Ambiguity, Low Risk-Free Rates, and Consumption Inequality

Yulei Luo*
The University of Hong Kong

Jun Nie†
Federal Reserve Bank of Kansas City

Eric R. Young‡
University of Virginia and Zhejiang University

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Abstract

The failure of macroeconomists to predict the Great Recession suggests possible misspecification of existing macroeconomic models. If agents bear in mind this misspecification, how their optimal decisions changed and how large are the associated welfare costs? To shed light on these questions, we develop a tractable continuous-time recursive utility (RU) version of the Huggett (1993) model to study the effects of model uncertainty due to a preference for robustness (RB, or ambiguity aversion). We show that RB reduces the equilibrium interest rate and the relative dispersion of consumption to income, making them closer to the data, but our benchmark model cannot match the observed relative dispersion. An extension to a RU-RB model with a risky asset is successful at matching this dimension. Our analysis implies the welfare costs of model uncertainty are sizable: a typical consumer in equilibrium would be willing to sacrifice about 15 percent of his initial wealth to remove the model uncertainty he faces.

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*Faculty of Business and Economics, The University of Hong Kong, Hong Kong. Email address: yulei.luo@gmail.com.

†Research Department, Federal Reserve Bank of Kansas City. E-mail: jun.nie@kc.frb.org.

‡Department of Economics, University of Virginia, and Academy of Financial Research, Zhejiang University. E-mail: ey2d@virginia.edu.

1 Introduction

The failure of macroeconomists to predict the 2007−2009 Great Recession suggests possible misspecifications of existing macroeconomic models. The recent literature suggests that model uncertainty, the "unknown unknowns," was a crucial factor in the recent economic and financial crises. For example, Caballero and Krishnamurthy (2008) argued that the common aspects of investor behavior during the financial crisis – reevaluation of models, conservatism, and disengagement from risky investment – put emphasis on agents' optimal decisions on tail outcomes and worst-case scenarios, which means that these activities involved Knightian uncertainty and not merely an increase in risk exposure. In general, when making consumption-saving decisions, individual households not only face uncertainty about their future income, but also face uncertainty about the model generating the data. While the uncertainty about future income is modeled as risk that the consumers understand in the traditional consumption-saving models (e.g., Hall 1978, Caballero 1990), uncertainty about the data-generating process represents agents' pessimism about their ability to identify the correct model. This type of uncertainty is called model uncertainty (or Knightian uncertainty) due to ambiguity aversion and preferences for robustness.[1] There is direct experimental evidence for ambiguity aversion of this sort; for example, Ahn, Choi, Gale, and Kariv (2014) used experimental data to estimate a portfolio-choice model and found that about 40 percent of subjects display either statistically significant pessimism or ambiguity aversion.[2]

In this paper we ask general questions on how model uncertainty affects agents' consumption-saving choices in a general equilibrium framework, how it will change the equilibrium interest rate and the relative dispersion/inequality of individual consumption to income, and how large the welfare costs are coming from model uncertainty.[3] To answer these questions, we construct a tractable continuous-time dynamic stochastic general equilibrium (DSGE) heterogeneous-agent model with recursive utility and model uncertainty. The basic idea of model uncertainty is based on Hansen and Sargent (1995) who first introduced a preference for robustness (RB) into linear-quadratic-Gaussian (LQG) economic models to capture an agent's concern that their model is misspecified (a form of ambiguity aversion).[4] In RB models, agents are concerned about the

[1] In this paper we use both the terms, ambiguity (ambiguity aversion) and robustness (preferences for robustness), and the terms, Knightian uncertainty and model uncertainty, interchangebly.

[2] See Greenspan (2004) and Bernanke (2007) for discussions on the importance of model uncertainty (or Knightian uncertainty) for the central bank in making optimal monetary policy. Barlevy (2011) reviews and discusses the design of macroeconomic policies in the face of model uncertainty. Caballero (2010) also emphasizes the importance of understanding model uncertainty for macroeconomics after the $2007-2009$ financial crisis.

[3] In this paper we use "inequality" and "dispersion" interchangebly to describe the cross-sectional distributions of consumption and income.

[4] See Hansen and Sargent (2007) for a textbook treatment on robustness. It is worth noting that we can use either robust decision-making or recursive multiple-prior utility (Chen and Epstein 2002, Ju and Miao 2012) due to ambiguity aversion to capture the same idea that the decision maker is concerned that their model is misspecified and thus considers a range of models when making decisions (Gilboa and Schmeidler 1989). We follow Hansen and

possibility that their true model is misspecified in a manner that is difficult to detect statistically. Consequently, they have in mind a reference model that represents their best estimate of the model governing the dynamics of state variables. However, due to preference for robustness, they are worried that this reference model is incorrect in some way, and they make their optimal decisions as if the subjective distribution over shocks is chosen by an evil agent whose aim is to minimize their expected lifetime utility.

As shown in Hansen, Sargent, and Tallarini (1999) and Luo and Young (2010), RB models generate precautionary savings even within the class of discrete-time LQG models, which leads to analytical simplicity. Unfortunately, if we consider problems outside the discrete-time LQG setting (e.g., when the utility function is constant-absolute-risk-averse, CARA, or constant-relative-risk-averse, CRRA), RB-induced worst-case distributions are generally non-Gaussian which usually renders the model analytically intractable.[5][6] Furthermore, the models used in the existing literature are not clear about the separation of attitudes towards deterministic variation in consumption (the elasticity of intertemporal substitution, EIS), risk aversion, and uncertainty aversion.

This paper therefore fills the gap by developing a tractable continuous-time DSGE model in which consumers have an aversion to ambiguity, have recursive utility representations that disentangle risk and intertemporal attitudes, and face uninsurable labor income.[7] Key to our analytical results is the use of recursive exponential preferences, which use negative exponential functions to characterize both risk aversion and intertemporal substitution and lead to linear decision rules. We use this analytical framework to explore the general equilibrium implications of robustness for the risk-free rate, the cross-sectional dispersion of consumption (relative to income), and welfare gains from eliminating model uncertainty.[8] We investigate both the theoretical mechanism (how RB changes the equilibrium interest rate and the relative dispersion of consumption to income) and the empirical performance (whether plausibly calibrated values of RB lead to a better fit of the model to the data). The model proposed in this paper is the first, to our knowledge, to successfully capture this broad set of features. Our paper also contributes to the rapidly growing literature of using continuous-time heterogeneous-agent models to address inequality issues including Benhabib,

Sargent (2007) because it is technically easier.

[5]See Chapter 1 of Hansen and Sargent (2007) for discussions on the computational difficulties in solving non-LQG RB models, and Bidder and Smith (2012) and Young (2012) for numerical methods to compute the worst-case distributions.

[6]CRRA utility functions are more common in macroeconomics, mainly due to balanced-growth requirements. CRRA utility would greatly complicate our analysis because the intertemporal consumption model with CRRA utility and stochastic labor income has no explicit solution and leads to non-linear consumption rules. See Kasa and Lei (2017) for a recent application of RB in a continuous-time Blanchard-Yaari model with CRRA utility and wealth heterogeneity.

[7]See Cagetti, Hansen, Sargent, and Williams (2002), Anderson, Hansen, and Sargent (2003), Maenhout (2004), Kasa (2006), and Kasa and Lei (2017) for the applications of robustness in continuous-time models.

[8]For applications of recursive utility (RU) in intertemporal consumption-portfolio chice and asset pricing, see, for example, Epstein and Zin (1989), Campbell (2003), Vissing-Jorgensen and Attanasio (2003), and Guvenen (2006).

Bisin, and Zhu (2011), Gabaix, Lasry, Lions, and Moll (2016), and Kasa and Lei (2017). The key difference between their works and this paper is that they focus on income or wealth distributions while this paper is mainly about consumption inequality and equilibrium asset returns. Related to the discrete-time heterogenous-agent models (such as Huggett (1993) and Aiyagari (1994)), we provide analytical solutions in a heterogenous-agent model to help illustrate the key mechanisms through which model uncertainty influences the key results.

Our analysis has provided four main findings and contributions. As the first contribution of this paper, we find that the effective coefficient of absolute risk aversion ($\widetilde{\gamma}$) is determined by the interaction between the true coefficient of absolute risk aversion (γ), the EIS (ψ), and the degree of RB (ϑ) via the following formula:

$$\widetilde{\gamma} = \gamma + \frac{\vartheta}{\psi}.$$

It is clear from this expression that the EIS affects individual consumption-saving-portfolio rules jointly with the degree of robustness. As explained below, this finding sheds light on how the EIS influences the interest rate in the general equilibrium. This result plays an important role in understanding portfolio choice, where in the absence of risk aversion alone determines the allocation between risky and riskless assets.

Second, we show that a general equilibrium under RB can be constructed in the vein of Bewley (1986) and Huggett (1993) and we characterize how the equilibrium interest rate is affected by the three aspects of preferences.[9] An increase in the EIS affects the equilibrium interest rate through two distinct channels: (i) high EIS increases the relative importance of the impatience-induced dissaving effect (*the direct channel*) and (ii) reduces the precautionary saving amount by reducing the effect of RB (*the indirect channel*). In general equilibrium, the stronger the aversion to ambiguity the greater the amount of model uncertainty, leading to strong precautionary savings effects and therefore low interest rates. In addition, we show that the relative dispersion of consumption to income is determined only by the equilibrium interest rate and the persistence coefficient of the income process; the relative dispersion therefore decreases with RB if the income process is stationary.

Third, we show the model succeeds quantitatively in explaining the low equilibrium interest rate and the high relative dispersion of consumption to income. In the US economy the real risk-free interest rate averaged 1.87 percent between 1981 and 2010, and falls to 1.37 percent if the sample is extended to 2015.[10] A rational expectations model without model uncertainty would require the coefficient of risk aversion parameter to be 24 to match the rate of 1.87 percent if the EIS is

[9] Wang (2003) constructs a general equilibrium under rational expectations in the same Bewley-Huggett type model economy with CARA expected utility. Angeletos and Calvet (2006) characterize a closed-form recursive general equilibrium in a neoclassical growth model with idiosyncratic production risk and incomplete markets.

[10] Here the numbers are computed using CPI to measure inflation. Using PCE leads to similar results. See Table 1 for different measures of the risk-free rates.

0.8, and requires the coefficient to be 15 if the EIS is 0.5.[11] In contrast, when consumers take into account model uncertainty, the model can generate a low equilibrium interest rate with much lower values for the coefficient of risk aversion.[12] In addition, as income uncertainty increases, the relative dispersion of consumption to income decreases through the general equilibrium interest rate channel we noted above.[13] However, we also find that if the benchmark model generates the observed low risk-free rate, the relative dispersion of consumption to income is well below the empirical counterpart. The reason was noted in the previous paragraph – the relative dispersion depends only on the interest rate and the persistence of income changes, and this persistence is too low relative to the low interest rate to generate adequate consumption dispersion.

To correct this anomaly we extend our benchmark model to include an (idiosyncratically) risky asset. The presence of the risky asset affects equilibrium precautionary saving through two channels: (i) the risky asset can be used to hedge the labor income risk and (ii) it increases the amount of total uncertainty when the net supply of the risky asset is positive. We find that the relative dispersion of consumption to income is increasing in the supply of the risky asset and the risk-free rate is decreasing. For plausibly calibrated parameter values of RB, we find that the extended model can simultaneously generate the observed low risk free rate and high relative dispersion of consumption to income in the US economy.

Our paper thus contributes to a literature that examines the mechanisms that could explain the low interest rates in the US. Summers (2014) and Blanchard, Furceri, and Pescatori (2014) also argue that increases in global savings could be a reason for a lower equilibrium real interest rate in the US and other advanced economies. These explanations for higher savings rely on either demographic trends (such as an aging population) or capital flows from emerging economies to advanced economies, in contrast to our story about enhanced effective risk aversion. Similar to our paper, Hall (2017) emphasizes the shifting of wealth to higher risk-averse agents (like China), which would raise average effective risk aversion; Hall's evidence for this shift is indirect – he notes the rise in the volume of assets and liabilities held by "risk-splitting intermediaries".[14] A

[11] We normalize the mean consumption level to be 1, so the coefficient of relative risk aversion is equal to the coefficient of absolute risk aversion. While estimates of the EIS are all over the place (Havránek 2015), in general risk aversion coefficients about 10 are not considered reasonable in the macroeconomic literature; finance is a different ballgame, where very high risk aversion is seemingly accepted without comment.

[12] Barillas, Hansen, and Sargent (2009) showed that most of the observed high market price of risk in the U.S. can be reinterpreted as a market price of model uncertainty and the risk-aversion parameter can thus be reinterpreted as measuring the representative agent's doubts about the model specification.

[13] This theoretical result provides an explanation for the empirical evidence documented in Blundell, Pistaferri, and Preston (2008, henceforth BPP) that income and consumption inequality diverged over the sampling period they study. It is worth noting that they use the variances of log consumption and log income to measure consumption and income inequality. Since our paper adopts the CARA-Gaussian setting and the consumption process is non-stationary, we use the standard deviations of changes in consumption and income to measure the cross-sectional dispersions/inequality of changes in consumption and income, respectively.

[14] King (2016) raises a similar idea that uncertainty/ambiguity has risen but does not incorporate the insight into

related literature has tried to understand how the supply of safe assets matters for low interest rates (Barro, Fernández-Villaverde, Levintal, and Mollerus 2017, Caballero and Farhi 2014); our two-asset extension should be viewed as grappling with this question as well.

Finally, we use our preferred model to assess the welfare gains associated with eliminating model uncertainty – what would an agent pay in order to find out exactly (and with complete confidence) the stochastic process affecting his income? We find that these numbers are large – the cost can be as large as 15 percent of initial wealth. These costs are increasing in the aversion to model uncertainty and decreasing in the elasticity of intertemporal substitution (EIS). We also provide formulas to evaluate, at the margin, the welfare costs/gains of changes in the degree of model uncertainty and labor income volatility. We find that, under our calibrated parameter values, a 10-percent increase in the degree of model uncertainty leads to a welfare cost equivalent to a 1.23 percent reduction in initial income and this welfare cost is significantly larger in more volatile environment (e.g., a larger income volatility). Our analysis shows that a macro policy that reduces the income variance by 10 percent could lead to a welfare gain equivalent to about a 16 percent increase in initial income, while the gain is much smaller if there is no model uncertainty.

This paper is organized as follows. Section 2 presents a robustness version of the Caballero-Bewley-Huggett type model with incomplete markets and precautionary savings. Section 3 discusses the general equilibrium implications of RB for the interest rate and the joint dynamics of consumption and income. Section 4 presents our quantitative results after estimating the income process and calibrating the RB parameter. Section 5 considers the extension to the multiple-asset case. Section 6 examines the welfare implications. Section 7 concludes and briefly discusses future research.

2 A Continuous-time Heterogeneous-Agent Economy with Robustness

2.1 The Rational Expectations Model with Recursive Utility and Precautionary Savings

In this section, we first consider a rational expectations (RE) recursive utility model with labor income and precautionary savings. Although the expected power utility model has many attractive features, that model implies that the elasticity of intertemporal substitution is the reciprocal of the coefficient of relative risk aversion. Conceptually risk aversion (attitudes towards atemporal risks) and intertemporal substitution (attitudes towards shifts in consumption over time) capture two distinct aspects of decision-making and need not be so tightly connected.[15] In contrast, the class

a formal model.

[15] Risk aversion describes the agent's reluctance to substitute consumption across different states of the world and is meaningful even in a static setting. In contrast, intertemporal substitution describes the agent's willingness to

of recursive utility functions (Kreps and Porteus 1978; Epstein and Zin 1989; Duffie and Epstein 1992) enable one to disentangle risk aversion from intertemporal substitution. In this paper, we assume that agents in our model economy have the Kreps-Porteus type preference with *recursive exponential utility* (REU): for every stochastic consumption stream, $\{c_t\}_{t=0}^{\infty}$, the utility stream, $\{f(U_t)\}_{t=0}^{\infty}$, is recursively defined by[16]

$$f(U_t) = \left(1 - e^{-\delta\Delta t}\right) f(c_t) + e^{-\delta\Delta t} f(CE_t[U_{t+\Delta t}]) . \tag{1}$$

where Δt is the time interval, $\delta > 0$ is the agent's subjective discount rate, $f(U_t) = (-\psi)\exp(-U_t/\psi)$, $f(c_t) = (-\psi)\exp(-c_t/\psi)$,

$$CE_t[U_{t+\Delta t}] = g^{-1}(E_t[g(U_{t+\Delta t})]) , \tag{2}$$

is the certainty equivalent of U_{t+1} conditional on the period t information, and $g(U_{t+\Delta t}) = -\exp(-\gamma U_{t+\Delta t})/\gamma$. In (1), $\psi > 0$ governs the elasticity of intertemporal substitution (EIS), while $\gamma > 0$ governs the coefficient of absolute risk aversion (CARA).[17] A high value of ψ corresponds to a strong willingness to substitute consumption over time, and a high value of γ implies a low willingness to substitute consumption across states of nature. Note that when $\psi = 1/\gamma$, the functions f and g are the same and the recursive utility reduces to the standard time-separable expected utility function used in Caballero (1990) and Wang (2003).[18]

We assume that there is only one risk-free asset in the model economy and there are a continuum of consumers who face uninsurable labor income. The evolution of the financial wealth (w_t) of a typical consumer is

$$dw_t = (rw_t + y_t - c_t)\, dt; \tag{3}$$

r is the return to the risk-free asset and c_t and y_t are consumption and labor income at time t,

substitute consumption over time and is meaningful even in a deterministic setting.

