\sigma_s^2 - \overline{\alpha}^2\sigma_e^2\right), \widetilde{\alpha}_0 = \frac{\delta}{\widetilde{r}^*} - 1 - \frac{1}{2}\frac{\widetilde{r}^*}{\psi}\left(\gamma + \frac{\vartheta}{\psi}\right)\left(\widetilde{\sigma}_s^2 - \overline{\alpha}^2\sigma_e^2\right),$$

and r^* and $\widetilde{r}^*$ are the equilibrium interest rates in the RE and RB economies, respectively.[35] The following proposition summarizes the result about how RB affects the welfare costs in general equilibrium.

[34]Epaulard and Pommeret (2003) also use this approach to examine the welfare cost of volatility in a representative-agent model with recursive utility. In their model, the total welfare cost of volatility is defined as the percentage of capital the representative agent is ready to give up at the initial period to be as well off in a certain economy as he is in a stochastic one.

[35]When we compare welfare in these two economies, we assume that the asset supply is the same across the two economies. See Online Appendix C for the derivation of the value functions. Note that $\Delta = 0$ if $\vartheta = 0$.

Proposition 12. *The welfare costs due to model uncertainty are given by:*

$$\Delta = \frac{s_0 \left(\widetilde{\alpha}_1 - \alpha_1 \right) - \ln \left(\widetilde{\alpha}_1 / \alpha_1 \right) + \left(\widetilde{\alpha}_0 - \alpha_0 \right)}{\widetilde{\alpha}_1 s_0} = \left(1 - \frac{r^*}{\widetilde{r}^*} \right) + \frac{\psi}{\widetilde{c}_0} \ln \left(\frac{\widetilde{r}^*}{r^*} \right) + \frac{\psi \left(\widetilde{\alpha}_0 - \alpha_0 \right)}{\widetilde{c}_0}, \tag{58}$$

where $\widetilde{c}_0 = \widetilde{r}^* s_0$ is optimal consumption under RE.

Proof. Substituting the equilibrium condition (25) into the expressions of α_0 and $\widetilde{\alpha}_0$ in the value functions under RE and RB, we obtain that:

$$\alpha_0 = \frac{1}{\psi} \overline{\alpha} \pi, \widetilde{\alpha}_0 = \frac{1}{\psi} \overline{\alpha} \widetilde{\pi}.$$

Combining these results with (56) yields (58). ∎

Note that when $\overline{\alpha} = 0$, our multiple-asset model reduces to the benchmark model and $\Delta = \left(1 - \frac{r^*}{\widetilde{r}^*} \right) + \frac{\psi}{\widetilde{c}_0} \ln \left(\frac{\widetilde{r}^*}{r^*} \right)$. To understand how the welfare cost varies with the degree of uncertainty aversion, we note that:

$$\frac{\partial \Delta}{\partial \vartheta} = \frac{\partial \Delta}{\partial r^*} \frac{\partial r^*}{\partial \vartheta}.$$

The second term is negative, $\partial r^* / \partial \vartheta < 0$, for the reasons we have already discussed. The first term is

$$\frac{\partial \Delta}{\partial r^*} = -\frac{1}{\widetilde{r}^*} \left(1 - \frac{\psi}{\widetilde{c}_0} \right) - \frac{\overline{\alpha}}{\widetilde{c}_0} \widetilde{\gamma} \sigma_e \left(\rho_{ye} \sigma_s + \overline{\alpha} \sigma_e \right);$$

for reasonable values, we expect this term to be negative, so that higher model uncertainty leads to larger welfare costs.

To provide some quantitative results, we set $c_0 = \widetilde{r}^* s_0 = 1$ as we did in our calibrated model. Figure 7 illustrates how the welfare cost of model uncertainty varies with ϑ for different values of $\overline{\alpha}$ and ρ_{ye}. The left panel of the figure shows that the welfare costs of model uncertainty are nontrivial. For example, when $\vartheta = 2.9$, the value we calibrate to match the data (see Table 3), the welfare cost of model uncertainty Δ is 15.1 percent. That is, a typical consumer is willing to scarify 15.1 percent of his initial wealth in order to get rid of such model uncertainty. Furthermore, the welfare cost rises with the degree of model uncertainty. For instance, if ϑ increases by 50% (from 2.9 to 4.35), Δ increases by about 25% (from 15.1 percent to 19.0 percent). It is worth noting that these values of welfare costs are not directly comparable to that obtained in Lucas (1987) for because our welfare calculations are based on removing all model uncertainty when individual households make optimal consumption-saving-portfolio choices, while Lucas-type calculations are based on removing business cycles fluctuations in an *endowment* representative agent conomy.[36]

[36] Lewis (2000) finds that the welfare gains from international risk sharing based on equity returns are around 10 to 50 percent. He argues that the large difference is due to the high volatility of equity returns and the implied intertemporal substitution in the marginal utility.

In addition, the same figure shows that the welfare cost decreases with $\overline{\alpha}$, given the value of ϑ. The reason behind this result is that the risky asset provides a hedging tool for the consumer as long as $\rho_{ye} \neq 0$, and higher supply of the asset means that agents' inefficient precautionary savings motives are weaker. The right panel of Figure 7 shows that the welfare cost of model uncertainty also decreases with the correlation between the equity return and labor income. The intuition behind this result is that the higher correlation between the two risks can make the consumers better hedge the fundamental risk, which reduces the welfare cost of uncertainty.

6.2. The Local Welfare Effects of Model Uncertainty

In this section, we examine the *local* effects of RB on welfare costs. Rather than removing all uncertainty as we did in Section 6.1, here we define a cost that measures the welfare benefits from reducing model uncertainty at the margin. This marginal analysis is useful because most economic policies would not be designed to eliminate uncertainty entirely and thus calculating the potential benefits at the marginal may be useful in and of itself.

To examine the *local* effects of RB on welfare, we follow Barro (2009) and Luo and Young (2010) to compute the marginal welfare costs due to model uncertainty at different degrees of robustness (ϑ). The basic idea of this calculation is to use the value function (57) to calculate the effects of RB on the expected lifetime utility and compare them with those from proportionate changes in the initial income level. Specifically, following Barro (2009), the marginal welfare costs (mwc) due to RB can be written as:

$$\text{mwc } (\vartheta) = -\frac{\partial f_t / \partial \vartheta}{(\partial f / \partial y_t)\, y_t}|_{y_t = y_0} = \frac{1}{2\psi} \frac{\sigma_y^2 - \overline{\alpha}^2 \sigma_e^2 (r+\rho)^2}{(r+\rho)\, y_0}, \tag{59}$$

where $\partial f / \partial \vartheta$ and $\partial f / \partial y_t$ are evaluated in equilibrium for given y_0.[37] The value of mwc provides the proportionate increase in initial income that compensates, at the margin, for an increase in the degree of robustness–in the sense of keeping the level of lifetime utility unchanged. From (57), it is clear that this compensating income change depends on the EIS, the properties of the income process, and the equilibrium interest rate.

