Homework #4

Mathematical Methods II

Spring 2009

1. This problem studies the consumption/savings problem faced by a consumer who must allocate his savings between a variety of assets with differing risk characteristics. Let W_t be the consumer's wealth at the beginning of period t. For simplicity, suppose that there are two assets, asset A and asset B. Let Q_{jt}, $j = A, B$, denote the amount that the consumer invests in asset j in period t. The consumer's wealth evolves according to the following equation:

$$W_{t+1} = R_{A,t+1}Q_{At} + R_{B,t+1}Q_{Bt}$$

where $R_{j,t+1}$ is the gross rate of return on asset j between periods t and $t+1$. Define $R_t = [R_{At}\ R_{Bt}]'$. Assume that $\{R_t\}_{t=1}^{\infty}$ is a sequence of independently and identically distributed random vectors, each with cumulative distribution function F. Although the consumer does not observe the vector of returns R_{t+1} when he makes his period t savings decisions, he does know the cdf which governs realizations of this vector.

Let

$$c_t \equiv W_t - Q_{At} - Q_{Bt}$$

be the consumer's level of consumption in period t. The consumer's preferences over streams of consumption $\{c_t\}_{t=0}^{\infty}$ are given by

$$E_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right]$$

where $0 < \beta < 1$ and $u(c)$ satisfies continuous differentiability, strict increasing monotonicity, and strict concavity. The consumer seeks to solve the following dynamic optimization problem:

$$\max_{\{Q_{At},Q_{Bt}\}_{t=0}^{\infty}} E_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right]$$

subject to the wealth evolution equation, given the cdf F and an initial level of wealth W_0.

   1. Display the Bellman equation for the problem.

The Bellman equation is

$$v(W) = \max_{Q_A, Q_B} \left\{ u(W - Q_A - Q_B) + \beta E \left[v\left(R'_A Q_A + R'_B Q_B \right) \right] \right\}.$$

2. Derive the two Euler equations for this problem.

   The first-order conditions for the choice of assets are

$$\begin{aligned} -u'(W - Q_A - Q_B) + \beta E \left[v'\left(R'_A Q_A + R'_B Q_B \right) R'_A \right] &= 0 \\ -u'(W - Q_A - Q_B) + \beta E \left[v'\left(R'_A Q_A + R'_B Q_B \right) R'_B \right] &= 0. \end{aligned}$$

   The envelope condition is

$$v'(W) = u'(W - Q_A - Q_B).$$

   The resulting Euler equations are

$$\begin{aligned} -u'(W - Q_A - Q_B) + \beta E \left[u'\left(R'_A Q_A + R'_B Q_B - Q'_A - Q'_B \right) R'_A \right] &= 0 \\ -u'(W - Q_A - Q_B) + \beta E \left[u'\left(R'_A Q_A + R'_B Q_B - Q'_A - Q'_B \right) R'_B \right] &= 0. \end{aligned}$$

3. Use your answer from part (b) to show that

$$E \left[u'\left(c' \right) \left(R'_B - R'_A \right) \right] = 0$$

   and that

$$-u'(c)(Q_A + Q_B) + \beta E \left[u'\left(c' \right) W' \right] = 0.$$

   Setting both Euler equations equal to each other yields

$$-u'(c) + \beta E \left[u'\left(c' \right) R'_A \right] = -u'\left(c' \right) + \beta E \left[u'\left(c' \right) R'_B \right]$$

   which implies

$$E \left[u'\left(c' \right) \left(R'_B - R'_A \right) \right] = 0.$$

Multiplying the Euler equations by Q_A and Q_B respectively yields

$$\begin{aligned}
-u'(c)\,Q_A + \beta E\left[u'\left(c'\right)R'_A Q_A\right] &= 0 \\
-u'(c)\,Q_B + \beta E\left[u'\left(c'\right)R'_B Q_B\right] &= 0.
\end{aligned}$$

Adding these together and using

$$W' = R'_A Q_A + R'_B Q_B$$

yields

$$-u'(c)\,(Q_A + Q_B) + \beta E\left[u'\left(c'\right)W'\right] = 0.$$

4. Let $u(c) = \log(c)$. Let asset A be riskless with gross return $R_{At} = R\ \forall t$. Let $R_{Bt} = R + z_t$, where $\{z_t\}$ is an iid sequence of random variables with known cdf. Define

$$s_t \equiv \frac{Q_{At} + Q_{Bt}}{W_t}$$

and

$$\theta_t = \frac{Q_{Bt}}{Q_{At} + Q_{Bt}};$$

s_t is the savings rate in period t and θ_t is the fraction of savings invested in the risky asset in period t. Use the results from part (c) to show that s_t and θ_t are both constant over time. Determine the constant value of s_t and find an equation that determines the constant value of θ_t.