[16] Skiadas (Chapter 6, 2009) axiomatizes and systematically characterizes this type of recursive exponential utility (or transition-invariant recursive utility.) Skiadas (2009) also compares this type of recursive utility with the scale-invariant (SI) Kreps-Porteus recursive utility (e.g., the Epstein-Zin-Weil parametric utility form). See also Angeletos and Calvet (2006) for an application of REU in a business cycles model.

[17] It is well-known that the CARA utility specification is tractable for deriving optimal policies and constructing general equilibrium in different settings. See Caballero (1990), Calvet (2001), Wang (2003, 2009), and Angeletos and Calvet (2006).

[18] Another widely-used recursive utility used in the consumption-saving literature is the Weil (1993)-type utility with a proportional aggregator and a log-exponential conditional certainty equivalent function. We do not consider Weil-type RU because that model makes counterfactual predictions regarding equilibrium interest rates. See Online Appendix for a detailed discussion. Note that this inconsistency does not exist in Weil (1993) because the original Weil model is a partial equilibrium model.

respectively. Uninsurable labor income (y_t) follows an Ornstein-Uhlenbeck process:[19]

$$dy_t = \rho\left(\overline{y} - y_t\right)dt + \sigma_y dB_t, \tag{4}$$

where $\overline{y}$ is the unconditional mean of y_t, σ_y is the unconditional volatility of the income change over an incremental unit of time, $\sigma_y^2/\left(2\rho\right)$ is the unconditional variance of y_t, the persistence coefficient ρ governs the speed of convergence or divergence from the steady state, and B_t is a standard Brownian motion on the real line $\mathcal{R}$. To present the model more compactly, we define a new state variable, s_t:

$$s_t \equiv w_t + h_t,$$

where h_t is human wealth at time t and is defined as the expected present value of current and future labor income discounted at the risk-free interest rate r,

$$h_t \equiv E_t\left[\int_t^\infty \exp\left(-r\left(s-t\right)\right)y_s ds\right].$$

For the given the income process, (4), $h_t = y_t/\left(r+\rho\right) + \overline{y}/\left(r\left(r+\rho\right)\right)$.[20] Using s_t as the unique state variable, we can rewrite (3) as

$$ds_t = \left(rs_t - c_t\right)dt + \sigma_s dB_t, \tag{5}$$

where $\sigma_s = \sigma_y/\left(r+\rho\right)$ is the unconditional variance of the innovation to s_t.[21]

The optimization problem can thus be written as

$$f\left(J_t\right) = \max_{c_t}\left\{\left(1-e^{-\delta\Delta t}\right)f\left(c_t\right) + e^{-\delta\Delta t}f\left(CE_t\left[J_{t+\Delta t}\right]\right)\right\}, \tag{6}$$

subject to (5). An educated guess is that $J_t = As_t + A_0$, where A and A_0 are undetermined coefficients. The J function at t time $t+\Delta t$ can thus be written as

$$J\left(s_{t+\Delta t}\right) = As_{t+\Delta t} + A_0 \approx As_t + A\left(rs_t - c_t\right)\Delta t + A\sigma_s\Delta B_t + A_0,$$

[19] In this paper, we abstract from income growth. It is worth noting that higher income growth generates higher risk-free rates. However, within our REU-OU framework, assuming constant income growth leads to time-varying risk-free rates, which greatly complicates our model. The detailed proof is available from the corresponding author by request.

[20] If $\rho > 0$, the income process is stationary and deviations of income from the steady state are temporary; if $\rho \leq 0$, income is non-stationary. The last case captures the essence of Hall and Mishkin (1982)'s specification of individual income that includes a non-stationary component. The $\rho = 0$ case corresponds to a simple Brownian motion without drift. The larger ρ is, the less y tends to drift away from $\overline{y}$. As ρ goes to ∞, the variance of y goes to 0. We need to impose the restriction that $r > -\rho$ to guarantee the finiteness of human wealth.

[21] In the next section, we will introduce robustness directly into this "reduced" precautionary savings model. It is not difficult to show that the reduced univariate model and the original multivariate model are equivalent in the sense that they lead to the same consumption and saving functions, because the financial wealth part of total wealth is deterministic between periods. The detailed proof is available from the corresponding author by request.

where $\Delta s_t \equiv s_{t+\Delta t} - s_t$ and $\Delta s_t \approx (rs_t - c_t)\,\Delta t + \sigma_s \Delta B_t$ where $\Delta B_t = \sqrt{\Delta t}\epsilon$ and ϵ is a standard normal innovation.

In the benchmark RE model, we assume that the consumer trusts the model, i.e., no model uncertainty. The Hamilton-Jacobi-Bellman (HJB) equation is then

$$\delta f(J_t) = \sup_{c_t \in \mathcal{C}} \{\delta f(c_t) + \mathcal{D} f(s_t)\} \tag{7}$$

where

$$\mathcal{D} f(s_t) = f'(J_t)\left(A(rs_t - c_t) - \frac{1}{2}\gamma A^2 \sigma_s^2\right), \tag{8}$$

$\mathcal{C}$ is the set of admissible values for the consumption choice, and the transversality condition, $\lim_{t\to\infty}\{E\,|\exp(-\delta t)\, f_t|\} = 0$, holds at the optimum. Solving the HJB equation subject to (5) leads to the consumption function

$$c_t = rs_t + \Psi(r) - \Gamma(r), \tag{9}$$

where

$$\Psi(r) \equiv \psi\left(\frac{\delta}{r} - 1\right) \tag{10}$$

is the savings demand due to relative patience (if $\delta < r$, this term is negative and so savings rises) and

$$\Gamma(r) \equiv \frac{1}{2} r\gamma\sigma_s^2, \tag{11}$$

is the consumer's precautionary saving demand.[22] From (10), it is clear that if the consumer is impatient relative to the market ($\delta > r$), the higher the EIS, the stronger the demand for consumption. If $\delta > r$ households want consumption to fall over time, and a higher EIS implies that consumption will be allowed to fall faster for a given value of $\frac{\delta}{r}$; as a result, consumption must initially be high. Following the literature on precautionary savings, we measure the demand for precautionary saving as the amount of saving induced by the combination of uninsurable labor income risk and risk aversion. From (11), one can see that the precautionary saving demand is larger for a higher value of the coefficient of absolute risk aversion (higher γ), a more volatile income innovation (higher σ_y), and a larger persistence coefficient (lower ρ).[23] Holding other parameters constant, we can see from (9) to (11) that intertemporal substitution and risk aversion have opposing effects on consumption and saving decisions if $\delta > r$ (which will be the case in general equilibrium).[24]

[22] See Appendix 8.2 for the derivations.

[23] As argued in Caballero (1990) and Wang (2003, 2009), a more persistent income shock takes a longer time to wear off and thus induces a stronger precautionary saving demand by a prudent forward-looking consumer.

[24] As a side note, incomplete markets generally imply that aggregate dynamics depend on the wealth distribution, this "curse of dimensionality" is circumvented by our CARA-Gaussian specification since savings functions are linear.

2.2 Incorporating Fear of Model Uncertainty

To introduce aversion to model uncertainty into our model (and thus generate a demand for robust decision rules), we follow the continuous-time methodology proposed by Anderson, Hansen, and Sargent (2003) (henceforth, AHS) and adopted in Maenhout (2004). Households take Equation (5) as the approximating model. The corresponding set of distorting models can thus be obtained by adding endogenous distortions $\upsilon(s_t)$ to (5):

$$ds_t = (rs_t - c_t)\,dt + \sigma_s\left(\sigma_s \upsilon(s_t)\,dt + dB_t\right). \tag{12}$$

As shown in AHS (2003), the objective $\mathcal{D}J$ defined in (8) can be thought of as $E[dJ]/dt$ and plays a key role in generating robustness. Consumers accept (5) as the best approximating model, but are still concerned that the model is misspecified. They therefore want to consider a range of models (the distorted models (12)) surrounding the approximating model when computing the continuation payoff. A preference for robustness (ambiguity aversion) manifests by having the agent guard against the distorting model that is reasonably close to the approximating model. The drift adjustment $\upsilon(s_t)$ is chosen to minimize the sum of (i) the expected continuation payoff adjusted to reflect the additional drift component in (12) and (ii) an entropy penalty:

$$\inf_{\upsilon}\left[\mathcal{D}f + f'(J_t)\,A\upsilon(s_t)\,\sigma_s^2 + \frac{1}{\vartheta_t}\mathcal{H}\right], \tag{13}$$

where the first two terms are the expected continuation payoff when the state variable follows (12), i.e., the alternative model based on drift distortion $\upsilon(s_t)$, $\mathcal{H} = (\upsilon(s_t)\,\sigma_s)^2/2$ is the relative entropy or the expected log likelihood ratio between the distorted model and the approximating model and measures the distance between the two models, and $1/\vartheta_t$ is the weight on the entropy penalty term.[25] ϑ_t is fixed and state independent in AHS (2003), whereas it is state-dependent in Maenhout (2004). The role of the state-dependent counterpart to ϑ_t in Maenhout (2004) is to assure the homotheticity or scale invariance of the decision problem under a CRRA utility function.[26] Note that the evil agent's minimization problem, (13), is invariant to the scale of total resources s_t under the state-dependent specification for $\vartheta_t(s_t)$, which we use as well so that the demand for robustness does not disappear as the value of total wealth increases.

We can then obtain the HJB equation for the RB model:

$$\delta f(J_t) = \sup_{c_t \in \mathcal{C}} \inf_{\upsilon_t}\left\{\delta f(c_t) + \mathcal{D}f(s_t) + \upsilon(s_t)\,\sigma_s^2 J_s + \frac{1}{\vartheta(s_t)}\mathcal{H}\right\}. \tag{14}$$

[25] The last term in (13) is due to the consumer's preference for robustness. Note that the $\vartheta_t = 0$ case corresponds to the standard expected utility case. This robustness specification is called the *multiplier (or penalty)* robust control problem. It is worth noting that this multiplier preference of RB expresses *ambiguity* with a multiplier that penalizes deviations from the approximating model as measured by relative entropy, and they express *ambiguity aversion* with the minimization operator.

[26] See Maenhout (2004) for detailed discussions on the appealing features of "homothetic robustness".

Solving first for the infimization part of (14) yields

$$\upsilon(s_t)^* = -\vartheta(s_t) f_s,$$

where $\vartheta(s_t) = -\vartheta/f(s_t) > 0$ (see Appendix 8.3 for the derivation). Following Uppal and Wang (2003) and Liu, Pan, and Wang (2005), here we can also define "$1/f(s_t)$" in the $\vartheta(s_t)$ specification as a *normalization* factor that is introduced to convert relative entropy (the distance between the approximating model and a given distorted model) into units of utility so that it is consistent with the units of the expected future value function evaluated with the distorted model. Adopting a slightly more general specification, $\vartheta(s_t) = -\varphi\vartheta/f(s_t)$ where φ is a constant, does not affect the main results of the paper, as we can just define a new constant, $\widetilde{\vartheta} = \varphi\vartheta$, and $\widetilde{\vartheta}$, rather than ϑ, will enter the decision rules. It is worth noting that this state-dependent robustness parameter follows a geometric Brownian motion in general equilibrium. (See Section 3.2 for the details.) This resulting process is similar to the AR(1) ambiguity shocks proposed in Bhandari, Borovička, and Ho (2016). They identified AR(1) ambiguity shocks using survey data from the Surveys of Consumers and the Survey of Professional Forecasters, and found that in the data, the ambiguity shocks are an important source of variation in labor market variables.

Since $\vartheta(s_t) > 0$, the perturbation adds a negative drift term to the state transition equation because $J_s > 0$. Substituting for υ^* in (14) gives

$$\delta f(U_t) = \sup_{c_t \in \mathcal{C}} \left\{ \delta f(c_t) + f'(J_t) A \left(r s_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 + \frac{\vartheta}{f(U_t)} A f'(U_t) \sigma_s^2 \right) - \frac{\vartheta}{2 f(J_t)} A^2 \left(f'(J_t)\right)^2 \sigma_s^2 \right\}. \tag{15}$$

2.3 The Robust Consumption Function and Precautionary Saving

We can now solve (15) and obtain the consumption rule under robustness. The following proposition summarizes the solution.

Proposition 1 *Under robustness, the consumption function and the saving function are*

$$c_t^* = r s_t + \Psi(r) - \Gamma(\vartheta, r), \tag{16}$$

and

$$d_t^* = x_t + \Gamma - \Psi, \tag{17}$$

respectively, where $x_t \equiv \rho(y_t - \overline{y})/(r+\rho)$ is the demand for savings "for a rainy day",

$$\Psi(r) \equiv \psi\left(\frac{\delta}{r} - 1\right) \tag{18}$$

captures the saving demand of relative patience,

$$\Gamma(\vartheta, r) \equiv \frac{1}{2} r \widetilde{\gamma} \sigma_s^2 \tag{19}$$

is the demand for precautionary savings due to the interaction of income uncertainty, intertemporal substitution, and risk and uncertainty aversion, and

$$\widetilde{\gamma} \equiv \gamma + \frac{\vartheta}{\psi} \tag{20}$$

is the effective coefficient of absolute risk aversion. The corresponding value function is

$$f_t = -\frac{\delta\psi}{r} \exp\left(-\left(\frac{\delta}{r} - 1 - \frac{1}{2}\frac{r}{\psi}\widetilde{\gamma}\sigma_s^2\right) - \frac{r}{\psi}s_t\right). \tag{21}$$

Finally, the worst possible distortion is

$$\upsilon^* = -r\frac{\vartheta}{\psi}. \tag{22}$$

Proof. See Appendix 8.3. ■

From (16), it is clear that robustness does not change the marginal propensity to consume out of permanent income (MPC), but does affect the amount of precautionary savings (Γ). In continuous time, consumption is less sensitive to unanticipated income shocks than in the discrete-time robust LQG-PIH model of Hansen, Sargent, and Tallarini (1999) (henceforth, HST); in discrete time the MPC increases with the amount of model uncertainty, causing consumption to become respond more to changes in permanent income (as noted in Luo and Young 2010, robust control exacerbates the excess sensitivity puzzle). Expression (19) shows that the precautionary savings demand now depends on the effective coefficient of risk aversion $\widetilde{\gamma}$ which is a function of the EIS (ψ), the CARA (γ), and the degree of robustness (ϑ). Specifically, it increases with γ and ϑ, whereas it decreases with ψ.

Another interesting question here is the relative importance of RB (ϑ) and CARA (γ) in determining the precautionary savings demand, holding other parameters constant. We can use the elasticities of precautionary saving as a measure of their importance.

Proposition 2 *The relative sensitivity of precautionary saving to risk aversion (γ), intertemporal substitution (ψ), robustness (ϑ) can be measured by*

$$\mu_{\gamma\vartheta} \equiv \frac{e_\gamma}{e_\vartheta} = \frac{\gamma}{\vartheta/\psi}, \tag{23}$$

$$\mu_{\psi\vartheta} \equiv \frac{e_\psi}{e_\vartheta} = -1. \tag{24}$$

respectively, where $e_\gamma \equiv \frac{\partial\Gamma/\Gamma}{\partial\gamma/\gamma}$, $e_\psi \equiv \frac{\partial\Gamma/\Gamma}{\partial\psi/\psi}$, and $e_\vartheta \equiv \frac{\partial\Gamma/\Gamma}{\partial\vartheta/\vartheta}$ are the elasticities of the precautionary saving demand to CARA, EIS, and RB, respectively.