To provide some quantitative results, we use the same set of parameter values in the above calibrated model: $y_0 = 1$, $\gamma = 4.5$, $\psi = 0.5$, $\rho = 0.083$, $\sigma_y = 0.182$, $\overline{\alpha} = 10.5$, $\sigma_e = 0.16$, and $r = 1.91\%$. Based on these parameter values, we can calculate that mwc $= 0.193$, which suggests that a 10 percent increase in ϑ (from 2.9 to 3.2) requires an increase in initial income by 1.23% (i.e., mwc $\cdot 0.1 \cdot \vartheta = 0.193 \cdot 0.1 \cdot 2.9 = 1.23$) to make his lifetime utility unchanged. In other words, a typical consumer in our model economy would be willing to sacrifice 1.23% of his initial income to reduce the degree of model uncertainty by 10%. In addition, from (59) we can see that mwc is an

[37]Note that here we use the facts that $s_t = w_t + y_t / (r+\rho) + \overline{y} / (r(r+\rho))$ and $\sigma_s^2 = \sigma_y^2 / (r+\rho)^2$.

increasing function of income volatility (σ_y), which means the marginal welfare cost will be larger if the economy is in a more volatile environment. To quantitatively see this point, let's assume σ_y increases by 20% from 0.182 to 0.22. Following the same calculation above, under this more volatile environment, the welfare costs of a 10 percent increase in ϑ leads to a welfare loss equivalent to a 5.61% decline in initial income, significantly larger than the 1.23% under the low volatility environment.[38] This highlights the potentially larger welfare losses due to model uncertainty when the economy is facing larger income volatility, such as during economic crises. It is worth noting that the higher the supply of the risky asset, the less the welfare costs of model uncertainty. The main reason behind this result is that the presence of the risky asset can hedge the income risk and help the consumers better insure the risk and uncertainty they face. In addition, the marginal welfare cost is also decreasing with the value of the EIS because the higher the value of EIS, the less the impact of RB on total welfare of uncertainty-averse households.

In addition, we can also examine the *local* effects of income uncertainty on the welfare cost of model uncertainty for different degrees of RB. Specifically, the marginal welfare costs (mwc) due to income uncertainty can be written as:

$$\text{mwc}\left(\sigma_y^2\right) = -\frac{\partial f_t/\partial\sigma_y^2}{\left(\partial f/\partial y_t\right) y_t}|_{y_t=y_0} = \frac{1}{2}\frac{\widetilde{\gamma}}{\left(r+\rho\right) y_0}, \tag{60}$$

where $\partial f/\partial\sigma_y^2$ and $\partial f/\partial y_t$ are evaluated in equilibrium for given y_0. The value of mwc gives us the proportionate increase in y_0 to compensate for a small increase in σ_y^2 in the sense of keeping the level of lifetime utility unchanged. This formula can help us evaluate the importance of economic policies that aim to reduce income uncertainty of households. It is also clear from (60) that the higher the effective coefficient of risk aversion ($\widetilde{\gamma}$), the greater the welfare costs of income uncertainty. The intuition behind this result is that more risk- and uncertainty-averse consumers suffer more from income uncertainty.

Let's consider the following simple policy experiment. The government is implementing a macro policy to reduce the variance of household income by 10 percent, from σ_y^2 to $0.9\cdot\sigma_y^2$. Holding all the other parameters fixed, we can calculate that the 10 percent reduction in income variance leads to a welfare improvement equivalent to a 16.9 percent increase in the initial income. In other words, a typical household is willing to reduce his initial income by 16.9% to reduce the variance of his income process by 10 percent. As a comparison, this welfare gain is only 7.4% of the initial income if there is no model uncertainty. One policy implication stemming from this finding is that macro policies aiming to reduce income volatility and inequality are more beneficial in an economy in which consumers have a greater aversion to model uncertainty, both because

[38]Reducing the value of ρ can generate a similar result because either an increase in σ_y or a reduction in ρ lead to greater income uncertainty facing the consumers.

they reduce risk and because they mitigate costly precautionary saving.[39] Finally, although our benchmark model has no business cycle dynamics, the above welfare calculations can still help us infer some insight about the welfare costs of business cycles under RB. Note that one key fact about the US business cycles is that income volatility is countercyclical.[40] Specifically, when the economy moves from an expansion into a recession, σ_y will increase. At the same time, consumers may become more concerned about model misspecification and will thus suffer more from model uncertainty. Consequently, a macro policy aiming to reduce the aggregate fluctuations would be more beneficial to this economy.

7. Conclusions and Future Research

This paper has developed a tractable continuous-time recursive utility version of the Huggett (1993) model to explore how the preference for robustness interacts with intertemporal substitution and risk aversion and then affects the interest rate, the dynamics of consumption and income, and the welfare costs of model uncertainty in general equilibrium. We find that for moderate risk aversion and plausibly calibrated parameter values of robustness, our benchmark model can generate the observed low risk-free rate in the US economy. However, the model cannot generate the observed high relative dispersion of consumption to income. But if we allow for a positive net supply of a risky asset, our model is able to reconcile low interest rates, moderate risk aversion, and relatively high dispersion of consumption to income. The resulting model implies that the welfare costs of model uncertainty are large.

To better illustrate the key effects of robustness on the equilibrium interest rate and relative consumption dispersion, we choose a framework that can analytically show the key mechanisms and mathematically is as simple as possible. However, our key insights can also be carried to more complicated cases. For example, our framework can be extended to study implications of robustness in a hidden-state model in which the consumers cannot perfectly observe the growth of their stochastic labor income. As discussed in Hansen and Sargent (Chapters 17 and 18, 2007), in this case, agents' preference for robustness not only affects their optimal control problem but also affects their optimal filtering problem. The effect of robustness on optimal filtering also provides additional information that could be used to further distinguish ambiguity aversion (a preference for robustness) from risk aversion. We leave this extension for future research.

[39] Ellison and Sargent (2015) find that idiosyncratic consumption risk has a greater effect on the cost of business cycles when agents fear model misspecification. In addition, they showed that endowing agents with fears about misspecification leads to greater welfare costs caused by the idiosyncratic consumption risk. The underlying reasons are the same: the enhanced risk aversion created by uncertainty aversion.

[40] As discussed in Bloom (2014), unemployment rises during a recession, so the volatility of income at the household level will increase as well.

8. Appendix

8.1. Proof of the Existence and Uniqueness of the Equilibrium

Proof. There exists at least one equilibrium interest rate $r^* \in (0, \delta)$ in the our benchmark RB model; if $\delta < \rho$, the equilibrium interest rate is unique on $(0, \delta)$. If $r > \delta$, both $\Gamma(\vartheta, r)$ and $\Psi(r)$ in the expression for total savings $D(\vartheta, r)$ are positive, which contradicts the equilibrium condition $D(\vartheta, r) = 0$. Since $\Gamma(\vartheta, r) - \Psi(r) < 0$ (> 0) when $r = 0$ $(r = \delta)$, the continuity of the expression for total savings implies that there exists at least one interest rate $r^* \in (0, \delta)$ such that $D(\vartheta, r^*) = 0$. To establish the conditions under which this equilibrium is unique, we take the derivative:

$$\frac{\partial D(\vartheta, r)}{\partial r} = \left(\gamma + \frac{\vartheta}{\psi}\right) \frac{\sigma^2}{(r+\rho)^2} \left(\frac{1}{2} - \frac{r}{r+\rho}\right) + \frac{\delta \psi}{r^2}$$

and note a sufficient condition for this derivative to be positive for any $r > 0$ is

$$\frac{1}{2} - \frac{r}{r+\rho} > 0 \Leftrightarrow r < \rho.$$

Therefore, if $\rho > \delta$, there is only one equilibrium in $(0, \delta)$. From Expression (13), we can obtain the individual's optimal consumption rule under RB in general equilibrium as $c_t^* = r^* s_t$. Substituting (26) into (3) yields (27). Using (5) and (26), we can obtain (28).

