Lucas and Stokey "Optimal Growth with Many Consumers"

Uses recursive utility

$$u(x) = W(x_0, u({}_1x)),$$

which includes elastic discounting of the Uzawa type as well as non-separable recursive utilities discussed in Koopmans.

$${}_1x = (x_1, x_2, ...)$$

is the sequence of consumption from tomorrow onward. W is continuous, increasing, concave, and satisfies a discounting property:

$$|W(x,z) - W(x,z')| \leq \beta |z - z'|$$

$\forall x, z, z'$ and some $\beta \in (0,1)$. Roughly speaking, x is current consumption and z is utility promised from tomorrow onward. Thinking recursively, the function u is the unique fixed point of the operator

$$(T_W v)(x) = W(x, v({}_1x))$$

and thus uniquely represents preferences over infinite streams of consumption. The discounting assumption is sufficient to establish that T_W is a contraction mapping.

One example of recursive preferences is the Lucas-Uzawa type

$$u({}_0x) = \sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} \beta(x_j) \right) u(x_t);$$

the cumulative discount factor is a function of the history of consumption in a log-additive way. Another example are Epstein-Zin preferences:

$$u({}_0x) = \{x_0^\rho + \mu({}_1x)^\rho\}^{\frac{1}{\rho}}$$

where ρ measures the intertemporal elasticity of substitution and μ is a certainty equivalent function. Of course standard time-additive preferences also work – Backus, Routledge, and Zin (2004) contains a survey of recursive utility.

Let the Pareto weights be $\theta \in \Delta^{n-1}$. Define the set of feasible (x_t, k_{t+1}) given k_t by $B(k_t)$. The set of feasible consumption good sequences is then

$$Y(k_0) = \left\{ x \in \updownarrow_+^m : (x_t, k_{t+1}) \in B(k_t) \;\; \forall t, \text{ some } k_t \in \updownarrow_+^P \right\}.$$

Let $y = k'$ and $w = \theta'$. Make enough assumptions such that B is closed, convex, and compact.

The Bellman equation defining a Pareto optimum is

$$v(k,\theta) = \max_{z,x,y} \left\{ \sum\nolimits_{i=1}^{n} \theta_i W_i(x_i, z_i) \right\}$$

subject to

$$\begin{aligned} z_i &\geq 0 \\ (x,y) &\geq B(k) \\ \min_{w\in\Delta^{n-1}}\left\{v(y,w)-\sum\nolimits_{i=1}^{n} w_i z_i\right\} &\geq 0. \end{aligned}$$

We can show that this procedure defines a contraction mapping.

Lemma 1 *There is a unique $v : \mathcal{R}_+^P \times \Delta^{n-1} \to \mathcal{R}$ that satisfies the dynamic program. v is increasing and concave in k and convex in θ.*

Proof. *Let F be the Banach space of continuous, bounded functions $f : \mathcal{R}_+^P \times \Delta^{n-1} \to \mathcal{R}$ with norm*

$$||f|| = \sup_{k,\theta}\{|f(k,\theta)|\}.$$

Let T be defined as the operator defined as

$$(Tv)(k,\theta) = \max_{z,x,y}\left\{\sum\nolimits_{i=1}^{n}\theta_i W_i(x_i,z_i)\right\}$$

subject to

$$\begin{aligned} z_i &\geq 0 \\ (x,y) &\geq B(k) \\ \min_{w\in\Delta^{n-1}}\left\{v(y,w)-\sum\nolimits_{i=1}^{n} w_i z_i\right\} &\geq 0. \end{aligned}$$

Because the objective is continuous and the constraint set is compact, $(Tf)(k,\theta)$ is well-defined $\forall f \in F$. Since B and f are continuous, (Tf) is continuous. It is clearly bounded, so $T : F \to F$.

We now verify Blackwell's conditions. Let $f, f' \in F$ be such that $f \leq f'$. Since f' enlarges the constraint set, $Tf \leq Tf'$. Monotonicity is verified.

Let $a > 0$ be a constant and fix $f \in F$, $k \in \mathcal{R}_+^P$, $\theta \in \Delta^{n-1}$. Let (x^a, y^a, z^a) attain $T(f+a)$ at (k,θ). Define the sets

$$\begin{aligned} U &= \left\{z \in \mathcal{R}_+^n : f(y^a,w) - wz \geq 0 \ \forall w \in \Delta^{n-1}\right\} \\ U^a &= \left\{z \in \mathcal{R}_+^n : f(y^a,w) + a - wz \geq 0 \ \forall w \in \Delta^{n-1}\right\} \\ B &= \left\{z \in \mathcal{R}_+^n : z \leq z' + a \text{ for some } z' \in U\right\}. \end{aligned}$$

We will show that $U^a = B$.

Since $\sum_i w_i = 1$, it is clear that $z' \in U$ implies $z' + a \in U^a$, so that $B \subseteq U^a$. Suppose $z \in U^a$ and $z \notin B$. It follows that if $z' \in \mathcal{R}_+^n$ satisfies $z' + a \geq z$, then $z' \notin U$. Hence there exists $w \in \Delta^{n-1}$ such that $f(y^a,w) - wz' < 0$. This in turn implies $f(y^a,w) - w(z-a) < 0$, so $z \notin U^a$. Since this is a contradiction, $U^a \subseteq B$ and so $U^a = B$.

If (x^a, y^a, z^a) attains $T(f+a)$, then $z^a \in U^a = B$, so that $z^a \geq z' + a$ for some $z' \in U$. Then $(x^a, y^a, z' + a)$ also attains $T(f+a)$, since W_i functions *are increasing in their last argument. Then, since* (x^a, y^a, z') *is feasible for the problem* $(Tf)(k,\theta)$*,*

$$(Tf)(k,\theta) \geq \sum_{i=1}^{n} \theta_i W_i\left(x_i^a, z_i'\right),$$

it follows that

$$\begin{aligned}
(T(f+a))(k,\theta) - (Tf)(k,\theta) &\leq \sum_{i=1}^{n} \theta_i \left[W_i\left(x_i, z_i' + a\right) - W_i\left(x_i, z_i'\right)\right] \\
&\leq \sum_{i=1}^{n} \theta_i \beta_i a \\
&\leq \max_i \{\beta_i a\} \\
&< a.
\end{aligned}$$

Therefore discounting is verified.

Since T *is a contraction, there exists a unique* $v = Tv$*. To see that* v *is increasing in* k*, note that* $k_1 \leq k_2$ *implies* $B(k_1) \subseteq B(k_2)$*, so the constraint set is weakly smaller at* k_1*. To prove that* v *is concave in* k *and convex in* θ*, we need only show that* T *preserves these properties, which is immediate.* ■

Thus, we can iterate on the above procedure to compute the value function for a Pareto problem with agents who have recursive utility.