Proof. The proof is straightforward. ■

The interpretation of (23) is that the precautionary savings demand is more sensitive to the actual coefficient of (absolute) risk aversion (γ) than it is to RB (ϑ) if the actual CARA is greater than RB amplified by the inverse of the EIS, i.e., $\gamma > \vartheta/\psi$. Of course, it is not exactly clear how to interpret a proportional change in either parameter since they do not have units, but we report this result to show that risk aversion does not clearly dominate the motives of the agents in the model.

HST (1999) showed that the discount factor and the concern about robustness are observationally equivalent in the sense that they lead to the same consumption and investment decisions in a discrete-time LQG representative-agent permanent income model. The reason for this result is that introducing a concern about robustness increases savings in the same way as increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RB on consumption and investment.[27] In contrast, in our continuous-time CARA-Gaussian model, we have a more general observational equivalence result between δ, γ, and ϑ:

Proposition 3 *Let*

$$\gamma^{fi} = \gamma + \frac{\vartheta}{\psi}, \tag{25}$$

where γ^{fi} is the coefficient of absolute risk aversion in the FI-RE model. Then consumption and savings are identical in the FI-RE and RB models, holding other parameter values constant. Furthermore, let $\delta = r$ in the RB model, and

$$\delta^{fi} = r - \frac{1}{2}\vartheta \left(\frac{r}{\psi}\right)^2 \sigma_s^2, \tag{26}$$

where δ^{fi} is the discount rate in the FI-RE model. Then consumption and savings are identical in the FI-RE and RB models, ceteris paribus.

Proof. Using (16) and (19), the proof is straightforward. ■

Expression (25) means that a consumer with a preference for robustness (ϑ) and recursive utility with EIS (ψ) and CARA (γ) is observationally equivalent to a consumer with full-information and recursive utility with EIS (ψ) and CARA ($\gamma + \vartheta/\psi$). In contrast, within a Merton model with recursive utility, Maenhout (2004) showed that an agent with a preference for robustness and Epstein-Zin recursive utility with EIS (ψ) and CRRA (γ) is observationally equivalent to an agent with full-information and recursive utility with EIS (ψ) and CARA ($\gamma + \vartheta$). In other words, in Maenhout's model, the effective coefficient of relative risk aversion ($\gamma + \vartheta$) does not depend on the EIS (ψ).

[27] As shown in HST (1999), the two models have different implications for asset prices because continuation valuations would alter as one alters the values of the discount factor and the robustness parameter within the observational equivalence set.

3 General Equilibrium Implications of RB

3.1 Definition of the General Equilibrium

As in Huggett (1993) and Wang (2003), we assume that the economy is populated by a continuum of *ex ante* identical, but *ex post* heterogeneous agents, with each agent having the saving function, (19). In addition, we also assume that the risk-free asset in our model economy is a pure-consumption loan and is in zero net supply.[28] The key insights can be also obtained in a CARA-Gaussian production economy with incomplete markets (as in Angeletos and Calvet 2006) using a neoclassical production function with capital and bonds as saving instruments. We consider the simpler Huggett-type endowment economy for two reasons. First, in the endowment economy, we can directly compare the model's predictions on the dynamics of individual consumption and income with its empirical counterpart, and do not need to infer the idiosyncratic productivity shock process. Second, the endowment economy allows us to solve the models explicitly, and thus helps us identify distinct channels via which RB interacts with risk aversion, discounting, and intertemporal substitution and affects the consumption-saving behavior.

In the model economy, the initial cross-sectional distribution of income is assumed to be its stationary distribution $\Phi(\cdot)$. By the law of large numbers in Sun (2006), provided that the spaces of agents and the probability space are constructed appropriately, aggregate income and the cross-sectional distribution of permanent income $\Phi(\cdot)$ will be constant over time.

Proposition 4 *The total savings demand "for a rainy day" in the precautionary savings model with RB equals zero for any positive interest rate. That is,* $F_t(r) = \int_{y_t} x_t(r)\, d\Phi(y_t) = 0$*, for* $r > 0$*.*

Proof. Given that labor income is a stationary process, the LLN can be directly applied and the proof is the same as that in Wang (2003). ■

This proposition states that the total savings "for a rainy day" is zero, at any positive interest rate; with a constant income distribution and linear decision rules, agents in the stationary wealth distribution follow the 'American dream and American nightmare' path, where any rise in income today is eventually offset by a future decline. Therefore, from (17), for $r > 0$, the expression for total savings under RB in the economy at time t can be written as

$$D(\vartheta, r) \equiv \Gamma(\vartheta, r) - \Psi(r). \tag{27}$$

where $\Psi(r)$ and $\Gamma(\vartheta, r)$ are defined in (18) and (19), respectively. We can now define a general equilibrium.

Definition 5 *Given (27), a general equilibrium under RB is defined by an interest rate* r^* *satisfying*

$$D(\vartheta, r^*) = 0. \tag{28}$$

[28] We can easily generalize to fixed positive net savings, as in a Lucas-style tree model. Nothing would change.

3.2 Theoretical Results

The following proposition shows that an equilibrium exists and provides a sufficient condition for uniqueness. We also show that, in any equilibrium, the PIH is satisfied.

Proposition 6 *There exists at least one equilibrium interest rate* $r^* \in (0, \delta)$ *in the precautionary-savings model with RB; if* $\delta < \rho$ *the equilibrium interest rate is unique on* $(0, \delta)$. *In equilibrium, each consumer's optimal consumption is described by the PIH, in that*

$$c_t^* = r^* s_t. \tag{29}$$

Furthermore, the evolution equations of wealth and consumption are

$$dw_t^* = x_t dt, \tag{30}$$

$$dc_t^* = \frac{r^*}{r^* + \rho} \sigma_y dB_t, \tag{31}$$

respectively, where $x_t = \rho (y_t - \overline{y}) / (r^* + \rho)$. *Finally, in general equilibrium,* ϑ_t *follows a geometric Brownian motion:*

$$\frac{d\vartheta_t}{\vartheta_t} = \frac{1}{2} (r^* \sigma_s)^2 dt + (r^* \sigma_s) dB_t, \tag{32}$$

if the true economy is governed by the approximating model, where $\vartheta_t = -\vartheta / f(s_t)$ *and* $f(s_t)$ *is provided in (21).*

Proof. If $r > \delta$, both $\Gamma(\vartheta, r)$ and $\Psi(r)$ in the expression for total savings $D(\vartheta, r)$ are positive, which contradicts the equilibrium condition $D(\vartheta, r) = 0$. Since $\Gamma(\vartheta, r) - \Psi(r) < 0$ (> 0) when $r = 0$ $(r = \delta)$, the continuity of the expression for total savings implies that there exists at least one interest rate $r^* \in (0, \delta)$ such that $D(\vartheta, r^*) = 0$. To establish the conditions under which this equilibrium is unique, we take the derivative

$$\frac{\partial D(\vartheta, r)}{\partial r} = \left(\gamma + \frac{\vartheta}{\psi}\right) \frac{\sigma^2}{(r + \rho)^2} \left(\frac{1}{2} - \frac{r}{r + \rho}\right) + \frac{\delta \psi}{r^2}$$

and note a sufficient condition for this derivative to be positive for any $r > 0$ is

$$\frac{1}{2} - \frac{r}{r + \rho} > 0 \Leftrightarrow r < \rho.$$

Therefore, if $\rho > \delta$ there is only one equilibrium in $(0, \delta)$. From Expression (16), we can obtain the individual's optimal consumption rule under RB in general equilibrium as $c_t^* = r^* s_t$. Substituting (29) into (3) yields (30). Using (5) and (29), we can obtain (31).

In general equilibrium, the state transition equation is $ds_t = \sigma_s dB_t$ if the true economy is governed by the approximating model. Using the definition of $\vartheta_t = -\vartheta / f(s_t)$, we have

$$\ln \vartheta_t = \ln \left(\frac{\vartheta r^*}{\delta \psi}\right) + \frac{r^*}{\psi} s_t \text{ or } d \ln \vartheta_t = \frac{r^* \sigma_s}{\psi} dB_t,$$

which means that

$$\frac{d\vartheta_t}{\vartheta_t} = \frac{1}{2}\left(\frac{r^*\sigma_s}{\psi}\right)^2 dt + \left(\frac{r^*\sigma_s}{\psi}\right) dB_t. \tag{33}$$

Similarly, we can also show that ϑ_t follows a similar geometric Brownian motion with a different drift coefficient when the true economy is governed by the distorted model. ■

The intuition behind this proposition is similar to that in Wang (2003). With an individual's constant total precautionary savings demand $\Gamma(\vartheta, r)$, for any $r > 0$, the equilibrium interest rate r^* must be at a level with the property that individual's dissavings demand due to impatience is exactly balanced by their total precautionary-savings demand, $\Gamma(\vartheta, r^*) = \Psi(r^*)$. We can see from (28) that EIS affects the equilibrium interest rate via two channels: (i) the precautionary saving channel and (ii) the impatience-induced dissaving channel. As EIS decreases, it increases the precautionary saving demand via increasing the effective coefficient of risk aversion and also reduces the impatience-induced dissaving effect; both channels drive down the equilibrium interest rate. It is also clear from (28) that a high value of ψ would amplify the relative importance of the dissaving effect $\Psi(r)$ for the equilibrium interest rate. The intuition behind this result is simple. When ψ is higher, consumption growth responds less to changes in the interest rate. In order to clear the market, the consumer must be offered a higher equilibrium risk free rate in order to be induced to save more and making his consumption tomorrow even more in excess of what it is today (less smoothing).

From the equilibrium condition,

$$\frac{1}{2}r^*\left(\gamma + \frac{\vartheta}{\psi}\right)\frac{\sigma^2}{(r^*+\rho)^2} - \psi\left(\frac{\delta}{r^*} - 1\right) = 0, \tag{34}$$

it is straightforward to show that

$$\frac{dr^*}{d\vartheta} = -\frac{r^*\sigma_s^2}{\psi}\left(\widetilde{\gamma}\sigma_s^2\frac{\rho - r^*}{\rho + r^*} + \frac{2\delta\psi}{r^{*2}}\right)^{-1}; \tag{35}$$

if $\rho > \delta > r^*$ then this derivative is negative, so that r^* is decreasing in the degree of RB, ϑ. In addition, it is straightforward to see that

$$\frac{dr^*}{d\gamma} < 0 \text{ and } \frac{dr^*}{d\psi} > 0.$$

That is, the equilibrium interest rate decreases with the degree of risk aversion and increases with the degree of intertemporal substitution. From (30) and (31), we can conclude that although both the CARA model and the LQ model lead to the PIH in general equilibrium, both risk aversion and intertemporal substitution play roles in affecting the dynamics of consumption and wealth in the CARA model via the equilibrium interest rate channel. From (33), it is clear that ϑ_t follows a geometric Brownian motion in general equilibrium. This result is comparable to the AR(1) process proposed in Bhandari, Borovička, and Ho (2017). They argue that the ambiguity shocks identified using survey data can help account for important business cycle facts.

Following the literature on precautionary savings (e.g., Caballero 1991 and Wang 2003), we set $\gamma = 3$. In addition, we set $\psi = 0.5$, $\rho = 0.083$, and $\sigma_y = 0.182$.[29] Figure 1 shows that the aggregate saving function $D(\vartheta, r)$ is increasing with the interest rate for different values of ϑ when $\delta = 0.036$, and there exists a unique interest rate r^* for every given ϑ such that $D(\vartheta, r^*) = 0$.[30]

The magnitude of the EIS (ψ) is an open and unresolved question, as the literature has found a very wide range of values. Vissing-Jorgensen and Attanasio (2003) estimate the EIS to be well in excess of one, while Campbell (2003) estimate a value well below one (and possibly zero). Guvenen (2006) finds that stockholders have a higher EIS (around 1.0) than non-stockholders (around 0.1). Havránek (2015) surveys the vast literature and suggests that a range around $0.3-0.4$ is appropriate after correcting for selective reporting bias, while Crump, Eusepi, Tambalotti, and Topa (2015) find that the EIS is precisely and robustly estimated to be around 0.8 in the general population using the newly released FRBNY Survey of Consumer Expectations (SCE). Here we choose $\psi = 0.5$ for illustrative purposes and will examine how EIS affects the general equilibrium under RI in Section 4.

Note that mathematically, the cross-sectional dispersion of consumption (relative to income) can be measured by the relative volatility of consumption to income, as our model satisfies a mixing condition in the steady state. The following result is then immediate.

Proposition 7 *The relative dispersion of consumption growth to income growth is*

$$\mu \equiv \frac{\mathrm{sd}(dc_t^*)}{\mathrm{sd}(dy_t)} = \frac{r^*}{r^* + \rho}. \tag{36}$$

Figure 1 also shows how RB (ϑ) affects the equilibrium interest rate (r^*). It is clear from the figure that the stronger the preference for robustness, the lower the equilibrium interest rate. From (36), we can see that RB will affect the dispersion of consumption by reducing the equilibrium interest rate. The following proposition summarizes the results about how the persistence coefficient of income changes the effect of RB on μ.

Proposition 8 *Using (36), we have*

$$\frac{\partial \mu}{\partial \vartheta} = \frac{\rho}{(r^* + \rho)^2} \frac{\partial r^*}{\partial \vartheta} < 0$$

because $\rho > 0$ *and* $\partial r^* / \partial \vartheta < 0$.

Proof. The proof is straightforward. ■

[29] In Section 4.1, we will provide more details about how to estimate the income process using the U.S. panel data. The main result here is robust to the choices of these parameter values.

[30] We ignore negative interest rate equilibria because the resulting consumption function does not make economic sense. It is easy to see that D has the same zeroes as a cubic function, so that there exist conditions under which the equilibrium is globally unique, but these conditions are rather impenetrable.

4 Quantitative Analysis

In this section, we first describe how we estimate the income process and calibrate the robustness parameter. We then present quantitative results on how RB affects the equilibrium interest rate and relative dispersion of consumption to income.

4.1 Estimation of the Income Process

To implement the quantitative analysis, we need to first estimate ρ and σ_y in the income process specification (4). We use micro data from the Panel Study of Income Dynamics (PSID). Following BPP (2008), we define the household income as total household income (including wage, financial, and transfer income of head, wife, and all others in household) minus financial income (defined as the sum of annual dividend income, interest income, rental income, trust fund income, and income from royalties for the head of the household only) minus the tax liability of non-financial income. This tax liability is defined as the total tax liability multiplied by the non-financial share of total income. Tax liabilities after 1992 are not reported in the PSID and so we estimate them using the TAXSIM program from the NBER. Details on sample selection are reported in Appendix 8.1.

To exclude extreme outliers, following Floden and Lindé (2001) we normalize both income and consumption measures as ratios of the mean of each year, and exclude households in the bottom and top 1 percent of the distribution of those ratios. To eliminate possible heteroskedasticity in the income measures, we regress each on a series of demographic variables to remove variation caused by differences in age and education. We next subtract these fitted values from each measure to create a panel of income residuals. We then use this panel to estimate the household income process as a stationary AR(1) process with Gaussian innovations:

$$y_t = \phi_0 + \phi_1 y_{t-1} + \sigma\varepsilon_t,\ t \geq 1,\ \ |\phi_1| < 1, \tag{37}$$

where $\varepsilon_t \sim N(0,1)$, $\phi_0 = (1-\phi_1)\,\overline{y}$, $\overline{y}$ is the mean of y_t, and the initial level of labor income y_0 are given. Once we have estimates of ϕ_1 and σ, we can recover the drift and diffusion coefficients in the Ornstein-Uhlenbeck process specified in (4) by rewriting (37) in the time interval $[t, t+\Delta t]$ as

$$y_{t+\Delta t} = \phi_0 + \phi_1 y_t + \sigma\sqrt{\Delta t}\varepsilon_{t+\Delta t}, \tag{38}$$

where $\phi_0 = \kappa\left(1-\exp\left(-\rho\Delta t\right)\right)/\left(\rho\Delta t\right)$, $\phi_1 = \exp\left(-\rho\Delta t\right)$, $\sigma = \sigma_y\sqrt{\left(1-\exp\left(-2\rho\Delta t\right)\right)/\left(2\rho\Delta t\right)}$, and $\varepsilon_{t+\Delta t}$ is the time-$(t+\Delta t)$ standard normal distributed innovation to income.[31] As the time interval, Δt, converges to 0, (38) reduces to the Ornstein-Uhlenbeck process, (4). The estimation results and the recovered persistence and volatility coefficients in (4) are reported in Table 2.

[31] Note that here we use the fact that $\Delta B_t = \varepsilon_t\sqrt{\Delta t}$, where ΔB_t represents the increment of a Wiener process.