In general equilibrium, the state transition equation is $ds_t = \sigma_s dB_t$ if the true economy is governed by the approximating model. Using the definition of $\theta_t = -\vartheta / f(s_t)$, we have:

$$\ln \theta_t = \ln\left(\frac{\vartheta r^*}{\delta \psi}\right) + \frac{r^*}{\psi} s_t \text{ or } d \ln \theta_t = \frac{r^* \sigma_s}{\psi} dB_t,$$

which means that

$$\frac{d\theta_t}{\theta_t} = \frac{1}{2}\left(\frac{r^* \sigma_s}{\psi}\right)^2 dt + \left(\frac{r^* \sigma_s}{\psi}\right) dB_t. \tag{61}$$

■

In the extended model with a risky asset, the proof of the equilibrium existence and uniqueness is the same as that for our benchmark model except that we replace σ^2 with $\left(1 - \rho_{ye}^2\right) \sigma^2$. Similarly, if $\rho > \delta$ and $\rho_{ye} \geq 0$, this equilibrium is unique.

8.2. The Calibrating Procedure

The general idea of the calibration procedure is to find a value of ϑ such that $p(\vartheta)$ equals a given value after simulating model P, (5), and model Q, (9).[41] The detailed procedure is as follows. First,

[41] The number of periods used in the calculation, N, is set to be 31, the actual length of the data $(1980 - 2010)$.

define p_P as:

$$p_P = \text{Prob}\left(\ln\left(\frac{L_Q}{L_P}\right) > 0 \middle| P\right), \tag{62}$$

where $\ln\left(\frac{L_Q}{L_P}\right)$ is the log-likelihood ratio. (62) means that when model P generates the data, p_P measures the probability that a likelihood ratio test selects model Q. In this case, we call p_P the probability of the model detection error. Similarly, when model Q generates the data, we can define p_Q as:

$$p_Q = \text{Prob}\left(\ln\left(\frac{L_P}{L_Q}\right) > 0 \middle| Q\right). \tag{63}$$

Given initial priors of 0.5 on each model and the length of the sample is N, the detection error probability, p, can be written as:

$$p(\vartheta) = \frac{1}{2}(p_P + p_Q), \tag{64}$$

where ϑ is the robustness parameter used to generate model Q. Given this definition, we can see that $1 - p$ measures the probability that econometricians can distinguish the approximating model from the distorted model.

In the continuous-time model with the iid Gaussian specification, $p(\vartheta)$ can be easily computed. Since both models P and Q are arithmetic Brownian motions with constant drift and diffusion coefficients, the log-likelihood ratios are Brownian motions. The logarithm of the Radon-Nikodym derivative of the distorted model (Q) with respect to the approximating model (P) can be written as:

$$\ln\left(\frac{L_Q}{L_P}\right) = \int_0^t \overline{b} dB_s - \frac{1}{2}\int_0^t \overline{b}^2 ds, \tag{65}$$

where

$$\overline{b} \equiv b^* \sigma_s = -\frac{\vartheta}{\psi} r^* \sigma_s. \tag{66}$$

Similarly, the logarithm of the Radon-Nikodym derivative of the approximating model (P) with respect to the distorted model (Q) is:

$$\ln\left(\frac{L_P}{L_Q}\right) = -\int_0^t \overline{b} dB_s + \frac{1}{2}\int_0^t \overline{b}^2 ds. \tag{67}$$

References

[1] Aiyagari, S Rao (August 1994), "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics* 109 (3): 659–684.

[2] Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent (2003), "A Quartet of Semigroups for Model Specification, Robustness, Prices of Risk, and Model Detection," *Journal of the European Economic Association*, 1(1), 68-123.

[3] Angeletos, George-Marios and Laurent-Emmanuel Calvet (2006), "Idiosyncratic Production Risk, Growth and the Business Cycle," *Journal of Monetary Economics* 53(6), 1095-1115.

[4] Barillas, Francisco, Lars Peter Hansen, and Thomas J. Sargent (2009), "Doubts or Variability?" *Journal of Economic Theory* 144(6), 2388–2418.

[5] Barro, Robert J. (2009), "Rare Disasters, Asset Prices, and Welfare Costs," *American Economic Review* 99(1), 243-264.

[6] Barro, Robert J., Jesús Fernández-Villaverde, Oren Levintal, and Andrew Mollerus (2017), "Safe Assets" Working Paper 20652, National Bureau of Economic Research.

[7] Benhabib, Jess, Alberto Bisin, and Shenghao Zhu (2011), "The Distribution of Wealth and Fiscal Policy in Economies With Finitely Lived Agents," *Econometrica* 79(1), 123-157.

[8] Bhandari, Anmol, Jaroslav Borovička, and Paul Ho (2017), "Survey Data and Subjective Beliefs in Business Cycle Models," manuscript, University of Minnesota.

[9] Blanchard, Olivier, Davide Furceri, and Andrea Pescatori (2014), "A Prolonged Period of Low Real Interest Rates?," in *Secular Stagnation: Facts, Causes and Cures*, Coen Tuulings and Richard Baldwin (eds.), CEPR Press.

[10] Bloom, Nicholas (2014), "Fluctuations in Uncertainty," *Journal of Economic Perspectives* 28(2), 153-176.

[11] Blundell, Richard, Luigi Pistaferri, and Ian Preston (2008), "Consumption Inequality and Partial Insurance," *American Economic Review* 98(5), 1887-1921.

[12] Caballero, Ricardo (1990), "Consumption Puzzles and Precautionary Savings," *Journal of Monetary Economics* 25(1), 113-136.

[13] Caballero, Ricardo and Arvind Krishnamurthy (2008), "Collective Risk Management in a Flight to Quality Episode," *Journal of Finance* 63(5), 2195-2230.

[14] Campbell, John Y. (2003) "Consumption-Based Asset Pricing." In *Handbook of the Economics of Finance Vol. 1B*, edited by George Constantinides, Milton Harris, and Rene Stulz, pp. 803-887. North-Holland Press.

[15] Chen, Zengjing and Larry Epstein (2002), "Ambiguity, Risk, and Asset Returns in Continuous Time," *Econometrica* 70(4), 1403-1443.

[16] Crump, Richard K., Stefano Eusepi, Andrea Tambalotti, and Giorgio Topa (2015), "Subjective Intertemporal Substitution," Federal Reserve Bank of New York Staff Reports, Number 734.

[17] Davis, Steven and Paul Willen (2000), "Using Financial Assets to Hedge Labor Income Risks: Estimating the Benefits," manuscript, University of Chicago and Boston Fed.

[18] Duffie, Darrell and Epstein, Larry (1992), “Stochastic Differential Utility,” *Econometrica* 60(2), 353-94.

[19] Ellison, Martin and Thomas J. Sargent (2015), “Welfare Cost of Business Cycles in Economies with Individual Consumption Risk,” *American Economic Journal: Macroeconomics* 7(2), 40-57.

[20] Epaulard, Anne and Aude Pommeret (2003), “Recursive Utility, Growth, and the Welfare Cost of Volatility,” *Review of Economic Dynamics* 6(3), 672-684.

[21] Epstein, Larry G. and Stanley E. Zin (1989), “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework,” *Econometrica* 57(4), 937-969.

[22] Fischer, Stanley (2016), “Monetary Policy, Financial Stability, and the Zero Lower Bound,” *American Economic Review* 106(5), 39-42.

[23] Floden, Martin and Jesper Lindé (2001), “Idiosyncratic Risk in the United States and Sweden: Is There a Role for Government Insurance?” *Review of Economic Dynamics* 4, 406-437.

[24] Gabaix, Xavier, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll (2016), “The Dynamics of Inequality,” *Econometrica* 84, 2071-2111.

[25] Guvenen, Fatih (2006), “Reconciling Conflicting Evidence on the Elasticity of Intertemporal Substitution: A Macroeconomic Perspective,” *Journal of Monetary Economics* 53(7), 1451-1472.

[26] Hall, Robert E. (2017), “The Role of the Growth of Risk-Averse Wealth in the Decline of the Safe Real Interest Rate,” manuscript, Stanford University.