Let

$$\begin{aligned}
W_{t+1} &= RQ_{At} + RQ_{Bt} + z_{t+1}Q_{Bt} \\
c_t &= W_t - Q_{At} - Q_{Bt} \\
c_{t+1} &= (R-1)\,Q_{At} + (R-1)\,Q_{Bt} + z_{t+1}Q_{Bt}.
\end{aligned}$$

From part (c) we have

$$\begin{aligned} E_t\left[\frac{z_{t+1}}{(R-1)\,Q_{At}+(R-1)\,Q_{Bt}+z_{t+1}Q_{Bt}}\right] &= 0 \\ \frac{-\left(Q_{At}+Q_{Bt}\right)}{W_t-Q_{At}-Q_{Bt}}+\beta E_t\left[\frac{RQ_{At}+RQ_{Bt}+z_{t+1}Q_{Bt}}{(R-1)\,Q_{At}+(R-1)\,Q_{Bt}+z_{t+1}Q_{Bt}}\right] &= 0. \end{aligned}$$

Note that

$$W_{t+1}=R\left(1-\theta_t\right)s_tW_t+\left(R+z_{t+1}\right)\theta_ts_tW_t.$$

We can rewrite the problem as

$$\max_{\{S_t,\theta_t\}_{t=0}^{\infty}} E_0\sum_{t=0}^{\infty}\beta^t\log\left(W_t-S_t\right)$$

subject to

$$W_{t+1}=R\left(1-\theta_t\right)S_t+\left(R+z_{t+1}\right)\theta_tS_t$$

where $S_t=s_tW_t$. The Euler equations for this problem are

$$\begin{aligned} -\frac{1}{W_t-S_t}+\beta E_t\left[\frac{R+z_{t+1}\theta_t}{W_{t+1}-S_{t+1}}\right] &= 0 \\ \beta E_t\left[\frac{S_tz_{t+1}}{W_{t+1}-S_{t+1}}\right] &= 0. \end{aligned}$$

We guess that

$$S_t=\pi W_t$$

where π is undetermined. This implies

$$E_t\left[\frac{S_tz_{t+1}}{W_{t+1}-\pi W_t}\right]=\frac{1}{1-\pi}E_t\left[\frac{S_tz_{t+1}}{W_{t+1}}\right]=0$$

or

$$E_t\left[\frac{z_{t+1}}{R+z_{t+1}\theta_t}\right]=0.$$

This equation determines $\theta_t=\theta^*\ \forall t$, since z_t is iid. We must now verify that $S_t=\pi W_t$. Inserting this into the other Euler equation yields

$$-\frac{1}{W_t-\pi W_t}+\beta E_t\left[\frac{R+z_{t+1}\theta^*}{W_{t+1}-\pi W_{t+1}}\right]=0$$

or

$$-\frac{1}{W_t} + \beta E_t \left[\frac{R + z_{t+1}\theta^*}{W_{t+1}} \right] = 0.$$

Using the definition of W_{t+1} we arrive at

$$-1 + \frac{\beta}{\pi} E_t \left[\frac{R + z_{t+1}\theta^*}{R + z_{t+1}\theta^*} \right] = 0$$

which implies

$$\pi = \beta.$$

Therefore, the solution is

$$S_t = \beta W_t$$

or

$$s_t = \beta.$$

5. Now suppose that $R = 1$ and that z_t is uniformly distributed on the interval $[-0.2, H]$, where $H \geq 0.2$. (Note that if a random variable is distributed uniformly on the interval $[a, b]$ then its density function is equal to $(b - a)^{-1}$ on this interval.) For what value of H does the fraction of savings invested in the risky asset equal 0.5? What happens to this fraction as H approaches 0.2? Give an intuitive explanation for your answer. Hint: The following fact may prove useful:

$$\int \frac{x}{ax + b} dx = \frac{1}{a} x - \frac{b}{a^2} \log(ax + b).$$

Start with the Euler relation

$$E_t \left[\frac{z_{t+1}}{R + z_{t+1}\theta^*} \right] = 0.$$

With $R = 1$ this implies

$$E_t \left[\frac{z_{t+1}}{1 + z_{t+1}\theta^*} \right] = 0.$$

Computing the expectation yields

$$
\begin{aligned}
\int_{-0.2}^{H} \frac{z_{t+1}}{1+z_{t+1}\theta^*} \frac{1}{H+0.2} dz_{t+1} &= 0 \\
\frac{1}{H+0.2} \left[\frac{z_{t+1}}{\theta^*} - \frac{1}{(\theta^*)^2} \log\left(1+\theta^* z_{t+1}\right) \right]_{-0.2}^{H} &= 0.
\end{aligned}
$$