4.2 Calibration of the Robustness Parameter

We adopt the calibration procedure outlined in Hansen, Sargent, and Wang (2002, henceforth HSW) and AHS (2003) to calibrate the value of the RB parameter (ϑ) that governs the degree of robustness. Specifically, we calibrate ϑ by using the method of detection error probabilities (DEP) that is based on a statistical theory of model selection. We can then infer what values of ϑ imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by p is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard many models, implying that the cloud of models surrounding the approximating model is large. In this case, it is easier for the consumer to distinguish the two models. The value of p is determined by the following procedure. Let model P denote the approximating model, (5) and model Q be the distorted model, (12). Define p_P as

$$p_P = \text{Prob}\left(\ln\left(\frac{L_Q}{L_P}\right) > 0 \middle| P \right), \tag{39}$$

where $\ln\left(\frac{L_Q}{L_P}\right)$ is the log-likelihood ratio. When model P generates the data, p_P measures the probability that a likelihood ratio test selects model Q. In this case, we call p_P the probability of the model detection error. Similarly, when model Q generates the data, we can define p_Q as

$$p_Q = \text{Prob}\left(\ln\left(\frac{L_P}{L_Q}\right) > 0 \middle| Q \right). \tag{40}$$

Given initial priors of 0.5 on each model and the length of the sample is N, the detection error probability, p, can be written as:

$$p(\vartheta; N) = \frac{1}{2}(p_P + p_Q), \tag{41}$$

where ϑ is the robustness parameter used to generate model Q. Given this definition, we can see that $1-p$ measures the probability that econometricians can distinguish the approximating model from the distorted model.

The general idea of the calibration procedure is to find a value of ϑ such that $p(\vartheta; N)$ equals a given value after simulating model P, (5), and model Q, (12).[32] In the continuous-time model with the iid Gaussian specification, $p(\vartheta; N)$ can be easily computed. Since both models P and Q are arithmetic Brownian motions with constant drift and diffusion coefficients, the log-likelihood ratios are Brownian motions. The logarithm of the Radon-Nikodym derivative of the distorted model (Q) with respect to the approximating model (P) can be written as

$$\ln\left(\frac{L_Q}{L_P}\right) = \int_0^t \overline{v} dB_s - \frac{1}{2}\int_0^t \overline{v}^2 ds, \tag{42}$$

[32] The number of periods used in the calculation, N, is set to be 31, the actual length of the data $(1980-2010)$.

where

$$\overline{\upsilon} \equiv \upsilon^* \sigma_s = -\frac{\vartheta}{\psi} r^* \sigma_s. \tag{43}$$

Similarly, the logarithm of the Radon-Nikodym derivative of the approximating model (P) with respect to the distorted model (Q) is

$$\ln\left(\frac{L_P}{L_Q}\right) = -\int_0^t \overline{\upsilon} dB_s + \frac{1}{2}\int_0^t \overline{\upsilon}^2 ds. \tag{44}$$

Using (39)-(44), it is straightforward to derive $p(\vartheta; N)$:

$$p(\vartheta; N) = \Pr\left(x < \frac{\overline{\upsilon}}{2}\sqrt{N}\right), \tag{45}$$

where x follows a standard normal distribution. From the expressions of $\overline{\upsilon}$, (43), and $p(\vartheta; N)$, (45), it is clear that the value of p is decreasing with the value of ϑ. Under the observational equivalence condition between the multiplier and constraint robustness formulations, (45) can be rewritten as $p(\vartheta; N) = \Pr\left(x < -\sqrt{2\eta}\sqrt{N}\right)$, where η is the upper bound on the distance between the two models and measures the consumer's tolerance for model misspecification.

We first explore the relationship between the DEP (p) and the value of the RB parameter, ϑ. A general finding is a negative relationship between these two variables. The upper panels of Figure 2 illustrates how DEP (p) varies with the value of ϑ for different values of EIS (ψ) and CARA (γ).[33] We can see from the figures that the stronger the preference for robustness (higher ϑ), the less the DEP (p) is. For example, let $\gamma = 3$ and $\psi = 0.5$, then $p = 0.403$ and $r^* = 2.83$ percent when $\vartheta = 1$, while $p = 0.163$ and $r^* = 1.99$ percent when $\vartheta = 5$. Both values of p are reasonable as argued in AHS (2002), HSW (2002), Maenhout (2004), and Hansen and Sargent (Chapter 9, 2007). In other words, a value of ϑ below 5 is reasonable in this case in which $\gamma = 3$ and $\psi = 0.5$. Furthermore, from the two upper panels of Figure 2, we can also see that the DEP increases with both $\not\psi$ and γ for given values of ϑ, and the impact of a change in ψ is much larger than that of a change in γ. The intuition is that a change in the EIS has two channels to affect the values of $\overline{\upsilon}$ and p: (i) the direct channel and (ii) the indirect channel via affecting the general interest rate channel (r^*), and the direct channel dominates the indirect channel. In contrast, a change in CARA only affects the values of $\overline{\upsilon}$ and p via the indirect equilibrium interest rate channel, which is relatively weak. Furthermore, using (23), in this case, we have $\mu_{\gamma\vartheta} = 1.5$ and 0.3 when we set $\vartheta = 1$ and 5, respectively. That is, the relative importance of risk aversion to RB in determining the precautionary savings demand decreases with the value of ϑ, holding other parameters constant.

[33] Based on the estimation results, we set $\overline{y} = 1$, $\sigma_y = 0.182$, and $\rho = 0.083$. The implied coefficient of relative risk aversion (CRRA) in our CARA utility specification can be written as either γc or γy. Given that the value of the CRRA is very stable and $\overline{\upsilon}$ can be expressed as $r(\vartheta/\psi)\sigma_y/(r+\rho)$, proportional changes in the mean and standard deviation of y do not change our calibration results because their effects on γ and σ_y cancel.

The two lower panels of Figure 2 illustrate how DEP (p) varies with ϑ for different values of σ_y and ρ if $\psi = 0.5$ and $\gamma = 3$.[34] It also shows that the higher the value of ϑ, the less the DEP (p). In addition, to calibrate the same value of p, smaller values of σ_y (less volatile labor income processes) or higher values of ρ (less persistent income processes) lead to higher values of ϑ. The intuition behind this result is that σ_s and ϑ have opposite effects on $\overline{v}$ and then p (see (43)).

As emphasized in Hansen and Sargent (2007), in the robustness model, p is a measure of the amount of model uncertainty, whereas ϑ is a measure of the agent's aversion to model uncertainty. If we keep p constant when recalibrating ϑ for different values of γ, ρ, or σ_y, the amount of model uncertainty is held constant – that is, the set of distorted models with which we surround the approximating model does not change. In contrast, if we keep ϑ constant, p will change accordingly if the values of γ, ρ, or σ_y change; in this case, the amount of model uncertainty is "elastic" and will change accordingly as the agent's aversion to uncertainty changes.

4.3 Effects of RB on the Equilibrium Interest Rate and Consumption Dispersion

The equilibrium interest rate and relative dispersion of consumption to income are jointly determined by the degree of robustness, risk aversion, intertemporal substitution, and the income process. To better see how RB affects the equilibrium interest rate and the relative dispersion, we present two quantitative exercises here. The first exercise fixes the parameters of the income process at the estimated values and allows the risk aversion and intertemporal substitution parameters to change, while the second exercise fixes the risk aversion and intertemporal substitution parameters and allows the key income process parameter to vary.

Figure 3 shows that the equilibrium interest rate and the equilibrium relative consumption dispersion decrease with the calibrated value of ϑ for different values of ψ and γ when $\sigma_y = 0.182$ and $\rho = 0.083$. For example, if ϑ is increased from 1 to 5 (p decreases from 0.403 to 0.163), r^* falls from 2.63 percent to 1.98 percent and μ falls from 0.241 to 0.193, given $\psi = 0.5$ and $\gamma = 3$.[35] In addition, the figure also shows that the interest rate and the relative dispersion decrease with γ and increase with ψ for different values of ϑ.

Our model has the potential to explain the observed low real interest rate in the U.S. economy; see Laubach and Williams (2015) or Hall (2016) for evidence on low real rates. One of our theoretical results shows that a stronger aversion to model uncertainty lowers the equilibrium real interest rate. In the US, the average real risk-free interest rate has been about 1.87 percent between 1981 and 2010 if we use CPI to measure inflation, and about 1.96 percent if we use PCE to measure inflation.[36]

[34] Since $\sigma_s = \sigma_y/(r+\rho)$, both changes in the persistence coefficient (ρ) and changes in volatility coefficient (σ_y) will change the value of σ_s.

[35] In the RE case, $r^* = 2.92$ percent and $\mu = 0.26$.

[36] Following Campbell (2003), we calculate the average of the real 3-month Treasury yields. Here we choose the $1981-2010$ period because it is more consistent with our sample period of the panel data in estimating the joint

Therefore, depending on what inflation index is used, the risk-free rate is between 1.87 and 1.96 percent. In our following discussion, we set the risk free rate to be 1.91 percent which is the average of the two real interest rates under CPI and PCE. Using the equilibrium condition, we find that the FI-RE model without RB requires the coefficient of risk aversion parameter to be 23 to match this rate if $\psi = 0.8$, and requires the coefficient to be 14.5 if $\psi = 0.5$.[37]

In contrast, when consumers take into account model uncertainty, the model can generate an equilibrium interest rate of 1.91 percent with much lower values of the coefficient of risk aversion.[38] Figure 4 shows the relationship between γ and ϑ for interest rates equal to 1.91 percent for different values of ψ. For example, if $\psi = 0.5$, the RB model with $\gamma = 4.5$ and $\vartheta = 5$ leads to the same interest rate as in the RE model with $\gamma = 14.5$. Note that $\gamma = 4.5$ is much lower than the risk aversion levels used in most macro-asset pricing models. Using the same calibration procedure discussed in Section 4.1, we find that the corresponding DEP is $p = 0.171$. In other words, agents tolerate a 17.1 percent probability that they cannot distinguish the distorted model from the approximating model. We have summarized these results in Table 3. As argued in Hansen and Sargent (2007) and in Section 4.2, this value is viewed as reasonable in the literature.

The explanation that agents have become more concerned about model misspecification after the $2007 - 09$ financial crisis does not seem unreasonable given the long and deep recession which generated skepticism (at least in the popular press) about whether the standard macro models fully capture the key features of the economy.[39] To provide a numerical example, under our calibrated parameter values, $\psi = 0.5$, and $\gamma = 4.5$, an increase in model uncertainty reflected by a reduction in the DEP from $p = 0.297$ to $p = 0.171$ (an increase in ϑ from 2.5 to 5) leads to a reduction in the equilibrium interest rate from 2.21 percent to 1.91 percent.

To examine how RB affects the relative dispersion of consumption to income ($\mu = \text{sd}\left(dc_t^*\right) / \text{sd}\left(dy_t\right)$), we follow LNWY (2017) and construct a panel data set which contains both consumption and income at the household level.[40] Figure 5 shows the relative dispersion of consumption to income between 1980 and 2000.[41] From the figure, the average empirical value of the relative dispersion

consumption and income process. When we consider an extended period from 1981 to 2015, the real interest rate is 1.37 percent when using CPI and is 1.75 percent when using PCE. Hall (2016) finds that real rates (computed using TIPS) have been consistently falling for several decades, so we are overstating the current rate.

[37] Note that since we set the mean income level to be 1, the coefficient of relative risk aversion (CRRA) evaluated at this level is equal to the coefficient of absolute risk aversion (CARA).

[38] This result is comparable to that obtained in Barillas, Hansen, and Sargent (2009). They found that most of the observed high market price of uncertainty in the U.S. can be reinterpreted as a market price of model uncertainty rather than the traditional market price of risk.

[39] Boyarchenko (2012) used data on CDS spreads to show that the amount of ambiguity in the economy rose during the financial crisis but the aversion to ambiguity did not change significantly; in our model the same parameter controls both features; see also King (2016) for more historical discussion.

[40] Appendix 8.1 presents details on how the panel is constructed.

[41] See Appendix 8.1 for more details on how the panel was constructed.

(μ) is 0.377 for the $1980 - 1996$ period, and is 0.326 for the period from 1980 to 2010. The minimum and maximum values of the empirical relative dispersion from 1980 to 2010 are 0.195 (year 2006) and 0.55 (year 1982), respectively. From the expression for the equilibrium relative dispersion (36), we can see that when the real interest rate is low, it is impossible for the model to generate sufficiently high relative dispersion of consumption to income without using an implausible value for ρ. For example, when $r^* = 1.91$ percent we obtain $\mu = 0.19$, which is well below the average value $\mu = 0.326$; to get $\mu = 0.326$ we would need $\rho = 0.039$, a value that can be rejected given our estimated value of $\rho = 0.082$. Because this moment matters for the welfare calculations that are the focus of the paper, in the next section we will resolve the disparity between data and model using a risky asset in positive net supply.

5 Extension to an RU-RB Model with Multiple Financial Assets

5.1 Model Specification

In this section, we follow Maenhout (2004) and Wang (2009), and assume that consumers can assess two financial assets: one risk free asset and one risky asset. Our aim here is to resolve the anomaly from the benchmark model regarding the relative dispersion of consumption to income at low interest rates. Specifically, the consumer can purchase both a risk-free asset with a constant interest rate r and a risky asset (the market portfolio) with a risky return r_t^e. The instantaneous return dr_t^e of the risky market portfolio over dt is given by

$$dr_t^e = (r + \pi)\, dt + \sigma_e dB_{e,t}, \tag{46}$$

where π is the market risk premium; σ_e is the standard deviation of the market return; and $B_{e,t}$ is a standard Brownian motion. Let ρ_{ye} be the contemporaneous correlation between the labor income process and the return of the risky asset. If $\rho_{ye} = 0$, the labor income risk is purely idiosyncratic, so the risky asset does not provide a hedge against labor income declines. The agent's financial wealth evolution is then given by

$$dw_t = (rw_t + y_t - c_t)\, dt + \alpha_t (\pi dt + \sigma_e dB_{e,t}), \tag{47}$$

where α_t denotes the amount of wealth that the investor allocates to the market portfolio at time t.

As in the benchmark model, we define a new state variable, s_t: $s_t \equiv w_t + h_t$, where h_t is human wealth at time t and is defined as the expected present value of current and future labor income discounted at the risk-free interest rate r: $h_t \equiv E_t \left[\int_t^\infty \exp(-r(s-t))\, y_s ds\right]$. Following the same state-space-reduction approach used in the benchmark model, the budget constraint can be written as:

$$ds_t = (rs_t - c_t + \pi\alpha_t)\, dt + \sigma dB_t, \tag{48}$$

where $\sigma dB_t = \sigma_e \alpha_t dB_{e,t} + \sigma_s dB_{y,t}$, $\sigma_s = \sigma_y / (r + \rho)$, and

$$\sigma = \sqrt{\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t} \tag{49}$$

is the unconditional variance of the innovation to s_t.

5.2 Consumption and Saving Rules under RB

To introduce robustness into the above recursive utility model, we follow the same procedure as in Section 2.2 and write the distorting model by adding an endogenous distortion $\upsilon(s_t)$ to the law of motion of the state variable s_t, (48),

$$ds_t = (rs_t - c_t + \pi\alpha_t)\,dt + \sigma\left(\sigma\upsilon(s_t)\,dt + dB_t\right). \tag{50}$$

As in the benchmark model, the drift adjustment $\upsilon(s_t)$ is chosen to minimize:

$$\inf_{\upsilon}\left[\mathcal{D}f + f'(J_t)\,A\upsilon(s_t)\,\sigma^2 + \frac{1}{\vartheta_t}\mathcal{H}\right], \tag{51}$$

where $\mathcal{H} = (\upsilon(s_t)\,\sigma)^2/2$ is the relative entropy or the expected log likelihood ratio between the distorted model, and $1/\vartheta_t$ is the weight on the entropy penalty term. Following the same procedure we used in solving the benchmark model, we can also solve the multiple-asset case explicitly. The following proposition summarizes the solution to this dynamic program.