[27] Hansen, Lars Peter and Jianjun Miao (2018), “Aversion to Ambiguity and Model Misspecification in Dynamic Stochastic Environments," manuscript, University of Chicago and Boston University.

[28] Hansen, Lars Peter and Thomas J. Sargent (1995), “Discounted Linear Exponential Quadratic Gaussian Control,” *IEEE Transactions on Automatic Control* 40, 968-71.

[29] Hansen, Lars Peter and Thomas J. Sargent (2007), *Robustness*, Princeton University Press.

[30] Hansen, Lars Peter and Thomas J. Sargent (2011), “Robustness and Ambiguity in Continuous Time,” *Journal of Economic Theory* 146(3): 1195-1223.

[31] Hansen, Lars Peter, Thomas J. Sargent, and Thomas D. Tallarini, Jr. (1999), “Robust Permanent Income and Pricing,” *Review of Economic Studies* 66(4), 873-907.

[32] Havránek, Tomáš (2015), “Measuring Intertemporal Substitution: The Importance of Method Choices and Selective Reporting,” *Journal of the European Economic Association* 13(6), 1180-1204.

[33] Huggett, Mark (1993), "The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control* 17(5-6), 953-969.

[34] Kasa, Kenneth and Xiaowen Lei (2017), "Risk, Uncertainty, and the Dynamics of Inequality," *Journal of Monetary Economics* 94, 60-78.

[35] Krueger, Dirk and Fabrizio Perri (2006), "Does Income Inequality Lead to Consumption Inequality? Evidence and Theory," *Review of Economic Studies* 73(1), 163–193.

[36] Ju, Nengjiu, and Jianjun Miao (2012), "Ambiguity, Learning, and Asset Returns," *Econometrica* 80(2), 559–591.

[37] Laubach, Thomas and John C. Williams (2015), "Measuring the Natural Rate of Interest Redux," Federal Reserve Bank of San Francisco Working Paper 2015-16.

[38] Lewis, Karen (2000), "Why do Stocks and Consumption imply Such Different Gains from International Risk Sharing," *Journal of International Economics* 52, 1-35.

[39] Lucas, Robert E. (1987), *Models of Business Cycles*, New York: Basil Blackwell.

[40] Luo, Yulei and Eric R. Young (2010), "Risk-sensitive Consumption and Savings under Rational Inattention," *American Economic Journal: Macroeconomics* 2(4), 281-325.

[41] Luo, Yulei, Jun Nie, Gaowang Wang, and Eric R. Young (2017), "Rational Inattention and Dynamics of Consumption and Wealth in General Equilibrium," *Journal of Economic Theory* 172, 55-87.

[42] Maenhout, Pascal (2004), "Robust Portfolio Rules and Asset Pricing," *Review of Financial Studies* 17, 951-983.

[43] Obstfeld, Maurice (1994), "Risk-Taking, Global Diversification and Growth." *American Economic Review* 84, 1310-1329.

[44] Skiadas, Costis (2009), *Asset Pricing Theory*, Princeton University Press.

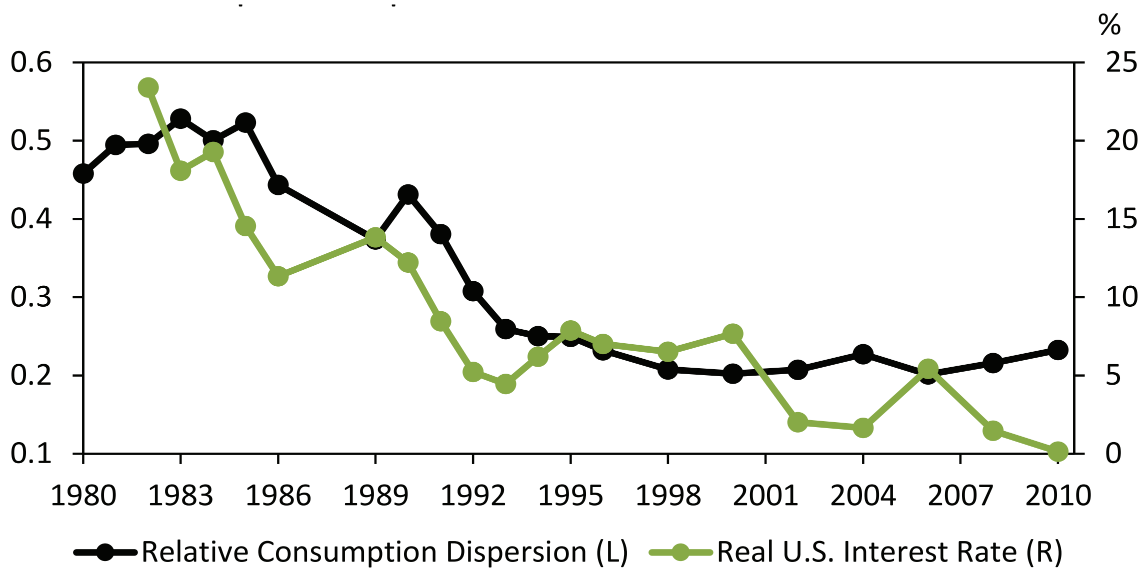
[45] Summers, Larry (2014), "U.S. Economic Prospects: Secular Stagnation, Hysteresis, and the Zero Lower Bound," *Business Economics* 49(2), 49-65.

[46] Tallarini, Thomas D. (2000), "Risk-sensitive Real Business Cycles," *Journal of Monetary Economics* 45(3), 507-532.

[47] Viceira, Luis. (2001) "Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income." *Journal of Finance*, 56, 433-470.

[48] Vissing-Jørgensen, Annette and Orazio P. Attanasio (2003), "Stock-Market Participation, Intertemporal Substitution, and Risk-Aversion," *American Economic Review* 93(2), 383-391.

[49] Wang, Neng (2003), "Caballero Meets Bewley: the Permanent-Income Hypothesis in General Equilibrium," *American Economic Review* 93(3), 927-936.



Note: "Relative consumption dispersion" is defined as the standard deviation of consumption changes to the standard deviation of income changes; "Real U.S. interest rate" is defined as the 3-month Treasury yield deflated by the PCE index

