Homework #5

Mathematical Methods II

Spring 2009

1. Consider a variant of the growth model in which households value consumption and housing services. The utility function is

$$\sum_{t=0}^{\infty} \beta^t \left[\log\left(c_t\right) + \theta \log\left(s_t\right)\right]. \tag{0.1}$$

The resource constraint for the planner is

$$c_t + h_{t+1} + k_{t+1} = Ak_t^{\alpha} + \left(1 - \delta_k\right) k_t + \left(1 - \delta_h\right) h_t, \tag{0.2}$$

where h_t is the stock of housing. Housing services are a multiple of the stock:

$$s_t = Bh_t. \tag{0.3}$$

1. Derive the equations that determine the steady state of this economy.

The Bellman equation is

$$v\left(k,h\right) = \max_{k',h'} \left\{\log\left(Ak^{\alpha} + \left(1-\delta_k\right)k + \left(1-\delta_h\right)h - h' - k'\right) + \theta \log\left(Bh\right) + \beta v\left(k',h'\right)\right\} \tag{0.4}$$

with first-order conditions

$$\begin{aligned}
\frac{1}{Ak^{\alpha} + \left(1-\delta_k\right)k + \left(1-\delta_h\right)h - h' - k'} &= \beta v_1\left(k',h'\right) \\
\frac{1}{Ak^{\alpha} + \left(1-\delta_k\right)k + \left(1-\delta_h\right)h - h' - k'} &= \beta v_2\left(k',h'\right).
\end{aligned}$$

The envelope conditions are

$$\begin{aligned}
v_1\left(k,h\right) &= \frac{\alpha A k^{\alpha-1} + 1 - \delta_k}{Ak^{\alpha} + \left(1-\delta_k\right)k + \left(1-\delta_h\right)h - h' - k'} \\
v_2\left(k,h\right) &= \frac{1 - \delta_h}{Ak^{\alpha} + \left(1-\delta_k\right)k + \left(1-\delta_h\right)h - h' - k'} + \frac{\theta}{h}.
\end{aligned}$$

The Euler equations are therefore

$$\begin{aligned}
\frac{1}{Ak^{\alpha}+(1-\delta_k)k+(1-\delta_h)h-h'-k'} &= \beta\frac{\alpha A(k')^{\alpha-1}+1-\delta_k}{A(k')^{\alpha}+(1-\delta_k)k'+(1-\delta_h)h'-h''-k''} \\
\frac{1}{Ak^{\alpha}+(1-\delta_k)k+(1-\delta_h)h-h'-k'} &= \beta\frac{1-\delta_h}{A(k')^{\alpha}+(1-\delta_k)k'+(1-\delta_h)h'-h''-k''}+\beta\frac{\theta}{h'}.
\end{aligned}$$

The steady state is then defined by the equations

$$\begin{aligned}
1 &= \beta\left(\alpha Ak^{\alpha-1}+1-\delta_k\right) \\
\frac{1}{c} &= \beta\frac{1-\delta_h}{c}+\beta\frac{\theta}{h} \\
c &= Ak^{\alpha}-\delta_k k-\delta_h h.
\end{aligned}$$

The first equation implies

$$k=\left(\frac{\alpha\beta}{1-\beta(1-\delta_k)}\right)^{\frac{1}{1-\alpha}}. \tag{0.5}$$

The second can be rearranged to

$$\begin{aligned}
\frac{1}{c}\left(1-\beta(1-\delta_h)\right) &= \frac{\beta\theta}{h} \\
c &= \frac{1-\beta(1-\delta_h)}{\beta\theta}h.
\end{aligned}$$

Inserting into the third yields

$$\frac{1-\beta(1-\delta_h)}{\beta\theta}h=A\left(\frac{\alpha\beta}{1-\beta(1-\delta_k)}\right)^{\frac{\alpha}{1-\alpha}}-\delta_k\left(\frac{\alpha\beta}{1-\beta(1-\delta_k)}\right)^{\frac{1}{1-\alpha}}-\delta_h h \tag{0.6}$$

or

$$h=\frac{A\left(\frac{\alpha\beta}{1-\beta(1-\delta_k)}\right)^{\frac{\alpha}{1-\alpha}}-\delta_k\left(\frac{\alpha\beta}{1-\beta(1-\delta_k)}\right)^{\frac{1}{1-\alpha}}}{\frac{1-\beta(1-\delta_h)}{\beta\theta}+\delta_h}. \tag{0.7}$$