Proposition 9 *Under robustness, the consumption function, the portfolio rule, and the saving function are*

$$c_t^* = r\left(s_t - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{r\sigma_e^2}\right) + \Psi(r) - \Gamma(\vartheta, r) + \Pi(\vartheta, r), \tag{52}$$

$$\alpha^* = \frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{53}$$

and

$$d_t^* = x_t + \Gamma - \Psi + \Pi, \tag{54}$$

respectively, where $x_t \equiv \rho(y_t - \overline{y})/(r + \rho)$ is the demand for savings "for a rainy day",

$$\Gamma(\vartheta, r) \equiv \frac{1}{2} r\widetilde{\gamma}\left(1 - \rho_{ye}^2\right)\sigma_s^2 \tag{55}$$

is the demand for precautionary savings due to the interactions of income uncertainty, intertemporal substitution, and risk and uncertainty aversion,

$$\Psi(r) \equiv \psi\left(\frac{\delta}{r} - 1\right) \tag{56}$$

captures the saving demand of relative patience,

$$\Pi(\vartheta, r) \equiv \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2} \tag{57}$$

is the additional saving demand due to the higher expected return of the risky asset, and $\widetilde{\gamma} \equiv \gamma + \vartheta/\psi$ *is the effective coefficient of absolute risk aversion. Finally, the worst possible distortion is* $\upsilon^* = -r(\vartheta/\psi)$.

Proof. See Appendix 8.4. ■

Expression (16) shows that the presence of the risky asset in the agent's investment opportunity has two effects on current consumption. First, it reduces the risk-adjusted certainty equivalent human wealth by $\pi\rho_{ye}\sigma_s\sigma_e/\left(r\sigma_e^2\right)$ because the agent faces more risk when holding the risky asset. Second, it increases current consumption because it offers a higher expected return (see Wang 2009). In general equilibrium, the second effect dominates the first effect. Furthermore, from the definition of individual saving, we can see that the presence of $\pi\alpha^*$ term has the potential to increase saving because it offers a higher expected return. Combining these two effects, it is easy to show that the net effect of the risky asset on current saving is governed by $\Pi > 0$ defined in (57). In addition, since the risky asset can be used to hedge labor income risk (provided the correlation is not zero), it will reduce the precautionary saving demand arising from income uncertainty by a factor $1 - \rho_{ye}^2 \in (0, 1)$. From (54), it is clear that there are four saving motives in the model with a risky asset. The first three saving motives, x_t, Γ, and Ψ, are the same as that mentioned in our benchmark model. The fourth term captures the additional saving demand due to the higher expected return of the risky asset and obviously does not appear in the benchmark model.

Since the effective coefficient of absolute risk aversion depends on both the EIS and the degree of RB, it is clear from (53) that even if the consumer only has a constant investment opportunity set, the optimal share invested in the risky asset not only depends on risk aversion, but also depends on intertemporal substitution if $\vartheta > 0$.[42] Wang (2009) showed in a FI-RE recursive utility model that when the investment opportunity set is constant, the optimal share invested in the risky asset depends on the risk aversion parameter, but not on the EIS.

5.3 General Equilibrium Implications

We first consider the equilibrium in the market for the risky asset. Assuming that the net supply of the risky asset is $\overline{\alpha} \geq 0$, the equilibrium condition in the market for the risky asset is

$$\overline{\alpha} = \frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} \tag{58}$$

for a given risk free rate, r.

Using the individual saving function (54) and following the same aggregation procedure used in the previous section, we have the following result on savings:

[42] A constant investment opportunity set means a constant interest rate, a constant expected return on risky assets, and a constant volatility the returns on risky assets.

Proposition 10 *The total demand of savings "for a rainy day" equals zero for any positive interest rate. That is,* $F_t(r) = \int_{y_t} x_t(r)\, d\Phi(y_t) = 0$*, for* $r > 0$.

Proof. The proof uses the LLN and is the same as that in Wang (2003). ■

Using this result, from (54), after aggregating across all consumers, the expression for total savings can be written as

$$D^{total}(\vartheta, r) \equiv \Gamma(\vartheta, r) - \Psi(r) + \Pi(\vartheta, r), \tag{59}$$

where $\Gamma(\vartheta, r)$, $\Psi(r)$, and $\Pi(\vartheta, r)$ are given in (55), (56), and (57), respectively. To compare $D(\vartheta, r)$ with the aggregate saving function obtained in the benchmark model, we rewrite $D(\vartheta, r)$ as follows:

$$D^{total}(\vartheta, r) = \widetilde{\Gamma}(\vartheta, r) - \Psi(r) + \widetilde{\Pi}(\vartheta, r), \tag{60}$$

where $\widetilde{\Gamma}(\vartheta, r) = r\widetilde{\gamma}\sigma_s^2/2$, $\widetilde{\Pi}(\vartheta, r) = \overline{\alpha} r\widetilde{\gamma}\rho_{ye}\sigma_s\sigma_e + \overline{\alpha}^2 r\widetilde{\gamma}\sigma_e^2/2$, and $\overline{\alpha}$ is determined by (58). Comparing the two aggregate saving functions, $\widetilde{\Pi}(\vartheta, r)$ is an additional term due to the positive net supply of the risky asset in this model. As in the benchmark model, we still assume that the net supply of the risk-free asset is zero in equilibrium, i.e., an equilibrium interest rate r^* satisfies:

$$D(\vartheta, r^*) \equiv \Gamma(\vartheta, r^*) - \Psi(r^*) = 0, \tag{61}$$

where $D(\vartheta, r^*)$ denotes the amount of saving in the risk-free asset. The following proposition proves that an equilibrium exists and that the PIH is satisfied.

Proposition 11 *There exists an equilibrium with an interest rate* $r^* \in (0, \delta)$ *and*

$$\pi^* = r^*\widetilde{\gamma}\sigma_e(\rho_{ye}\sigma_s + \overline{\alpha}\sigma_e), \tag{62}$$

and if $\rho > \delta$ *and* $\rho_{ye} \geq 0$ *this equilibrium is unique. In any such equilibrium, each consumer's optimal consumption-portfolio rules are described by:*

$$c_t^* = r^* s_t, \tag{63}$$

and

$$\alpha^* = \overline{\alpha}, \tag{64}$$

respectively. Furthermore, in this equilibrium, the evolution equation of s_t *is*

$$ds_t = \left(\frac{\pi^2}{r\widetilde{\gamma}\sigma_e^2}\right) dt + \sigma dB_t, \tag{65}$$

if the true economy is governed by the approximating model.

Proof. The equilibrium existence and uniqueness proof is the same as that for our benchmark model except that we replace σ^2 with $\left(1-\rho_{ye}^2\right)\sigma^2$. ■

Figure 6 shows that the aggregate saving function $D(\vartheta, r)$ is increasing with the interest rate for different values of ρ_{ye}.[43] It clearly shows that there exists a unique interest rate r^* for every given ρ_{ye} such that $D(\vartheta, r^*) = 0$, and a higher correlation between the equity return and labor income leads to a higher equilibrium interest rate given ϑ. The intuition behind this result is that the presence of the risky asset helps hedge labor income risk, leading to less precautionary savings. However, we can see from the figure that does not have significant effects on the equilibrium interest rate. The following result is an immediate implication on how the presence of the risky asset affects the relative dispersion of consumption growth to income growth under RB.

Proposition 12 *The relative dispersion of consumption growth to income growth is*

$$\mu \equiv \frac{\mathrm{sd}\left(dc_t^*\right)}{\mathrm{sd}\left(dy_t\right)} = r^* \sqrt{\left(\frac{\sigma_e}{\sigma_y}\right)^2 \overline{\alpha}^2 + \left(\frac{1}{r^*+\rho}\right)^2 + 2\frac{\rho_{ye}}{r^*+\rho}\frac{\sigma_e}{\sigma_y}\overline{\alpha}}. \tag{66}$$

Comparing (36) with (66), it is clear that the positive net supply of the risky asset will be helpful at increasing the relative dispersion of consumption to income while keeping the real interest rate at a low level. To quantitatively examine the effects of RB on the relative dispersion of consumption growth to income growth, we first use the observed risk premium of 7.2 percent to calibrate the value of $\overline{\alpha}$ using (58).[44] Estimating the correlation between individual labor income and the equity return is complicated by the lack of panel data on household portfolio choice, and we found several estimates in the literature: Viceira (2001) adopted $\rho_{ye} = 0.35$ when simulating his life-cycle consumption-portfolio choice model. Davis and Willen (2000) estimated that the correlation is between 0.1 and 0.3 for college-educated males, and is 0.25 or more for college-educated women, while Heaton and Lucas (1999) found that the correlation between entrepreneurial earnings and equity returns was about 0.2. Here we follow Viceira (2001) and set $\rho_{ye} = 0.35$. In our model, if $\gamma = 4.5$, $\psi = 0.5$, $\vartheta = 2.9$, $\delta = 0.03$, and $\rho_{ye} = 0.35$, the corresponding DEP (p) is 0.19, the equilibrium interest rate (r^*) is 1.91 percent, and the relative dispersion (μ) is 0.31, which equals the empirical counterpart for the sample from 1980 to 1996. These results are summarized in Table 3.

5.4 Explaining the Decline in the Relative Consumption Dispersion

To test the model's predictions on the effects of robustness on the dynamic relative consumption dispersion, we quantitatively examine how well a calibrated version of our extended model can

[43] Here we set $\gamma = 4.5$, $\psi = 0.5$, $\vartheta = 2.9$, and $\delta = 0.03$. The parameters in the income process are the same as before.

[44] In a recent survey paper, Mehra (2012) documented that the risk premium in the US is about 7.2 percent during $1926-2010$, and 6.7 percent during $1946-2010$.

explain the decline in the relative consumption dispersion from 1980 to 2010 (see Figure 5). To do this, we divide our sample into two periods, $1980-1995$ and $1996-2010$. We calibrate our model to the first period by choosing the robustness parameter (ϑ) and the aggregate asset supply parameter ($\overline{\alpha}$) to match the observed real interest rate (r) and the relative consumption dispersion (μ_{cy}). To focus on the effects of robustness, we fix the other parameters at the same values used in the previous section. Then, we let the robustness parameter vary to match the real interest rate in the second period. In other words, this is the amount of model uncertainty the model needs to explain the decline in the real interest rate from the first period to the second period. Finally, we check how much this change in the amount of model uncertainty can explain the decline in the relative consumption dispersion from the first period to the second period.

The results in Table 4 show the model does a good job of explaining the decline in the relative consumption dispersion. The first row shows that the model with $\overline{\alpha}=8.8$ and $\vartheta=0.57$ matches the average real interest rate of 3.1% and the average relative consumption dispersion of 0.39 in the first period, $1980-1995$. To generate a real interest rate of 1.5% in the second period, $1996-2010$, we need ϑ to increase from 0.57 to 19, which corresponds to a decrease in the DEP from 0.46 to 0.06 (third column in the table). Remember a larger DEP means that it is more difficult to distinguish the approximating model and the distorted model, or there is less model uncertainty. The calibrated results therefore suggest that both the degree of robustness and the amount of model uncertainty increased significantly from the first period to the second period. With this increase in the degree of robustness, the model predicts a decrease in the relative consumption dispersion from 0.39 to 0.21, which nearly matches the observed decrease in the data. This accurate out-of-sample prediction provides additional evidence that incorporating model uncertainty due to ambiguity and robustness can help the model to better explain the data.

6 The Welfare Cost of Model Uncertainty

The uncertainty about model specifications due to a preference for robustness or ambiguity aversion generates welfare losses. We measure the welfare cost of model uncertainty in a standard and intuitive way – the amount of income an average consumer is willing to pay to remove or reduce such uncertainty. In particular, we provide two approaches to calculate the welfare cost of model uncertainty from different angles. The first approach is based on Lucas (1987)'s elimination-of-risk method which tells us how much the consumers would pay to fully resolve all model uncertainty. The second approach is based on Barro (2009)'s local welfare analysis which allows us to answer questions like "how much consumers would pay to reduce partial uncertainty, such as 10 percent of model uncertainty, in order to keep the level of lifetime utility unchanged." We evaluate welfare costs based on our extended model which better accounts for data than the benchmark model and our conclusions also hold in our benchmark model. This is consistent with Barro's (2009) argument that welfare and policy analyses of the impacts of consumption uncertainty should be carried out within models that can at least roughly capture the stylized facts in the asset markets.

6.1 Total Welfare Gains from Eliminating Model Uncertainty

We follow Lucas's elimination-of-risk method (see Lucas 1987 and Tallarini 2000) to quantify the welfare cost of RB in the general equilibrium. We define the total welfare cost of model uncertainty as the percentage of initial wealth a typical consumer is willing to give up to be as well off in the FI-RE economy as he is in the RB economy.[45] That is, define

$$\widetilde{f}\left(s_0\left(1-\Delta\right)\right) = f\left(s_0\right), \tag{67}$$

where

$$\widetilde{f}\left(s_0\left(1-\Delta\right)\right) = -\frac{\delta}{\widetilde{\alpha}_1}\exp\left(-\widetilde{\alpha}_0 - \widetilde{\alpha}_1 s_0\left(1-\Delta\right)\right) \text{ and } f\left(s_0\right) = -\frac{\delta}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1 s_0\right)$$

are the value functions under FI-RE and RB, respectively, Δ is the compensating amount measured as a percentage of s_0, the initial wealth,

$$\alpha_1 = \frac{r^*}{\psi}, \widetilde{\alpha}_1 = \frac{\widetilde{r}^*}{\psi},$$
$$\alpha_0 = \frac{\delta}{r^*} - 1 - \frac{1}{2}\frac{r^*}{\psi}\left(\gamma + \frac{\vartheta}{\psi}\right)\left(\sigma_s^2 - \overline{\alpha}^2\sigma_e^2\right), \widetilde{\alpha}_0 = \frac{\delta}{\widetilde{r}^*} - 1 - \frac{1}{2}\frac{\widetilde{r}^*}{\psi}\left(\gamma + \frac{\vartheta}{\psi}\right)\left(\widetilde{\sigma}_s^2 - \overline{\alpha}^2\sigma_e^2\right),$$

and r^* and $\widetilde{r}^*$ are the equilibrium interest rates in the FI-RE and RB economies, respectively.[46] The following proposition summarizes the result about how RB affects the welfare costs in general equilibrium.

Proposition 13 *The welfare costs due to model uncertainty are given by*

$$\Delta = \frac{s_0\left(\widetilde{\alpha}_1 - \alpha_1\right) - \ln\left(\widetilde{\alpha}_1/\alpha_1\right) + \left(\widetilde{\alpha}_0 - \alpha_0\right)}{\widetilde{\alpha}_1 s_0} = \left(1 - \frac{r^*}{\widetilde{r}^*}\right) + \frac{\psi}{\widetilde{c}_0}\ln\left(\frac{\widetilde{r}^*}{r^*}\right) + \frac{\psi\left(\widetilde{\alpha}_0 - \alpha_0\right)}{\widetilde{c}_0}, \tag{68}$$

where $\widetilde{c}_0 = \widetilde{r}^* s_0$ *is optimal consumption under FI-RE.*

Proof. Substituting the equilibrium condition (28) into the expressions of α_0 and $\widetilde{\alpha}_0$ in the value functions under FI-RE and RB, we obtain that

$$\alpha_0 = \frac{1}{\psi}\overline{\alpha}\pi, \widetilde{\alpha}_0 = \frac{1}{\psi}\overline{\alpha}\widetilde{\pi}.$$

[45] This approach is also used in Epaulard and Pommeret (2003) to examine the welfare cost of volatility in a representative-agent model with recursive utility. In their model, the total welfare cost of volatility is defined as the percentage of capital the representative agent is ready to give up at the initial period to be as well off in a certain economy as he is in a stochastic one.

[46] When we compare welfare in these two economies, we assume that the asset supply is the same across the two economies. See Appendix 8.3 for the derivation of the value functions. Note that $\Delta = 0$ when $\vartheta = 0$.

Combining these results with (67) yields (68). ■

Note that when $\overline{\alpha} = 0$, our multiple-asset model reduces to the benchmark model and $\Delta = \left(1 - \frac{r^*}{\widetilde{r}^*}\right) + \frac{\psi}{\widetilde{c}_0} \ln\left(\frac{\widetilde{r}^*}{r^*}\right)$. To understand how the welfare cost varies with the degree of uncertainty aversion, we note that

$$\frac{\partial \Delta}{\partial \vartheta} = \frac{\partial \Delta}{\partial r^*} \frac{\partial r^*}{\partial \vartheta}.$$

The second term is negative, $\partial r^* / \partial \vartheta < 0$, for the reasons we have already discussed. The first term is

$$\frac{\partial \Delta}{\partial r^*} = -\frac{1}{\widetilde{r}^*}\left(1 - \frac{\psi}{\widetilde{c}_0}\right) - \frac{\overline{\alpha}}{\widetilde{c}_0} \widetilde{\gamma} \sigma_e \left(\rho_{ye} \sigma_s + \overline{\alpha} \sigma_e\right);$$

for reasonable values we expect this term to be negative, so that higher model uncertainty leads to larger welfare costs.