Figure 1. *Real Risk-free Rates and Relative Consumption Dispersion*

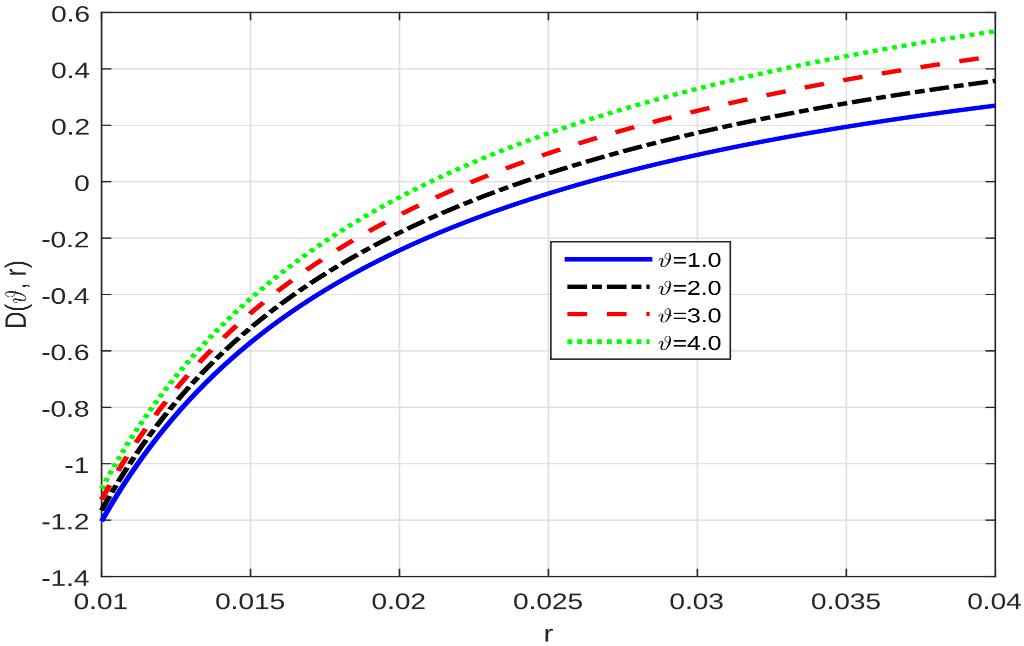


Figure 2. *Effects of RB on Aggregate Savings*

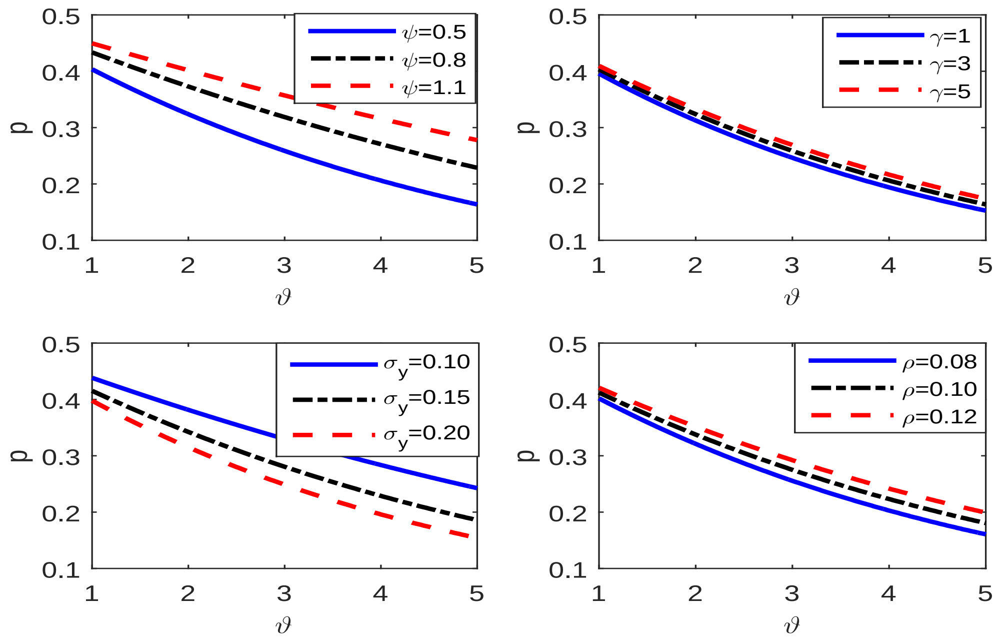


Figure 3. *Relationship between* ϑ *and* p

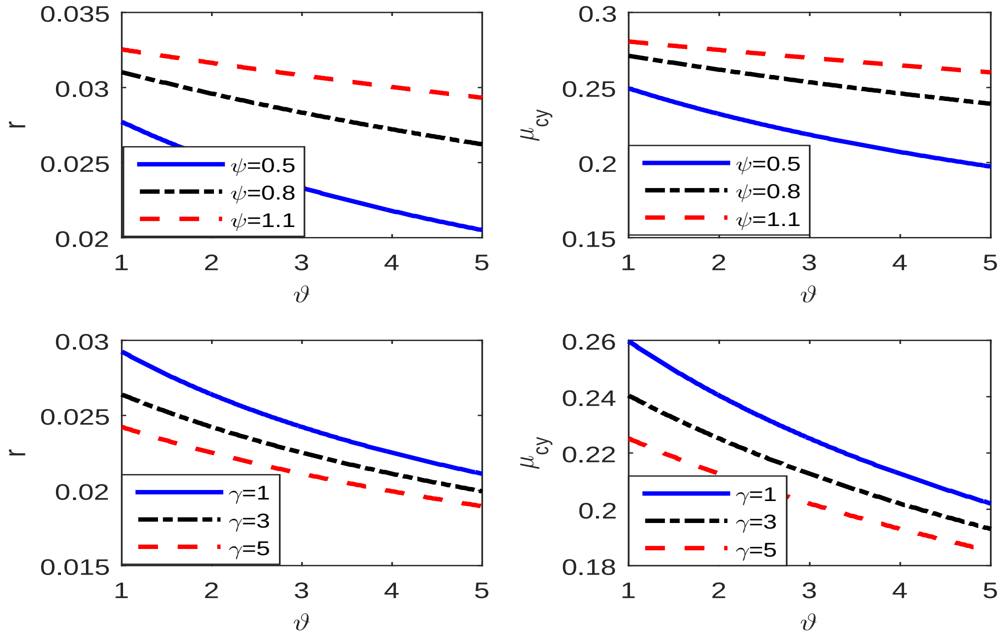


Figure 4. *Effects of RB on the Interest Rate and Consumption Volatility*

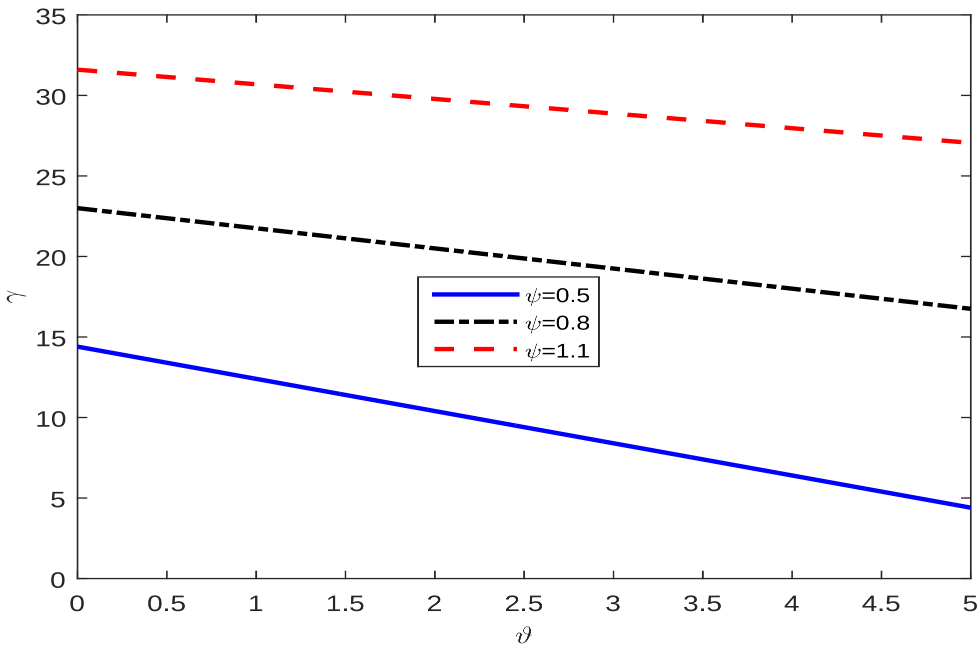


Figure 5. *Relationship between γ and ϑ.*

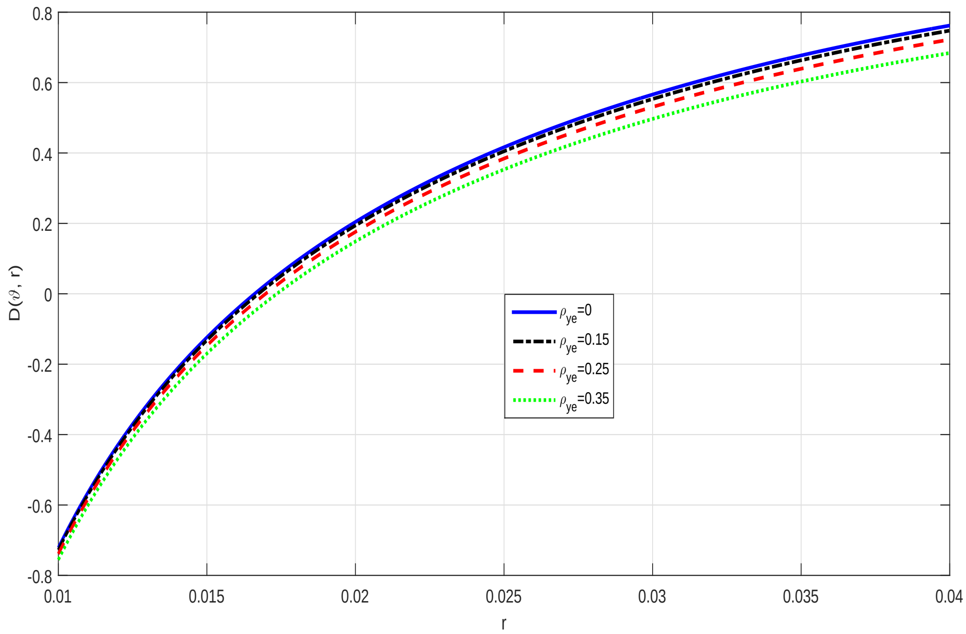


Figure 6. *Effects of RB on Aggregate Savings*

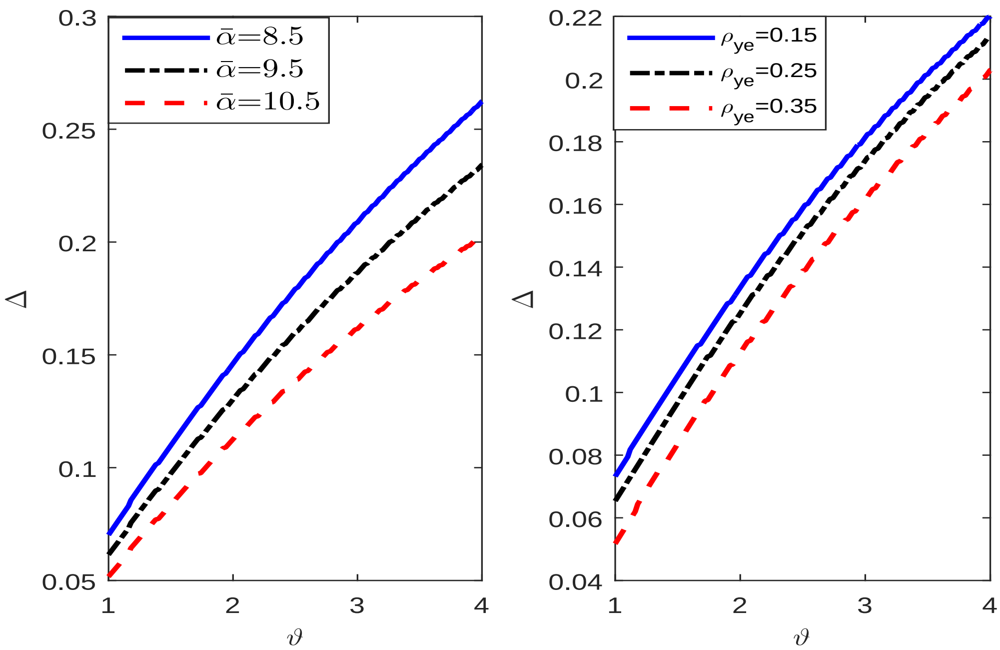


Figure 7. *Effects of RB on the Welfare Cost*

Table 1. *Measures of the Risk Free Rate*

	Three-month Nominal T-Bond	One-year Nominal T-Bond
CPI Inflation $(1981-2010)$	1.87%	2.33%
PCE Inflation $(1981-2010)$	1.96%	2.42%
CPI Inflation $(1981-2015)$	1.37%	1.78%
PCE Inflation $(1981-2015)$	1.75%	2.16%

Table 2. *Estimation and Calibration Results*

Labor Income	Parameter	Values
Discrete time specification		
Constant	ϕ_0	0.0005
Persistence	ϕ_1	0.919
Std. of shock	σ	0.175
Continuous-time specification		
Persistence	ρ	0.083
Std. of income changes	σ_y	0.182

Table 3. *Model Comparison with Key Parameter Values*

Parameters Moments	Data	RE	RB (Benchmark)	RB ($\overline{\alpha} = 10.5$)
γ		14.5	4.5	4.5
ϑ		0	5	2.9
p		0.5	0.17	0.19
r^*	1.91%	1.91	1.91	1.91
π^*	7.2%	n.a.	n.a.	7.2%
μ	0.31	0.19	0.19	0.31

__Table 4.__ Explaining the Decline in Relative Consumption Dispersion

Period	Key Parameter		DEP	Data		Model	