We then have

$$c=\left(\frac{1-\beta(1-\delta_h)}{\beta\theta}\right)\left(\frac{A\left(\frac{\alpha\beta}{1-\beta(1-\delta_k)}\right)^{\frac{\alpha}{1-\alpha}}-\delta_k\left(\frac{\alpha\beta}{1-\beta(1-\delta_k)}\right)^{\frac{1}{1-\alpha}}}{\frac{1-\beta(1-\delta_h)}{\beta\theta}+\delta_h}\right). \tag{0.8}$$

2. Set $\delta_k=0.025$, $\delta_h=0.01$, $\beta=0.99$, $\theta=0.5$, $\alpha=0.36$, and $A=1$. Compute the values of (c,h,k) at the steady state. Use these values to compute a quadratic approximation

to the utility function at those values, after substituting c using the resource constraint. The steady state values are

$$\begin{aligned} c &= 2.2057 \\ k &= 37.9893 \\ h &= 54.8649. \end{aligned}$$

3. Use the Ricatti equation to solve for the decision rules $k' = g_k(k,h)$ and $h' = g_h(k,h)$ and the value function $v(k,h)$. Is your value function concave?

   The computed decision rules are

$$\begin{aligned} k' &= 16.1347 + 0.2382k + 0.2334h \\ h' &= -14.5668 + 0.7524k + 0.7417h \end{aligned}$$

   which yields the steady state when evaluated at the steady state.

2. The above economy was Pareto-efficient, so we could use the planning problem to compute the allocations. Imagine that we did not know this, and instead set out to solve for the recursive competitive equilibrium. The household budget constraint is

$$c_t + k_{t+1} + h_{t+1} \leq (r_t + 1 - \delta_k) k_t + w_t + (1 - \delta_h) h_t. \tag{0.9}$$

   The firm rents capital and labor to produce output:

$$\max_{K_t, H_t} \left\{ AK_t^{\alpha} N_t^{1-\alpha} - r_t K_t - w_t N_t \right\}. \tag{0.10}$$

   Markets clear if

$$\begin{aligned} k_t &= K_t \\ N_t &= 1 \\ C_t + K_{t+1} + H_{t+1} &= AK_t^{\alpha} + (1 - \delta_k) K_t + (1 - \delta_h) H_t. \end{aligned}$$

   1. Write down the Bellman equation for the individual household.

The Bellman equation is

$$v(k,h,K,H) = \max_{k',h'} \left\{ \begin{array}{c} \log\left(\left(\alpha A K^{\alpha-1} + 1 - \delta_k\right) k + (1-\alpha) A K^{\alpha} + (1-\delta_h) h - k' - h'\right) + \\ \beta v(k',h',K',H') \end{array} \right\} \tag{0.11}$$

subject to laws of motion for K' and H'.

2. Assume that the laws of motion for the aggregates are linear. Take a quadratic approximation to the utility function around the equilibrium steady state.

   Obviously the steady state is the same as in Problem 1.

3. Use Kydland's method to obtain decision rules $k' = \widehat{g}_k(k,h,K,H)$ and $h' = \widehat{g}_h(k,h,K,H)$ and the value function $v(k,h,K,H)$. Verify that your decision rules aggregate to $K' = G_k(K,H)$ and $H' = G_h(K,H)$. Also verify that you obtain similar aggregate laws of motion as you obtained in Problem 1.

   The computed decision rules are

$$\begin{aligned} k' &= 16.1088 - 0.5975K - 0.5856H + 0.8359k + 0.8193h \\ h' &= -14.5451 + 0.5889K + 0.5772H + 0.1675k + 0.1641h. \end{aligned}$$

   The aggregate laws of motion are

$$\begin{aligned} K' &= 16.1088 + 0.2384K + 0.2337H \\ H' &= -14.5451 + 0.7564K + 0.7414H, \end{aligned}$$

   which are very close to the ones obtained from the planning problem (approximation error is the reason they're not identical). Evaluating the individual decision rules at the equilibrium values $k = K$ and $h = H$ yields the aggregate laws of motion.