To provide some quantitative results, we set $c_0 = \widetilde{r}^* s_0 = 1$ as we did in our calibrated model. Figure 7 illustrates how the welfare cost of model uncertainty varies with ϑ for different values of $\overline{\alpha}$ and ρ_{ye}. The left panel of the figure shows that the welfare costs of model uncertainty are nontrivial. For example, when $\vartheta = 2.9$, the value we calibrate to match the data (see Table 3), the welfare cost of model uncertainty Δ is 15.1 percent. That is, a typical consumer is willing to scarify 15.1 percent of his initial wealth in order to get rid of such model uncertainty. Furthermore, the welfare cost rises with the degree of model uncertainty. For instance, if ϑ increases by 50% (from 2.9 to 4.35), Δ increases by about 25% (from 15.1 percent to 19.0 percent).

In addition, the same figure shows that the welfare cost decreases with $\overline{\alpha}$, given the value of ϑ. The reason behind this result is that the risky asset provides a hedging tool for the consumer as long as $\rho_{ye} \neq 0$, and higher supply of the asset means that agents' inefficient precautionary savings motives are weaker. The right panel of Figure 7 shows that the welfare cost of model uncertainty also decreases with the correlation between the equity return and labor income. The intuition behind this result is that the higher correlation between the two risks can make the consumers better hedge the fundamental risk, which reduces the welfare cost of uncertainty.

6.2 The Local Welfare Effects of Model Uncertainty

To examine the *local* effects of RB on welfare, we follow Barro (2009) and Luo and Young (2010) to compute the marginal welfare costs due to model uncertainty at different degrees of robustness (ϑ). The basic idea of this calculation is to use the value function (21) to calculate the effects of RB on the expected lifetime utility and compare them with those from proportionate changes in the initial income level. Specifically, following Barro (2009), the marginal welfare costs (mwc) due to RB can be written as:

$$\text{mwc}\left(\vartheta\right) = -\frac{\partial f_t / \partial \vartheta}{\left(\partial f / \partial y_t\right) y_t}|_{y_t = y_0} = \frac{1}{2\psi} \frac{\sigma_y^2 - \overline{\alpha}^2 \sigma_e^2 \left(r + \rho\right)^2}{\left(r + \rho\right) y_0}, \tag{69}$$

where $\partial f/\partial\vartheta$ and $\partial f/\partial y_t$ are evaluated in equilibrium for given y_0.[47] The value of mwc provides the proportionate increase in initial income that compensates, at the margin, for an increase in the degree of robustness – in the sense of keeping the level of lifetime utility unchanged. From (21), it is clear that this compensating income change depends on the EIS, the properties of the income process, and the equilibrium interest rate.

To provide some quantitative results, we use the same set of parameter values in the above calibrated model: $y_0 = 1$, $\gamma = 4.5$, $\psi = 0.5$, $\rho = 0.083$, $\sigma_y = 0.182$, $\overline{\alpha} = 10.5$, $\sigma_e = 0.16$, and $r = 1.91\%$. Based on these parameter values, we can calculate that mwc $= 0.193$, which suggests that a 10-percent increase in ϑ (from 2.9 to 3.2) requires an increase in initial income by 1.23% (i.e., $mwc \cdot 0.1 \cdot \vartheta = 0.193 \cdot 0.1 \cdot 2.9 = 1.23$) to make his lifetime utility unchanged. In other words, a typical consumer in our model economy would be willing to sacrifice 1.23% of his initial income to reduce the degree of model uncertainty by 10%. In addition, from (69) we can see that mwc is an increasing function of income volatility (σ_y), which means the marginal welfare cost will be larger if the economy is in a more volatile environment. To quantitatively see this point, let's assume σ_y increases by 20% from 0.182 to 0.22. Following the same calculation above, under this more volatile environment, the welfare costs of a 10-percent increase in ϑ leads to a welfare loss equivalent to a 5.61% decline in initial income, significantly larger than the 1.23% under the low volatility environment. This highlights the potentially larger welfare losses due to model uncertainty when the economy is facing larger income volatility, such as during economic crises.

It is worth noting that we can also use the value function under robustness, (21), to examine the *local* effects of income uncertainty on the welfare cost of model uncertainty for different degrees of RB. Specifically, the marginal welfare costs (mwc) due to income uncertainty can be written as:

$$\text{mwc}\left(\sigma_y^2\right) = -\frac{\partial f_t/\partial\sigma_y^2}{\left(\partial f/\partial y_t\right) y_t}|_{y_t=y_0} = \frac{1}{2}\frac{\widetilde{\gamma}}{\left(r+\rho\right) y_0}, \tag{70}$$

where $\partial f/\partial\sigma_y^2$ and $\partial f/\partial y_t$ are evaluated in equilibrium for given y_0. The value of mwc gives us the proportionate increase in y_0 to compensate for a small increase in σ_y^2 in the sense of keeping the level of lifetime utility unchanged. This formula can help us evaluate the importance of economic policies that aim to reduce income uncertainty of households.

Let's consider the following simple policy experiment. The government is implementing a macro policy to reduce the variance of household income by 10 percent, from σ_y^2 to $0.9 \cdot \sigma_y^2$. Holding all the other parameters fixed, we can calculate that the 10 percent reduction in income variance leads to a welfare improvement equivalent to a 16.9% increase in the initial income. In other words, a typical household is willing to reduce his initial income by 16.9% to reduce the variance of his income process by 10 percent. As a comparison, this welfare gain is only 7.4% of the initial income if there is no model uncertainty. One policy implication stemming from this finding is that macro policies aiming to reduce income volatility and inequality are more beneficial in an economy in

[47] Note that here we use the facts that $s_t = w_t + y_t/\left(r+\rho\right) + \overline{y}/\left(r\left(r+\rho\right)\right)$ and $\sigma_s^2 = \sigma_y^2/\left(r+\rho\right)^2$.

which consumers have a greater aversion to model uncertainty, both because they reduce risk and because they mitigate costly precautionary saving.[48] Finally, although our benchmark model has no business cycle dynamics, the above welfare calculations can still help us infer some insight about the welfare costs of business cycles under RB. Note that one key fact about the US business cycles is that income volatility is countercyclical.[49] Specifically, when the economy moves from an expansion into a recession, σ_y will increase and consumers with higher degrees of RB will suffer more from model uncertainty. Consequently, a macro policy aiming to remove the aggregate fluctuations is more beneficial to this economy.

7 Conclusions and Future Research

This paper has developed a tractable continuous-time recursive utility version of the Huggett (1993) model to explore how the preference for robustness (RB) interacts with intertemporal substitution and risk aversion and then affects the interest rate, the dynamics of consumption and income, and the welfare costs of model uncertainty in general equilibrium. We found that for moderate risk aversion and plausibly calibrated parameter values of robustness, our benchmark model can generate the observed low risk free rate in the US economy. However, the model cannot generate the observed high relative dispersion of consumption to income. But if we allow for a positive net supply of a risky asset (interpreted as the market portfolio), our model is able to reconcile low interest rates, moderate risk aversion, and relatively high dispersion of consumption to income. The resulting model implies that the welfare costs of model uncertainty are large.

To better illustrate the key effects of robustness on the equilibrium interest rate and relative consumption dispersion, we choose a framework which can analytically show the key mechanisms and mathematically is as simple as possible. However, our key insights can also be carried to more complicated cases. For example, our framework can be extended to study implications of robustness in a hidden-state model in which the consumers cannot perfectly observe the growth of their stochastic labor income. As discussed in HSW (2002), Hansen and Sargent (2006), and Kasa (2006), in this case, agents' preference for robustness not only affects their optimal control problem but also affects their optimal filtering problem. The effect of robustness on optimal filtering also provides additional information that could be used to further distinguish ambiguity aversion (a preference for robustness) from risk aversion. We provide some preliminary discussions in Appendix 8.5 and leave this for future research.

[48] Ellison and Sargent (2015) found that idiosyncratic consumption risk has a greater effect on the cost of business cycles when agents fear model misspecification. In addition, they showed that endowing agents with fears about misspecification leads to greater welfare costs caused by the idiosyncratic consumption risk. The underlying reasons are the same: the enhanced risk aversion created by uncertainty aversion.

[49] As discussed in Bloom (2014), unemployment rises during a recession, so the volatility of income at the household level will increase as well.

It is worth noting that our baseline model can also be extended to include rare macroeconomic disasters (e.g., financial crises). Specifically, to model rare disasters, we can assume that the labor income process not only includes a diffusion component but also includes a Poisson jump component.[50] For example, the dynamics of income, (4), can be described by the following more general specification:

$$dy_t = \rho\left(\overline{y} - y_t\right)dt + \sigma_y dB_t - \varphi_t dq, \tag{71}$$

where B_t reflects "normal" economic fluctuations and q is a pure jump process with frequency parameter λ. This specification means that labor income is reduced by an amount φ_t at random points in time.[51] That is, $dq = 1$ when the jump happens and $dq = 0$ otherwise. It is not difficult to introduce this new income process into our baseline model and solve it explicitly when we assume that the jump size, φ_t, is constant. However, when φ_t is stochastic, the problem would become much more complicated because agents in this situation will have different degrees of concerns about the y_t and φ_t processes.[52] We also leave this for future research.

8 Appendix

8.1 Description of Data

This appendix describes the data we use to estimate the income process as well as the method we use to construct a panel of both household income and consumption for our empirical analysis.

We use micro data from the Panel Study of Income Dynamics (PSID). Our household sample selection closely follows that of Blundell, Pistaferri, and Preston (2008).[53] We exclude households in the PSID low-income and Latino samples. We exclude household incomes in years of family composition change, divorce or remarriage, and female headship. We also exclude incomes in years where the head or wife is under 30 or over 65, or is missing education, region, or income responses. We also exclude household incomes where non-financial income is less than \$1000, where year-over-year income change is greater than \$90,000, and where year-over-year consumption change is greater than \$50,000. Our final panel contains 7,220 unique households with 54,901 yearly income responses and 50,422 imputed nondurable consumption values.[54]

[50] Barro (2009) also considered an output jump process within a discrete-time Lucas-type asset pricing setting.

[51] Note that here φ_t can be constant or stochastic and here we can interpret φ as a fraction of the mean of income because we assumed that the mean level of income ($\overline{y}$) is 1.

[52] Liu, Pan, and Wang (2005) focused on concerns about jump uncertainty and restricted the representative agent to a set of alternative models that differ only in terms of the jump component.

[53] They create a new panel series of consumption that combines information from PSID and CEX, focusing on the period when some of the largest changes in income inequality occurred.

[54] There are more household incomes than imputed consumption values because food consumption - the main input variable in Guvenen and Smith's nondurable demand function - is not reported in the PSID for the years 1987 and 1988. Dividing the total income responses by unique households yields an average of $7-8$ years of responses per household. These years are not necessarily consecutive as our sample selection procedure allows households to be

The PSID does not include enough consumption expenditure data to create full picture of household nondurable consumption. Such detailed expenditures are found, though, in the Consumer Expenditure Survey (CEX) from the Bureau of Labor Statistics. However, households in the CEX are only interviewed for four consecutive quarters and thus do not form a panel. To create a panel of consumption to match the PSID income measures, we use an estimated demand function for imputing nondurable consumption created by Guvenen and Smith (2014). Using an IV regression, they estimate a demand function for nondurable consumption that fits the detailed data in the CEX. The demand function uses demographic information and food consumption which can be found in both the CEX and PSID. Thus, we use this demand function of food consumption and demographic information (including age, family size, inflation measures, race, and education) to estimate nondurable consumption for PSID households, creating a consumption panel.

In order to estimate the income process, we narrow the sample period to the years $1980 - 1996$, due to the PSID survey changing to a biennial schedule after 1996. To further restrict the sample to exclude households with dramatic year-over-year income and consumption changes, we eliminate household observations in years where either income or consumption has increased more than 200 percent or decreased more than 80 percent from the previous year.

8.2 Solving the RE Recursive Utility Model

The RE optimizing problem can be written as:

$$f(J_t) = \max_{c_t} \left\{ \left(1 - e^{-\delta\Delta t}\right) f(c_t) + e^{-\delta\Delta t} f\left(CE_t\left[J_{t+\Delta t}\right]\right) \right\}, \tag{72}$$

where $f(J_t)$ is the value function. An educated guess is that $J_t = As_t + A_0$. The J function at t time $t + \Delta t$ can thus be written as

$$J(s_{t+\Delta t}) = As_{t+\Delta t} + A_0 \approx As_t + A(rs_t - c_t)\Delta t + A\sigma_s \Delta B_t + A_0,$$

where $\Delta s_t \equiv s_{t+\Delta t} - s_t$ and $\Delta s_t \approx (rs_t - c_t)\Delta t + \sigma_s \Delta B_t$. (Here $\Delta B_t = \sqrt{\Delta t}\epsilon$ and ϵ is a standard normal distributed variable.)

Using the definition of the certainty equivalent of $J_{t+\Delta t}$, we have

$$\begin{aligned}\exp(-\gamma CE_t) &= E_t\left[\exp\left(-\gamma J\left(s_{t+\Delta t}\right)\right)\right] \\ &= \exp\left(-\gamma A E_t\left[s_{t+\Delta t}\right] + \frac{1}{2}\gamma^2 A^2 \operatorname{var}_t\left[s_{t+\Delta t}\right] - \gamma A_0\right) \\ &= \exp\left(-\gamma A\left[s_t + (rs_t - c_t)\Delta t\right] + \frac{1}{2}\gamma^2 A^2 \sigma_s^2 \Delta t - \gamma A_0\right),\end{aligned}$$

which means that

excluded in certain years but return to the sample if they later meet the criteria once again.

$$CE_t = A\left[s_t + \left(rs_t - c_t - \frac{1}{2}\gamma A\sigma_s^2\right)\Delta t\right] + A_0. \tag{73}$$

Substituting these expressions back into (72) yields:

$$0 = \max_{c_t}\left\{\delta f(c_t)\Delta t + f'(J_t)\left(A(rs_t - c_t) - \frac{1}{2}\gamma A^2\sigma_s^2\right)\Delta t - \delta\Delta t f(J_t)\right\}$$

where we use the facts that $e^{-\delta\Delta t} = 1 - \delta\Delta t$,

$$J_{t+\Delta t} \approx J_t + J_t'(rs_t - c_t)\Delta t = J_t + A(rs_t - c_t)\Delta t,$$

and

$$f\left(J_t + A(rs_t - c_t)\Delta t + \frac{1}{2}A^2\sigma_s^2\Delta t\right) \approx f(J_t) + f'(J_t)\left(A(rs_t - c_t) - \frac{1}{2}\gamma A^2\sigma_s^2\right)\Delta t.$$

Dividing both sides by Δt, the Bellman equation can then be simplified as:

$$\delta f(J_t) = \max_{c_t}\left\{\delta f(c_t) + f'(J_t)\left(A(rs_t - c_t) - \frac{1}{2}\gamma A^2\sigma_s^2\right)\right\}. \tag{74}$$

The FOC for c is then

$$\delta f'(c_t) = f'(J_t)A,$$

which implies that

$$c_t = -\psi\ln\left(\frac{A}{\delta}\right) + (As_t + A_0). \tag{75}$$

Substituting this expression for c back to the Bellman equation and matching the coefficients, we have:

$$A = r$$

and

$$A_0 = \psi\left(\frac{\delta - r}{r}\right) + \psi\ln\left(\frac{r}{\delta}\right) - \frac{1}{2}\gamma r\sigma_s^2 - \frac{\vartheta}{2\psi}r\sigma_s^2.$$

Substituting these coefficients into (75) gives the consumption function, (9), in the main text.

8.3 Solving the Benchmark RU-RB Model

Following Hansen and Sargent (2007), Uppal and Wang (2003), and Maenhout (2004), we introduce robustness into the above otherwise standard model as follows:

$$0 = \max_{c_t}\min_{\upsilon_t}\left\{\delta f(c_t)\Delta t + f'(J_t)\left(A(rs_t - c_t) - \frac{1}{2}\gamma A^2\sigma_s^2\right)\Delta t - \delta f(J_t)\Delta t + \frac{1}{2(\vartheta_t/\Delta t)}\upsilon_t^2\sigma_s^2\right\}$$

subject to the distorting equation, $\Delta s_t \approx (rs_t - c_t)\Delta t + \sigma_s(\sigma_s\upsilon_t\Delta t + \Delta B_t)$. It is worth noting that here following Hansen and Sargent (2011) and Kasa and Lei (2017), we scale the robustness parameter (ϑ_t) by the sampling interval (Δt), effectively making the consumer have stronger preference for robustness (or more ambiguity averse) as the sampling interval shrinks.