Evaluating this expression yields the equation

$$
H + 0.2 - \frac{1}{\theta^*} \left[\log\left(1+\theta^* H\right) - \log\left(1-0.2\theta^*\right) \right] = 0.
$$

We can solve this numerically using `fzero`; for $\theta^* = 0.5$ we find $H = 0.2143$. As $H \to 0.2$ $\theta^* \to 0$. Intuitively, as $H \to 0.2$ the risky asset approaches a mean preserving spread of the riskless asset. With concavity of the utility function, the agent will prefer the riskless asset.

2. Consider the dynamic program

$$
P(x) = \max_a \left\{ x^T Q x + a^T R a + 2a^T W x + \beta E\left[P\left(x'\right) | x \right] \right\}
$$

where Q, R, and W are matrices such that the return function is jointly strictly concave in (x, a) and

$$
\begin{aligned}
x' &= Ax + Ba + \epsilon \\
\epsilon &\sim N(0, \Sigma)
\end{aligned}
$$

for some positive semidefinite matrix Σ. Both Q and R are symmetric.

1. Prove that the policy function $a = \pi(x)$ is independent of Σ.

   Since the return function is quadratic, existence of the value function can be obtained by setting $\phi(x) = 1$ and assuming $\beta < 1$. Then monotonicity and discounting constants are immediate: if P_1 and P_2 are such that $P_1(x) \geq P_2(x)$ holds $\forall x$, then

$$
\begin{aligned}
(TP_1)(x) &= \max_a \left\{ x^T Q x + a^T R a + 2a^T W x + \beta E\left[P_1\left(x'\right) | x \right] \right\} \\
&\geq \max_a \left\{ x^T Q x + a^T R a + 2a^T W x + \beta E\left[P_2\left(x'\right) | x \right] \right\} \\
&= (TP_2)(x)
\end{aligned}
$$

and

$$\begin{aligned}
(T(P+Z))(x) &= \max_a \left\{x^T Qx + a^T Ra + 2a^T Wx + \beta E\left[P\left(x'\right) + Z|x\right]\right\} \\
&= \max_a \left\{x^T Qx + a^T Ra + 2a^T Wx + \beta E\left[P_1\left(x'\right)|x\right]\right\} + \beta Z \\
&= (TP)(x) + \beta Z.
\end{aligned}$$

Therefore, the value function is the unique solution to the Bellman equation. Guessing that

$$P(x) = x^T Px + d$$

we obtain

$$\begin{aligned}
x^T Px + d &= \max_a \left\{x^T Qx + a^T Ra + 2a^T Wx + \beta E\left[\left(x^T A^T + a^T B^T + \epsilon^T\right) P (Ax + Ba + \epsilon) + d|x\right]\right\} \\
&= \max_a \left\{ \begin{array}{c} x^T Qx + a^T Ra + 2a^T Wx + \\ \beta E \left[\begin{array}{c} x^T A^T PAx + x^T A^T PBa + x^T A^T P\epsilon + \\ a^T B^T PAx + a^T B^T PBa + a^T B^T P\epsilon + \epsilon^T PAx + \epsilon^T PBa + \epsilon^T P\epsilon + d|x \end{array} \right] \end{array} \right. \\
&= \max_a \left\{ \begin{array}{c} x^T Qx + a^T Ra + 2a^T Wx + \beta x^T A^T PAx + \\ \beta x^T A^T PBa + \beta a^T B^T PAx + \beta a^T B^T PBa \end{array} \right\} + \beta E\left[\epsilon^T P\epsilon\right] + \beta d \\
&= \max_a \left\{ \begin{array}{c} x^T Qx + a^T Ra + 2a^T Wx + \beta x^T A^T PAx + \\ \beta x^T A^T PBa + \beta a^T B^T PAx + \beta a^T B^T PBa \end{array} \right\} + \beta \operatorname{trace}(P\Sigma) + \beta d.
\end{aligned}$$