	$\overline{\alpha}$	ϑ	p	r	μ_{cy}	r^*	${\mu_{cy}}^*$
$1980-1995$	7.3	1	0.41	3.1%	0.39	3.1%	0.39
$1996-2010$	7.3	13.5	0.02	1.5%	0.22	1.5%	0.21

Online Appendix for "Ambiguity, Low Risk-Free Rates, and Consumption Inequality" (Not for Publication)

Yulei Luo[*]
The University of Hong Kong

Jun Nie[†]
Federal Reserve Bank of Kansas City

Eric R. Young[‡]
University of Virginia

January 7, 2019

[*]Faculty of Business and Economics, The University of Hong Kong, Hong Kong. Email address: yulei.luo@gmail.com.

[†]Research Department, Federal Reserve Bank of Kansas City. E-mail: jun.nie@kc.frb.org.

[‡]Department of Economics, University of Virginia, Charlottesville, VA 22904. E-mail: ey2d@virginia.edu.

1 Online Appendix A: Solving the Rational Expectations (RE) and Robustness (RB) Versions of the Recursive Utility Model

The RE optimizing problem can be written as:

$$f\left(J_t\right)=\max_{c_t}\left\{\left(1-e^{-\delta\Delta t}\right)f\left(c_t\right)+e^{-\delta\Delta t}f\left(CE_t\left[J_{t+\Delta t}\right]\right)\right\}, \tag{1}$$

where $f\left(J_t\right)$ is the value function. An educated guess is that $J_t=As_t+A_0$. The J function at t time $t+\Delta t$ can thus be written as

$$J\left(s_{t+\Delta t}\right)=As_{t+\Delta t}+A_0\approx As_t+A\left(rs_t-c_t\right)\Delta t+A\sigma_s\Delta B_t+A_0,$$

where $\Delta s_t\equiv s_{t+\Delta t}-s_t$ and $\Delta s_t\approx\left(rs_t-c_t\right)\Delta t+\sigma_s\Delta B_t$. (Here $\Delta B_t=\sqrt{\Delta t}\epsilon$ and ϵ is a standard normal distributed variable.)