Dividing both sides by Δt, the Bellman equation reduces to:

$$\delta f(J_t) = \max_{c_t} \min_{\upsilon_t} \left\{ \delta f(c_t) + f'(J_t) A \left(r s_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 + \upsilon_t \sigma_s^2 \right) + \frac{1}{2\vartheta_t} \upsilon_t^2 \sigma_s^2 \right\}, \tag{76}$$

subject to (12). Solving first for the infimization part of the problem yields

$$\upsilon^*(s_t) = -\vartheta_t A f'(J_t).$$

Given that $\vartheta(s_t) > 0$, the perturbation adds a negative drift term to the state transition equation because $f'(J_t) > 0$. Substituting it into the above HJB equation yields:

$$\delta f(J_t) = \max_{c} \left\{ \delta f(c_t) + f'(J_t) A \left(r s_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 - \vartheta_t A f'(U_t) \sigma_s^2 \right) + \frac{1}{2\vartheta_t} \left(\vartheta_t A f'(J_t) \right)^2 \sigma_s^2 \right\} \tag{77}$$

Following Uppal and Wang (2003) and Maenhout (2004), we assume that

$$\vartheta_t = -\frac{\vartheta}{f(U_t)}.$$

The HJB equation reduces to

$$\delta f(U_t) = \max_{c} \left\{ \delta f(c_t) + f'(J_t) A \left(r s_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 + \frac{\vartheta}{f(U_t)} A f'(U_t) \sigma_s^2 \right) - \frac{\vartheta}{2 f(J_t)} A^2 \left(f'(J_t) \right)^2 \sigma_s^2 \right\}.$$

The FOC for c is then

$$\delta f'(c_t) = f'(J_t) A,$$

which implies that

$$c_t = -\psi \ln\left(\frac{A}{\delta}\right) + (A s_t + A_0). \tag{78}$$

Substituting this expression for c back to the Bellman equation and matching the coefficients, we have:

$$A = r \text{ and } A_0 = \psi\left(\frac{\delta}{r} - 1\right) + \psi \ln\left(\frac{r}{\delta}\right) - \frac{1}{2}\gamma r \sigma_s^2 - \frac{\vartheta}{2\psi} r \sigma_s^2.$$

Substituting these coefficients into (78) gives the consumption function, (16), and the value function, (21), in the main text.

Finally, we check if the consumer's transversality condition (TVC),

$$\lim_{t\to\infty} E\left[\exp(-\delta t) \left| f(s_t) \right|\right] = 0, \tag{79}$$

is satisfied. Substituting the consumption function, c_t^*, into the state transition equation for s_t yields

$$ds_t = \widetilde{A} dt + \sigma_s dB_t,$$

where $\widetilde{A} = -\frac{\psi(\delta - r)}{r} + \frac{1}{2} r \widetilde{\gamma} \sigma_s^2$ under the approximating model. This Brownian motion with drift can be rewritten as

$$s_t = s_0 + \widetilde{A} t + \sigma (B_t - B_0), \tag{80}$$

where $B_t - B_0 \sim N(0, t)$. Substituting (80) into $E[\exp(-\delta t)|f(s_t)|]$ yields:

$$\begin{aligned} E[\exp(-\delta t)|f(s_t)|] &= \frac{1}{\alpha_1} E[\exp(-\delta t - \alpha_0 - \alpha_1 s_t)] \\ &= \frac{1}{\alpha_1} \exp\left(E[-\delta - \alpha_0 - \alpha_1 s_t] + \frac{1}{2}\operatorname{var}(\alpha_1 s_t)\right) \\ &= \frac{1}{\alpha_1} \exp\left(-\delta t - \alpha_0 - \alpha_1\left(s_0 + \widetilde{A}t\right) + \frac{1}{2}\alpha_1^2\sigma_s^2 t\right) \\ &= |J(s_0)| \exp\left(-\left(\delta + \alpha_1\widetilde{A} - \frac{1}{2}\alpha_1^2\sigma_s^2\right)t\right) \end{aligned}$$

where $|J(s_0)| = \frac{1}{\alpha_1}\exp(-\alpha_0 - \alpha_1 s_0)$ is a positive constant and we use the facts that $s_t - s_0 \sim N\left(\widetilde{A}t, \sigma_s^2 t\right)$. Therefore, the TVC, (79), is satisfied if and only if the following condition holds:

$$\delta + \alpha_1\widetilde{A} - \frac{1}{2}\alpha_1^2\sigma_s^2 = r + \frac{1}{2}\left(\frac{r}{\psi}\right)^2\left(\frac{\gamma}{\psi} - 1 + \vartheta\right)\sigma_s^2 > 0.$$

Given the parameter values we consider in the text, it is obvious that the TVC is always satisfied in both the FI-RE and RB models. It is straightforward to show that the TVC still holds under the distorted model in which $\widetilde{A} = -\frac{\psi(\delta - r)}{r} + \frac{1}{2}r\widetilde{\gamma}\sigma_s^2 - \frac{r}{\psi}\vartheta\sigma_s^2$ for plausible values of ϑ.

8.4 Solving the RU-RB Model with a Risky Asset

The robust HJB equation for the RU-RB model with multiple financial assets can be written as:

$$\delta f(J_t) = \max_c \min_\upsilon \left\{\delta f(c_t) + f'(U_t) A\left(rs_t - c_t + \pi\chi_t - \frac{1}{2}\gamma A\sigma^2 + \upsilon_t\sigma^2\right) + \frac{1}{2\vartheta_t}\upsilon_t^2\sigma^2\right\}, \quad (81)$$

subject to the distorting equation, (50). Solving first for the infimization part of the problem yields

$$\upsilon^*(s_t) = -\vartheta_t A f'(J_t).$$

Given that $\vartheta(s_t) > 0$, the perturbation adds a negative drift term to the state transition equation because $f'(J_t) > 0$. Substituting it into the above HJB equation yields:

$$\delta f(J_t) = \max_c \left\{\delta f(c_t) + f'(U_t) A\left(rs_t - c_t + \pi\alpha_t - \frac{1}{2}\gamma A\sigma^2 - \vartheta_t A f'(U_t)\sigma^2\right) + \frac{1}{2\vartheta_t}\left(\vartheta_t A f'(U_t)\right)^2\sigma^2\right\}$$

Following Uppal and Wang (2003) and Maenhout (2004), we assume that $\vartheta_t = -\vartheta/f(U_t)$. The HJB equation reduces to

$$\delta f(U_t) = \max_c \left\{\delta f(c_t) + f'(U_t) A\left(rs_t - c_t + \pi\alpha_t - \frac{1}{2}\gamma A\sigma^2 + \frac{\vartheta f'(U_t)}{f(U_t)}A\sigma^2\right) - \frac{\vartheta\left(f'(U_t)\right)^2}{2f(J_t)}A^2\sigma^2\right\}.$$

Using the fact that $f(U_t) = (-\psi)\exp(-U_t/\psi)$, the HJB reduces to

$$\delta f(U_t) = \max_c \left\{\delta f(c_t) + f'(U_t) A\left(rs_t - c_t + \pi\alpha_t - \frac{1}{2}\left(\gamma + \frac{\vartheta}{\psi}\right)A\sigma^2 + \frac{\vartheta f'(U_t)}{f(U_t)}A\sigma^2\right)\right\}.$$

The FOC for c is then

$$\delta f'(c_t) = f'(J_t) A,$$

which implies that

$$c_t = -\psi \ln\left(\frac{A}{\delta}\right) + (As_t + A_0). \tag{82}$$

The FOC for α_t is

$$\alpha_t = \frac{\pi}{r(\gamma + \vartheta/\psi)\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{83}$$

which is just (53). Substituting this expression for c back to the Bellman equation and matching the coefficients, we have:

$$A = r \tag{84}$$

and

$$A_0 = \psi\left(\frac{\delta}{r} - 1\right) - \frac{1}{2} r\widetilde{\gamma}\left(1 - \rho_{ye}^2\right)\sigma_s^2 + \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2} - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} + \psi \ln\left(\frac{r}{\delta}\right). \tag{85}$$

Substituting these coefficients into (82) gives the consumption function, (52) in the main text.

Given the optimal consumption-portfolio rules, the individual saving function can be written as

$$\begin{aligned}
d_t^* &= ra_t + y_t - c_t^* + \pi\alpha^* \\
&= ra_t + y_t - \left(rs_t + \Psi - \Gamma - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} + \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}\right) + \pi\left(\frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}\right) \\
&= \left[r\left(a_t + \frac{1}{r+\rho_1}y_t + \frac{\rho_1}{r(r+\rho_1)}\overline{y}\right) - r\left(\frac{1}{r+\rho_1}y_t + \frac{\rho_1}{r(r+\rho_1)}\overline{y}\right)\right] + y_t \\
&- \left(rs_t + \Psi - \Gamma - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} + \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}\right) + \pi\left(\frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}\right) \\
&= \frac{\rho}{r+\rho}(y_t - \overline{y}) + \Gamma - \Psi + \Pi,
\end{aligned}$$

where $\Pi = \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}$.

8.5 Identifying the Degree of Robustness: A Hidden-State Case

In this appendix, we follow Wang (2009) and assume that the income growth rate in the income process is unknown and stochastic. To make the robust control and filtering problem more tractable, in this section we assume that the mean of income growth is unknown and follows a continuous-state mean-reverting Ornstein-Uhlenbeck process (in this section we denote the stochastic income growth by $\varkappa_t$ in (4)):[55]

$$d\varkappa_t = \lambda(\overline{\varkappa} - \varkappa_t)\,dt + \sigma_\varkappa dB_{\varkappa,t}, \tag{86}$$

where $B_{\varkappa,t}$ is a standard Brownian motions defined over the complete probability space, λ and $\overline{\varkappa}$ are positive constants, and the correlation between $dB_{y,t}$ and $dB_{\varkappa,t}$ is $\rho_{y\varkappa}$. Hence, in this model

[55] Wang (2009) assumed that the unobservable income growth rate follows a two-state Markov chain.

both the mean of income growth and the actual income are stochastic. Consequently, the consumers need to estimate $\varkappa_t$ using their observations of the realized labor income. For simplicity, here we assume that the loss function due to incomplete information on $\varkappa_t$ is the mean square error. Given a Gaussian prior, finding the posterior distribution of $\varkappa_t$ becomes a standard Kalman-Bucy filtering problem.

As discussed in HSW (2002) and Kasa (2006), agents's preference for robustness not only affects their optimal control problem, but also affects their optimal filtering problem. We thus consider a situation in which a typical consumer pursues a robust Kalman gain when facing unknown $\varkappa_t$. To obtain a robust Kalman filter gain, the consumer considers the following distorted model:

$$d\varkappa_t = \left[\lambda\left(\overline{\varkappa} - \varkappa_t\right) + \sigma_{\varkappa} v_{1,t}\right] dt + \sigma_{\varkappa} d\widetilde{B}_{\varkappa,t}, \tag{87}$$

$$dy_t = \left(\varkappa_t - \rho y_t + \sigma_y v_{2,t}\right) dt + \sigma_y d\widetilde{B}_{y,t}, \tag{88}$$

where $\widetilde{B}_{\varkappa,t}$ and $\widetilde{B}_{y,t}$ are Wiener processes that are related to the approximating processes via

$$\widetilde{B}_{\varkappa,t} = B_{\varkappa,t} - \int_0^t v_{1,s} ds \text{ and } \widetilde{B}_{y,t} = B_{y,t} - \int_0^t v_{2,s} ds$$

and $\nu_{1,t}$ and $\nu_{2,t}$ are distortions to the conditional means of the two shocks, $\widetilde{B}_{\varkappa,t}$ and $\widetilde{B}_{y,t}$, respectively. Following the same procedure adopted in Kasa (2006), we can solve the robust filtering problem explicitly. The following proposition summarizes the results for this robust filtering problem:

Proposition 14 *If $\vartheta \geq \sigma_{\varkappa}^2/(\sigma_{\varkappa}^2+\lambda^2)$, there exists a unique solution for the robust filtering problem:*

$$dm_t = \lambda\left(\overline{\varkappa} - m_t\right) dt + \sigma_m dB_{m,t}, \tag{89}$$

$$\frac{d\Sigma_t}{dt} = -2\lambda\Sigma_t - \left(\frac{1}{\sigma_y^2} - \vartheta\right)\left(\eta + \Sigma_t\right)^2 + \sigma_{\varkappa}^2, \tag{90}$$

where $m_t = E_t\left[\varkappa_t - m_t\right]$ and $\Sigma_t = E_t\left[\left(\varkappa_t - m_t\right)^2\right]$ are the conditional mean and variance of $\varkappa_t$, respectively, $dB_{m,t} = dB_{y,t} + \left(\varkappa_t - m_t\right)/\sigma_y$ is the normalized unanticipated innovation of the income growth process, $K_t = \left(\eta + \Sigma_t\right)/\sigma_y^2$ is the robust Kalman gain, and $\eta = \rho_{y\varkappa}\sigma_y\sigma_{\varkappa}$. In the steady state, Σ_t converges to

$$\Sigma^* = -\eta + \frac{-\lambda\sigma_y^2 + \sigma_y\sqrt{\lambda^2\sigma_y^2 + \left(1 - \vartheta\sigma_y^2\right)\left(2\lambda\eta + \sigma_{\varkappa}^2\right)}}{1 - \vartheta\sigma_y^2} > 0, \tag{91}$$

which implies that

$$\sigma_m \equiv K\sigma_y = \frac{\eta + \Sigma^*}{\sigma_y} = \frac{-\lambda\sigma_y + \sqrt{\lambda^2\sigma_y^2 + \left(1 - \vartheta\sigma_y^2\right)\left(2\lambda\eta + \sigma_{\varkappa}^2\right)}}{1 - \vartheta\sigma_y^2}. \tag{92}$$

Proof. The proof is the same as that used in Section 4 of Kasa (2006). ■

Given that $2\lambda\eta + \sigma_{\varkappa}^2 > 0$, it is clear from (91) show that

$$\frac{\partial \sigma_m}{\partial \vartheta} > 0 \text{ and } \frac{\partial K}{\partial \vartheta} > 0.$$

That is, the robust Kalman gain, $K = \sigma_m/\sigma_y$, is increasing with the preference for pursuing robust Kalman filter. The following proposition summarizes the solution for this robust consumption-portfolio rule problem:

Proposition 15 *With unknown income growth, the robust consumption and portfolio rules are*

$$c_t^* = r\left(w_t + h_t + l_t\right) + \Psi\left(r\right) - \Gamma\left(\vartheta, r\right), \tag{93}$$

where $h_t = \frac{1}{r+\rho}\left(y_t + \frac{\overline{\varkappa}}{r}\right)$, $l_t = \frac{1}{(r+\rho)(\lambda+r)}\left(m_t - \overline{\varkappa}\right)$, $\Psi\left(r\right) = \left(\frac{\delta}{r} - 1\right)\psi$,

$$\Gamma\left(\vartheta, r\right) = \frac{1}{2} r\widetilde{\gamma}\left(1+x\right)^2 \left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_m}{\left(r+\rho\right)\left(r+\lambda\right)}\right]^2, \tag{94}$$

is the investor's precautionary saving demand, $\widetilde{\gamma} \equiv \gamma + \vartheta/\psi$ *is the effective coefficient of absolute risk aversion,* $\sigma_m = -\lambda\sigma_y + \sqrt{\lambda^2\sigma_y^2 + 2\lambda\eta + \sigma_{\varkappa}^2}$, $\eta = \rho_{y\varkappa}\sigma_y\sigma_{\varkappa}$, *and* $x \equiv \frac{K}{r+\lambda} > 0$.

Proof. See Online Appendix C for the detailed derivations. ■

It is clear from (89)-(91) and (93)-(94) that the consumption and precautionary saving rules are broadly similar to the ones in the benchmark model except that they include the robust learning effects. More specifically, the preference for robustness (ϑ) affects the robust consumption-saving rule via two channels: (i) the direct channel (*the robust control channel*) via $\widetilde{\gamma}$ and (ii) the indirect channel (*the robust filtering channel*) via the robust Kalman gain (K) and x. We can also see from these expressions that although γ and ϑ affect the robust decision rules via $\widetilde{\gamma} \equiv \gamma + \vartheta/\psi$, a fixed combination of γ and ϑ, only ϑ matters in the robust filtering problem. Consequently, the hidden-state specification can help provide a way via which we can distinguish risk aversion and ambiguity aversion. In other words, if we can observe the agents' filtering/learning and control decisions using some micro-level or experimental data, we can distinguish the degree of robustness from risk aversion.