The Bellman operator preserves concavity, so $x^T Px$ must be concave, meaning that P is negative definite. The first-order conditions are therefore necessary and sufficient:

$$\left(R + \beta B^T PB\right) a + \left(W + \beta B^T PA\right) x = 0.$$

The optimal action is

$$a^* = \pi(x) = -\left(R + \beta B^T PB\right)^{-1} \left(W + \beta B^T PA\right) x.$$

Substituting into the problem yields

$$\begin{aligned}
x^T P x + d &= x^T Q x + x^T \left(W^T + \beta A^T P B\right) \left(R + \beta B^T P B\right)^{-1} R \left(R + \beta B^T P B\right)^{-1} \left(W + \beta B^T P A\right) x - \\
&\quad 2 x^T \left(W^T + \beta A^T P B\right) \left(R + \beta B^T P B\right)^{-1} W x + \\
&\quad \beta x^T A^T P A x - \beta x^T A^T P B \left(R + \beta B^T P B\right)^{-1} \left(W + \beta B^T P A\right) x - \\
&\quad \beta x^T \left(W^T + \beta A^T P B\right) \left(R + \beta B^T P B\right)^{-1} B^T P A x + \\
&\quad \beta x^T \left(W^T + \beta A^T P B\right) \left(R + \beta B^T P B\right)^{-1} B^T P B \left(R + \beta B^T P B\right)^{-1} \left(W + \beta B^T P A\right) x + \\
&\quad \beta \operatorname{trace}\left(P\Sigma\right) + \beta d.
\end{aligned}$$

Matching matrices yields that P satisfies

$$\begin{aligned}
P &= Q + \left(W^T + \beta A^T P B\right) \left(R + \beta B^T P B\right)^{-1} R \left(R + \beta B^T P B\right)^{-1} \left(W + \beta B^T P A\right) - \\
&\quad 2 \left(W^T + \beta A^T P B\right) \left(R + \beta B^T P B\right)^{-1} W + \\
&\quad \beta A^T P A - \beta A^T P B \left(R + \beta B^T P B\right)^{-1} \left(W + \beta B^T P A\right) - \\
&\quad \beta \left(W^T + \beta A^T P B\right) \left(R + \beta B^T P B\right)^{-1} B^T P A + \\
&\quad \beta \left(W^T + \beta A^T P B\right) \left(R + \beta B^T P B\right)^{-1} B^T P B \left(R + \beta B^T P B\right)^{-1} \left(W + \beta B^T P A\right),
\end{aligned}$$

which is independent of Σ; therefore $\pi(x)$ is independent of Σ as well.

2. Suppose that Σ is diagonal. Prove that lifetime utility is decreasing in the variance of each element of ϵ.

   The constant term in the value function satisfies

$$d = \beta \operatorname{trace}\left(P\Sigma\right) + \beta d$$

   so that

$$d = \frac{\beta}{1-\beta} \operatorname{trace}\left(P\Sigma\right).$$

   P is negative definite and Σ is positive semidefinite, so $\operatorname{trace}(P\Sigma) < 0$. Let Σ_1 and Σ_2 be diagonal with $\Sigma_1(j,j) > \Sigma_2(j,j)$; that is, the variance of the jth element of ϵ is strictly higher under Σ_1 than Σ_2. Then the trace of $\operatorname{trace}(P\Sigma_1) < \operatorname{trace}(P\Sigma_2)$, since

$$\operatorname{trace}\left(P\Sigma\right) = \sum_i P(i,i)\,\Sigma(i,i)$$

whenever Σ is diagonal and $P(i,i) < 0$. Therefore, $P(x)$ is decreasing in the diagonal elements of Σ.

3. Prove that the decision rules in part a) are the same as the decision rules in a version of the problem where x' is replaced with $E[x'|x]$. This property is called **certainty equivalence**.

   The value function will again be quadratic, so the new problem is

$$\begin{aligned} x^T P x + d &= \max_a \left\{ \begin{array}{c} x^T Q x + a^T R a + 2a^T W x + \\ \beta \left(x^T A^T + a^T B^T + E\left[\epsilon^T | x\right] \right) P \left(Ax + Ba + E\left[\epsilon | x\right] \right) + \beta d \end{array} \right\} \\ &= \max_a \left\{ \begin{array}{c} x^T Q x + a^T R a + 2a^T W x + \beta x^T A^T P A x + \\ \beta x^T A^T P B a + \beta a^T B^T P A x + \beta a^T B^T P B a + \beta d \end{array} \right\}. \end{aligned}$$

   The first-order conditions are identical to the ones in part a), and so is the matrix equation that defines P. The only difference is that $d = 0$ is the constant term here.