Using the definition of the certainty equivalent of $J_{t+\Delta t}$, we have

$$\begin{aligned}\exp\left(-\gamma CE_t\right)&=E_t\left[\exp\left(-\gamma J\left(s_{t+\Delta t}\right)\right)\right]\\&=\exp\left(-\gamma AE_t\left[s_{t+\Delta t}\right]+\frac{1}{2}\gamma^2A^2\operatorname{var}_t\left[s_{t+\Delta t}\right]-\gamma A_0\right)\\&=\exp\left(-\gamma A\left[s_t+\left(rs_t-c_t\right)\Delta t\right]+\frac{1}{2}\gamma^2A^2\sigma_s^2\Delta t-\gamma A_0\right),\end{aligned}$$

which means that

$$CE_t=A\left[s_t+\left(rs_t-c_t-\frac{1}{2}\gamma A\sigma_s^2\right)\Delta t\right]+A_0. \tag{2}$$

Substituting these expressions back into (1) yields:

$$0=\max_{c_t}\left\{\delta f\left(c_t\right)\Delta t+f'\left(J_t\right)\left(A\left(rs_t-c_t\right)-\frac{1}{2}\gamma A^2\sigma_s^2\right)\Delta t-\delta\Delta tf\left(J_t\right)\right\}$$

where we use the facts that $e^{-\delta\Delta t}=1-\delta\Delta t$,

$$J_{t+\Delta t}\approx J_t+J_t'\left(rs_t-c_t\right)\Delta t=J_t+A\left(rs_t-c_t\right)\Delta t,$$

and

$$f\left(J_t+A\left(rs_t-c_t\right)\Delta t+\frac{1}{2}A^2\sigma_s^2\Delta t\right)\approx f\left(J_t\right)+f'\left(J_t\right)\left(A\left(rs_t-c_t\right)-\frac{1}{2}\gamma A^2\sigma_s^2\right)\Delta t.$$

Dividing both sides by Δt, the Bellman equation can then be simplified as:

$$\delta f\left(J_t\right)=\max_{c_t}\left\{\delta f\left(c_t\right)+f'\left(J_t\right)\left(A\left(rs_t-c_t\right)-\frac{1}{2}\gamma A^2\sigma_s^2\right)\right\}. \tag{3}$$

The FOC for c is then

$$\delta f'(c_t) = f'(J_t) A,$$

which implies that

$$c_t = -\psi \ln \left(\frac{A}{\delta} \right) + (A s_t + A_0) . \tag{4}$$

Substituting this expression for c back to the Bellman equation and matching the coefficients, we have:

$$A = r$$

and

$$A_0 = \psi \left(\frac{\delta - r}{r} \right) + \psi \ln \left(\frac{r}{\delta} \right) - \frac{1}{2} \gamma r \sigma_s^2 - \frac{\vartheta}{2\psi} r \sigma_s^2 .$$

Substituting these coefficients into (4) gives the consumption function

$$c_t = r s_t + \Psi(r) - \Gamma(r) , \tag{5}$$

where

$$\Psi(r) \equiv \psi \left(\frac{\delta}{r} - 1 \right) \tag{6}$$

is the savings demand due to relative patience (if $\delta < r$, this term is negative and so savings rises) and

$$\Gamma(r) \equiv \frac{1}{2} r \gamma \sigma_s^2 , \tag{7}$$

is the consumer's precautionary saving demand.

Following Hansen and Sargent (2007), Uppal and Wang (2003), and Maenhout (2004), we introduce robustness into the above otherwise standard model as follows:

$$0 = \max_{c_t} \min_{\upsilon_t} \left\{ \delta f(c_t) \Delta t + f'(J_t) \left(A (r s_t - c_t) - \frac{1}{2} \gamma A^2 \sigma_s^2 \right) \Delta t - \delta f(J_t) \Delta t + \frac{1}{2 (\theta_t / \Delta t)} \upsilon_t^2 \sigma_s^2 \right\}$$

subject to the distorting equation, $\Delta s_t \approx (r s_t - c_t) \Delta t + \sigma_s (\sigma_s \upsilon_t \Delta t + \Delta B_t)$. It is worth noting that here following Hansen and Sargent (2011) and Kasa and Lei (2017), we scale the robustness parameter (θ_t) by the sampling interval (Δt), effectively making the consumer have stronger preference for robustness (or more ambiguity averse) as the sampling interval shrinks.

Dividing both sides by Δt, the Bellman equation reduces to:

$$\delta f(J_t) = \max_{c_t} \min_{\upsilon_t} \left\{ \delta f(c_t) + f'(J_t) A \left(r s_t - c_t - \frac{1}{2} \gamma A \sigma_s^2 + b_t \sigma_s^2 \right) + \frac{1}{2\theta_t} \upsilon_t^2 \sigma_s^2 \right\} , \tag{8}$$

subject to (**??**). Solving first for the infimization part of the problem yields

$$b^*(s_t) = -\theta_t A f'(J_t) .$$

Given that $\theta_t > 0$, the perturbation adds a negative drift term to the state transition equation because $f'(J_t) > 0$. Substituting it into the above HJB equation yields:

$$\delta f(J_t) = \max_c \left\{ \delta f(c_t) + f'(J_t) A \left(r s_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 - \theta_t A f'(U_t) \sigma_s^2 \right) + \frac{1}{2\theta_t} \left(\theta_t A f'(J_t) \right)^2 \sigma_s^2 \right\} \tag{9}$$

Following Uppal and Wang (2003) and Maenhout (2004), we assume that

$$\theta_t = -\frac{\vartheta}{f(U_t)}.$$

The HJB equation reduces to

$$\delta f(U_t) = \max_c \left\{ \delta f(c_t) + f'(J_t) A \left(r s_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 + \frac{\vartheta}{f(U_t)} A f'(U_t) \sigma_s^2 \right) - \frac{\vartheta}{2 f(J_t)} A^2 \left(f'(J_t) \right)^2 \sigma_s^2 \right\}.$$

The FOC for c is then

$$\delta f'(c_t) = f'(J_t) A,$$

which implies that

$$c_t = -\psi \ln\left(\frac{A}{\delta}\right) + (A s_t + A_0). \tag{10}$$

Substituting this expression for c back to the Bellman equation and matching the coefficients, we have:

$$A = r \text{ and } A_0 = \psi\left(\frac{\delta}{r} - 1\right) + \psi \ln\left(\frac{r}{\delta}\right) - \frac{1}{2}\gamma r \sigma_s^2 - \frac{\vartheta}{2\psi} r \sigma_s^2.$$

Substituting these coefficients into (10) gives the consumption function and the value function in the main text.

Finally, we check if the consumer's transversality condition (TVC),

$$\lim_{t\to\infty} E\left[\exp(-\delta t) \left| f(s_t) \right| \right] = 0, \tag{11}$$

is satisfied. Substituting the consumption function, c_t^*, into the state transition equation for s_t yields

$$ds_t = \widetilde{A} dt + \sigma_s dB_t,$$

where $\widetilde{A} = -\frac{\psi(\delta - r)}{r} + \frac{1}{2} r \widetilde{\gamma} \sigma_s^2$ under the approximating model. This Brownian motion with drift can be rewritten as

$$s_t = s_0 + \widetilde{A} t + \sigma(B_t - B_0), \tag{12}$$

where $B_t - B_0 \sim N(0, t)$. Substituting (12) into $E[\exp(-\delta t)|f(s_t)|]$ yields:

$$\begin{aligned}
E[\exp(-\delta t)|f(s_t)|] &= \frac{1}{\alpha_1} E[\exp(-\delta t - \alpha_0 - \alpha_1 s_t)] \\
&= \frac{1}{\alpha_1} \exp\left(E[-\delta - \alpha_0 - \alpha_1 s_t] + \frac{1}{2}\operatorname{var}(\alpha_1 s_t)\right) \\
&= \frac{1}{\alpha_1} \exp\left(-\delta t - \alpha_0 - \alpha_1\left(s_0 + \widetilde{A}t\right) + \frac{1}{2}\alpha_1^2\sigma_s^2 t\right) \\
&= |J(s_0)| \exp\left(-\left(\delta + \alpha_1\widetilde{A} - \frac{1}{2}\alpha_1^2\sigma_s^2\right)t\right)
\end{aligned}$$

where $|J(s_0)| = \frac{1}{\alpha_1}\exp(-\alpha_0 - \alpha_1 s_0)$ is a positive constant and we use the facts that $s_t - s_0 \sim N\left(\widetilde{A}t, \sigma_s^2 t\right)$. Therefore, the TVC, (11), is satisfied if and only if the following condition holds:

$$\delta + \alpha_1\widetilde{A} - \frac{1}{2}\alpha_1^2\sigma_s^2 = r + \frac{1}{2}\left(\frac{r}{\psi}\right)^2\left(\frac{\gamma}{\psi} - 1 + \vartheta\right)\sigma_s^2 > 0.$$

Given the parameter values we consider in the text, it is obvious that the TVC is always satisfied in both the FI-RE and RB models. It is straightforward to show that the TVC still holds under the distorted model in which $\widetilde{A} = -\frac{\psi(\delta - r)}{r} + \frac{1}{2}r\widetilde{\gamma}\sigma_s^2 - \frac{r}{\psi}\vartheta\sigma_s^2$ for plausible values of ϑ.

