Homework #4

Mathematical Methods II

Spring 2009

1. This problem studies the consumption/savings problem faced by a consumer who must allocate his savings between a variety of assets with differing risk characteristics. Let $W_t$ be the consumer's wealth at the beginning of period $t$. For simplicity, suppose that there are two assets, asset $A$ and asset $B$. Let $Q_{jt}$, $j = A, B$, denote the amount that the consumer invests in asset $j$ in period $t$. The consumer's wealth evolves according to the following equation:

$$W_{t+1} = R_{A,t+1}Q_{At} + R_{B,t+1}Q_{Bt}$$

where $R_{j,t+1}$ is the gross rate of return on asset $j$ between periods $t$ and $t+1$. Define $R_t = [R_{At}\ R_{Bt}]'$. Assume that $\{R_t\}_{t=1}^{\infty}$ is a sequence of independently and identically distributed random vectors, each with cumulative distribution function $F$. Although the consumer does not observe the vector of returns $R_{t+1}$ when he makes his period $t$ savings decisions, he does know the cdf which governs realizations of this vector.

Let

$$c_t \equiv W_t - Q_{At} - Q_{Bt}$$

be the consumer's level of consumption in period $t$. The consumer's preferences over streams of consumption $\{c_t\}_{t=0}^{\infty}$ are given by

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

where $0 < \beta < 1$ and $u(c)$ satisfies continuous differentiability, strict increasing monotonicity, and strict concavity. The consumer seeks to solve the following dynamic optimization problem:

$$\max_{\{Q_{At}, Q_{Bt}\}_{t=0}^{\infty}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to the wealth evolution equation, given the cdf $F$ and an initial level of wealth $W_0$.

1. Display the Bellman equation for the problem.
2. Derive the two Euler equations for this problem.

3. Use your answer from part (b) to show that

$$E\left[u'\left(c'\right)\left(R'_B - R'_A\right)\right] = 0$$

and that

$$-u'\left(c\right)\left(Q_A + Q_B\right) + \beta E\left[u'\left(c'\right)W'\right] = 0.$$

4. Let $u\left(c\right) = \log\left(c\right)$. Let asset $A$ be riskless with gross return $R_{At} = R\ \forall t$. Let $R_{Bt} = R + z_t$, where $\{z_t\}$ is an iid sequence of random variables with known cdf. Define

$$s_t \equiv \frac{Q_{At} + Q_{Bt}}{W_t}$$

and

$$\theta_t = \frac{Q_{Bt}}{Q_{At} + Q_{Bt}};$$

$s_t$ is the savings rate in period $t$ and $\theta_t$ is the fraction of savings invested in the risky asset in period $t$. Use the results from part (c) to show that $s_t$ and $\theta_t$ are both constant over time. Determine the constant value of $s_t$ and find an equation that determines the constant value of $\theta_t$.

5. Now suppose that $R = 1$ and that $z_t$ is uniformly distributed on the interval $[-0.2, H]$, where $H \geq 0.2$. (Note that if a random variable is distributed uniformly on the interval $[a, b]$ then its density function is equal to $(b - a)^{-1}$ on this interval.) For what value of $H$ does the fraction of savings invested in the risky asset equal 0.5? What happens to this fraction as $H$ approaches 0.2? Give an intuitive explanation for your answer. Hint: The following fact may prove useful:

$$\int \frac{x}{ax + b}dx = \frac{1}{a}x - \frac{b}{a^2}\log\left(ax + b\right).$$

2. Consider the dynamic program

$$P\left(x\right) = \max_a \left\{x^T Q x + a^T R a + 2a^T W x + \beta E\left[P\left(x'\right)|x\right]\right\}$$

where $Q$, $R$, and $W$ are matrices such that the return function is jointly strictly concave in

$(x, a)$ and

$$x' = Ax + Ba + \epsilon$$
$$\epsilon \sim N(0, \Sigma)$$

for some positive semidefinite matrix $\Sigma$. Both $Q$ and $R$ are symmetric.

1. Prove that the policy function $a = \pi(x)$ is independent of $\Sigma$.
2. Suppose that $\Sigma$ is diagonal. Prove that lifetime utility is decreasing in the variance of each element of $\epsilon$.
3. Prove that the decision rules in part a) are the same as the decision rules in a version of the problem where $x'$ is replaced with $E[x'|x]$. This property is called **certainty equivalence**.