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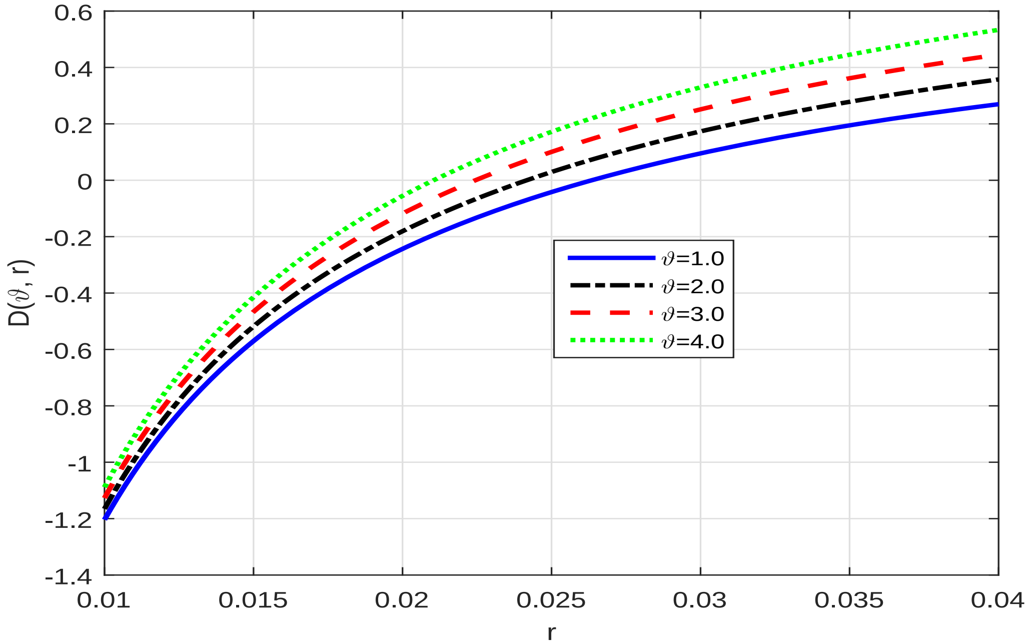

Figure 1: Effects of RB on Aggregate Savings

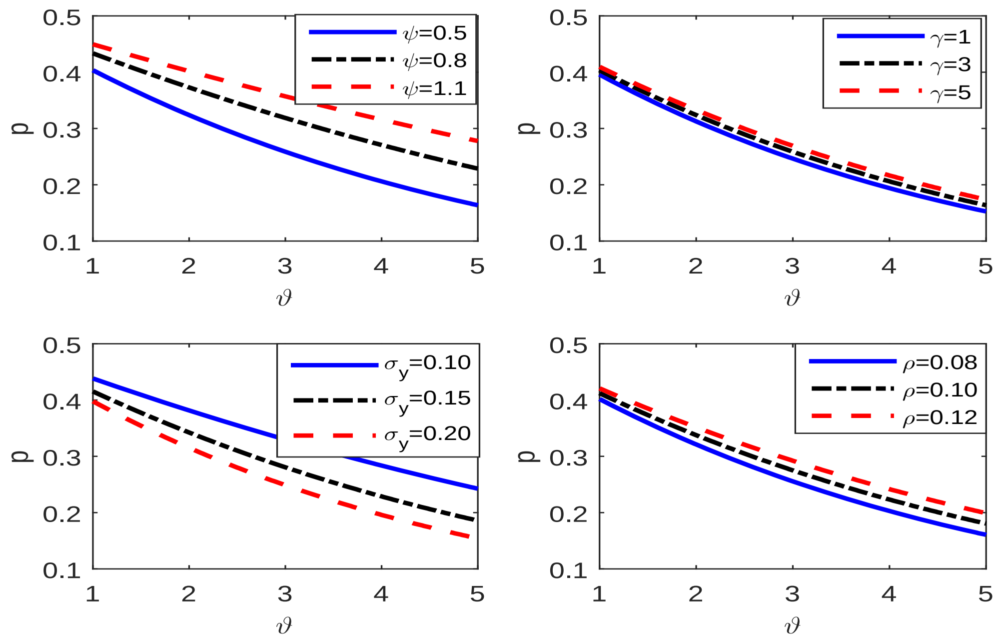

Figure 2: Relationship between ϑ and p

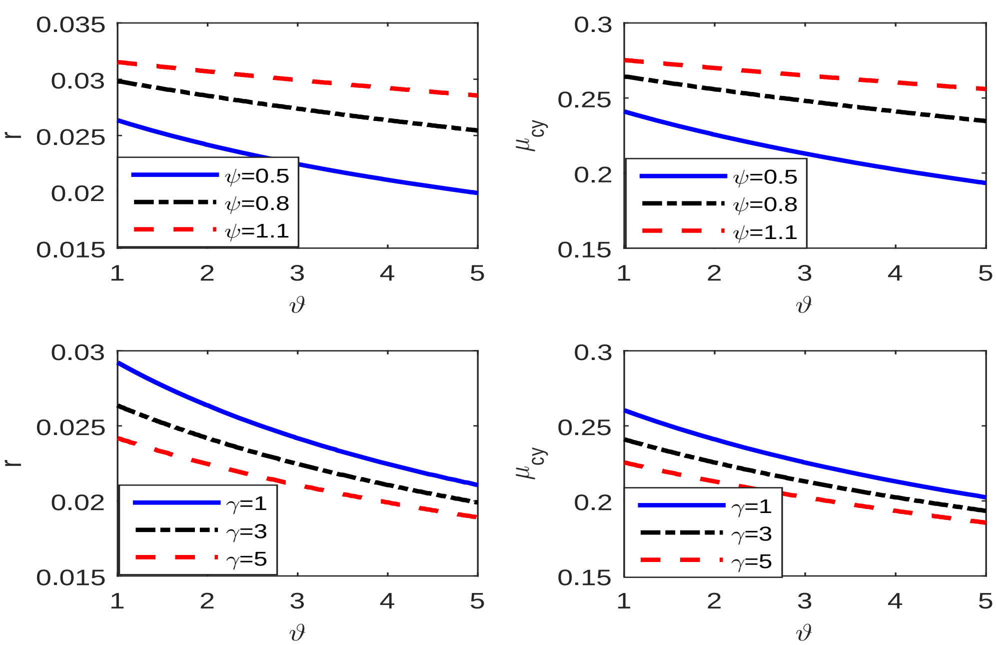

Figure 3: Effects of RB on the Interest Rate and Consumption Volatility

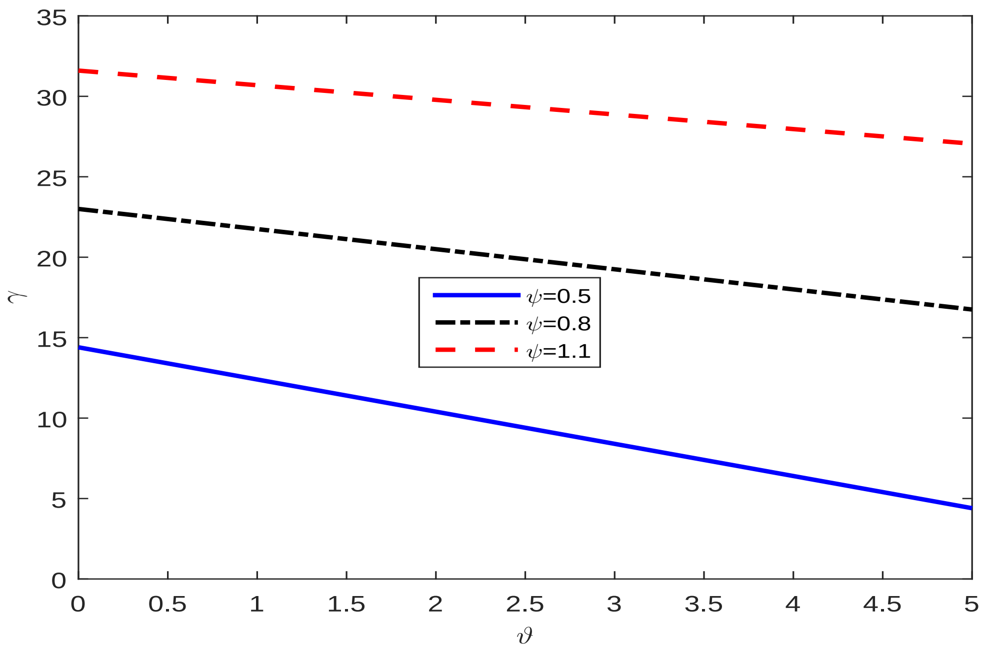

Figure 4: Relationship between γ and ϑ.

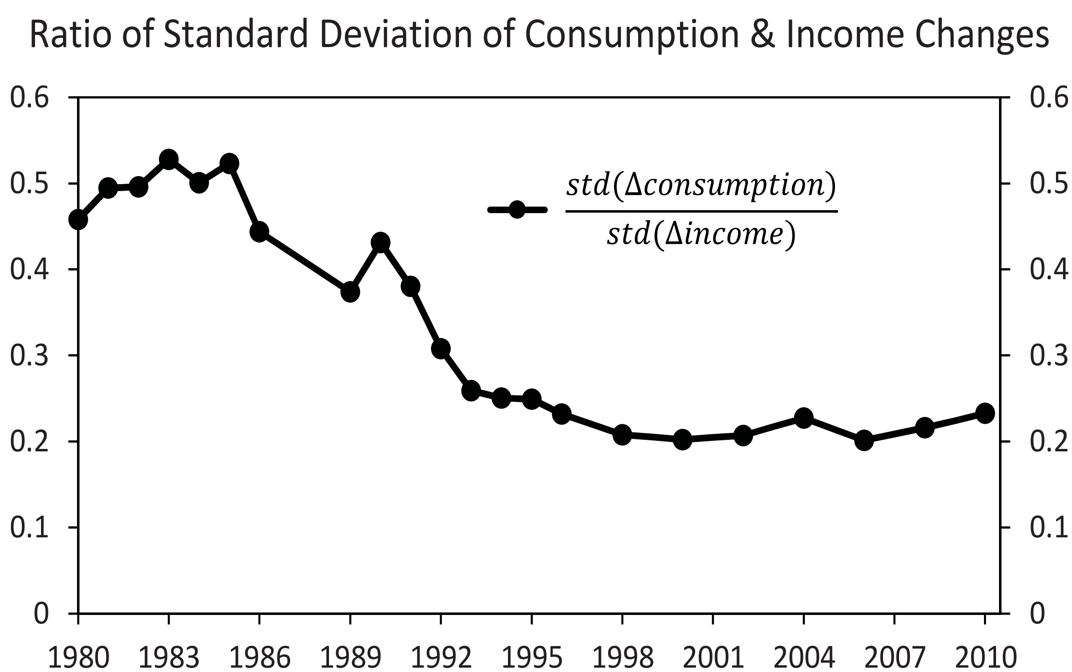

Figure 5: Relative Consumption Dispersion

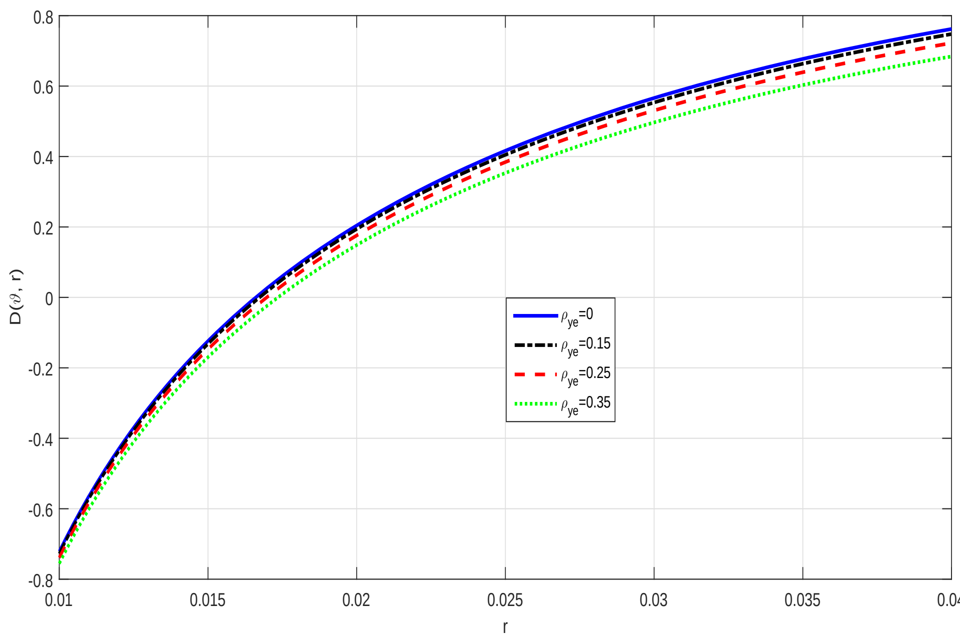

Figure 6: Effects of RB on Aggregate Savings

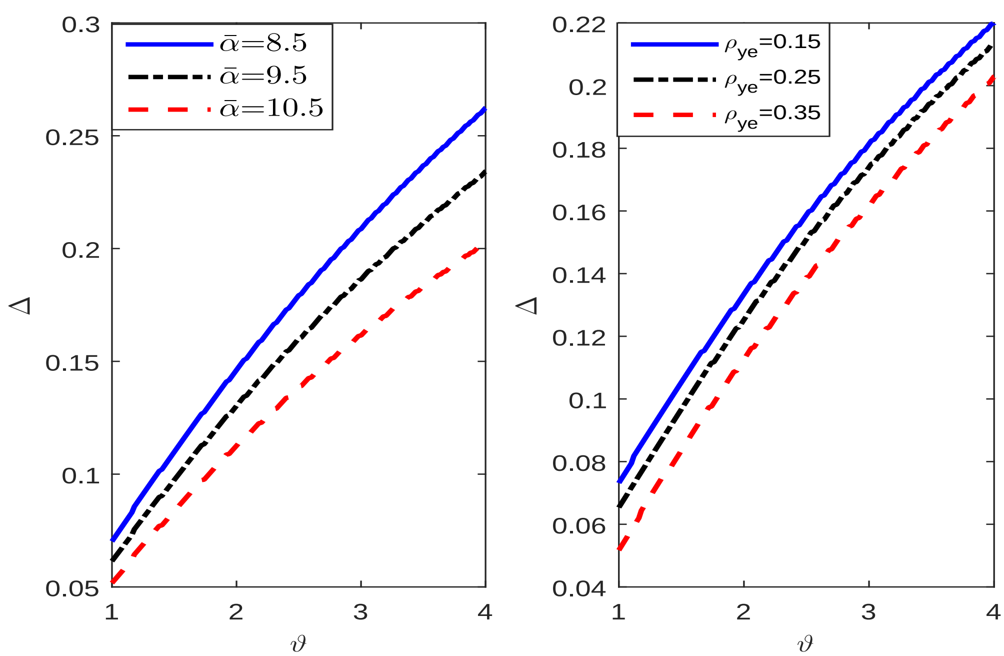

Figure 7: Effects of RB on the Welfare Cost

Table 1: Measures of the Risk Free Rate

	Three-month Nominal T-Bond	One-year Nominal T-Bond
CPI Inflation (1981 – 2010)	1.87%	2.33%
PCE Inflation (1981 – 2010)	1.96%	2.42%
CPI Inflation (1981 – 2015)	1.37%	1.78%
PCE Inflation (1981 – 2015)	1.75%	2.16%

Table 2: Estimation and Calibration Results

Labor Income	Parameter	Values
Discrete time specification		
constant	ϕ_0	0.0005
persistence	ϕ_1	0.919
std. of shock	σ	0.175
Continuous-time specification		
persistence	ρ	0.083
std. of income changes	σ_y	0.182

Table 3: Model Comparison with Key Parameter Values

Parameters Moments	Data	FI-RE	RB (Benchmark)	RB ($\overline{\alpha} = 10.5$)
γ		14.5	4.5	4.5
ϑ		0	5	2.9
p		0.5	0.17	0.19
r^*	1.91%	1.91	1.91	1.91
π^*	7.2%	n.a.	n.a.	7.2%
μ	0.31	0.19	0.19	0.31

Table 4: Explaining the Decline in Relative Consumption Dispersion

Period	Key Parameter		DEP	Data		Model	
	$\overline{\alpha}$	ϑ	p	r	μ_{cy}	r^*	${\mu_{cy}}^*$
$1980 - 1995$	7.3	1	0.41	3.1%	0.39	3.1%	0.39
$1996 - 2010$	7.3	13.5	0.02	1.5%	0.22	1.5%	0.21