2 Online Appendix B: Description of Data

This appendix describes the data we use to estimate the income process as well as the method we use to construct a panel of both household income and consumption for our empirical analysis.

We use micro data from the Panel Study of Income Dynamics (PSID). Our household sample selection closely follows that of Blundell, Pistaferri, and Preston (2008).[1] We exclude households in the PSID low-income and Latino samples. We exclude household incomes in years of family composition change, divorce or remarriage, and female headship. We also exclude incomes in years where the head or wife is under 30 or over 65, or is missing education, region, or income responses. We also exclude household incomes where non-financial income is less than \$1000, where year-over-year income change is greater than \$90,000, and where year-over-year consumption change is greater than \$50,000. Our final panel contains 7,220 unique households with 54,901 yearly income responses and 50,422 imputed nondurable consumption values.[2]

[1] They create a new panel series of consumption that combines information from PSID and CEX, focusing on the period when some of the largest changes in income inequality occurred.

[2] There are more household incomes than imputed consumption values because food consumption - the main input variable in Guvenen and Smith's nondurable demand function - is not reported in the PSID for the years 1987 and 1988. Dividing the total income responses by unique households yields an average of $7-8$ years of responses per household. These years are not necessarily consecutive as our sample selection procedure allows households to be excluded in certain years but return to the sample if they later meet the criteria once again.

The PSID does not include enough consumption expenditure data to create full picture of household nondurable consumption. Such detailed expenditures are found, though, in the Consumer Expenditure Survey (CEX) from the Bureau of Labor Statistics. However, households in the CEX are only interviewed for four consecutive quarters and thus do not form a panel. To create a panel of consumption to match the PSID income measures, we use an estimated demand function for imputing nondurable consumption created by Guvenen and Smith (2014). Using an IV regression, they estimate a demand function for nondurable consumption that fits the detailed data in the CEX. The demand function uses demographic information and food consumption which can be found in both the CEX and PSID. Thus, we use this demand function of food consumption and demographic information (including age, family size, inflation measures, race, and education) to estimate nondurable consumption for PSID households, creating a consumption panel.

In order to estimate the income process, we narrow the sample period to the years $1980-1996$, due to the PSID survey changing to a biennial schedule after 1996. To further restrict the sample to exclude households with dramatic year-over-year income and consumption changes, we eliminate household observations in years where either income or consumption has increased more than 200 percent or decreased more than 80 percent from the previous year.

3 Online Appendix C: Solving the RU-RB Model with a Risky Asset

The robust HJB equation for the RU-RB model with multiple financial assets can be written as:

$$\delta f\left(J_t\right)=\max_c \min_v \left\{\delta f\left(c_t\right)+f'\left(U_t\right)A\left(rs_t-c_t+\pi\chi_t-\frac{1}{2}\gamma A\sigma^2+b_t\sigma^2\right)+\frac{1}{2\theta_t}b_t^2\sigma^2\right\}, \tag{13}$$

subject to the distorting equation. Solving first for the infimization part of the problem yields

$$v^*\left(s_t\right)=-\theta_t A f'\left(J_t\right).$$

Given that $\theta_t>0$, the perturbation adds a negative drift term to the state transition equation because $f'\left(J_t\right)>0$. Substituting it into the above HJB equation yields:

$$\delta f\left(J_t\right)=\max_{\{c_t,\alpha_t\}}\left\{\delta f\left(c_t\right)+f'\left(U_t\right)A\left(rs_t-c_t+\pi\alpha_t-\frac{1}{2}\gamma A\sigma^2-\theta_t A f'\left(U_t\right)\sigma^2\right)+\frac{1}{2\theta_t}\left(\theta_t A f'\left(U_t\right)\right)^2\sigma^2\right\}$$

Following Uppal and Wang (2003) and Maenhout (2004), we assume that $\theta_t=-\vartheta/f\left(U_t\right)$. The HJB equation reduces to

$$\delta f\left(U_t\right)=\max_{\{c_t,\alpha_t\}}\left\{\delta f\left(c_t\right)+f'\left(U_t\right)A\left(rs_t-c_t+\pi\alpha_t-\frac{1}{2}\gamma A\sigma^2+\frac{\vartheta f'\left(U_t\right)}{f\left(U_t\right)}A\sigma^2\right)-\frac{\vartheta\left(f'\left(U_t\right)\right)^2}{2f\left(J_t\right)}A^2\sigma^2\right\}.$$

Using the fact that $f(U_t) = (-\psi)\exp(-U_t/\psi)$, the HJB reduces to

$$\delta f(U_t) = \max_{\{c_t,\alpha_t\}}\left\{\delta f(c_t) + f'(U_t)A\left(rs_t - c_t + \pi\alpha_t - \frac{1}{2}\left(\gamma + \frac{\vartheta}{\psi}\right)A\sigma^2 + \frac{\vartheta f'(U_t)}{f(U_t)}A\sigma^2\right)\right\}.$$

The FOC for c_t is then

$$\delta f'(c_t) = f'(J_t)A,$$

which implies that

$$c_t = -\psi\ln\left(\frac{A}{\delta}\right) + (As_t + A_0). \tag{14}$$

The FOC for α_t is

$$\alpha_t = \frac{\pi}{r(\gamma + \vartheta/\psi)\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{15}$$

which is just (**??**). Substituting this expression for c back to the Bellman equation and matching the coefficients, we have:

$$A = r \tag{16}$$

and

$$A_0 = \psi\left(\frac{\delta}{r} - 1\right) - \frac{1}{2}r\widetilde{\gamma}\left(1 - \rho_{ye}^2\right)\sigma_s^2 + \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2} - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} + \psi\ln\left(\frac{r}{\delta}\right). \tag{17}$$

Substituting these coefficients into (14) gives the consumption function, (**??**) in the main text.

Given the optimal consumption-portfolio rules, the individual saving function can be written as

$$\begin{aligned}
d_t^* &= ra_t + y_t - c_t^* + \pi\alpha^* \\
&= ra_t + y_t - \left(rs_t + \Psi - \Gamma - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} + \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}\right) + \pi\left(\frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}\right) \\
&= \left[r\left(a_t + \frac{1}{r+\rho_1}y_t + \frac{\rho_1}{r(r+\rho_1)}\overline{y}\right) - r\left(\frac{1}{r+\rho_1}y_t + \frac{\rho_1}{r(r+\rho_1)}\overline{y}\right)\right] + y_t \\
&- \left(rs_t + \Psi - \Gamma - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} + \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}\right) + \pi\left(\frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}\right) \\
&= \frac{\rho}{r+\rho}(y_t - \overline{y}) + \Gamma - \Psi + \Pi,
\end{aligned}$$

where $\Pi = \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}$.