Homework #2
Mathematical Methods II
Spring 2009

1. Suppose that we have the quadratic dynamic programming problem

$$x^T P x = \max_a \left\{ x^T Q x + a^T R a + x^T W a + a^T S x + \beta \left(x^T A^T + a^T B^T \right) P \left(Ax + Ba \right) \right\}$$

where x is an $n \times 1$ vector (with 1 as the first element) and a is a $k \times 1$ vector. The matrices are conformable to these vectors. Write a Matlab program that iterates on a functional equation in order to solve uniquely for P and the associated policy function $a^* = Fx$.

The first-order conditions for the problem yield

$$Ra + R^T a + W^T x + Sx + \beta B^T P^T Ax + \beta B^T PAx + \beta B^T P^T Ba + \beta B^T PBa = 0.$$

Solving for a yields

$$\begin{aligned} a^* &= -\left(R + R^T + \beta B^T \left(P + P^T \right) B \right)^{-1} \left(W^T + S + \beta B^T P^T A + \beta B^T PA \right) x \\ &= Fx. \end{aligned}$$

Substitution into the Bellman equation generates the recursive matrix equation

$$P = Q + F^T R F + W F + F^T S + \beta \left(A^T + F^T B^T \right) P \left(A + BF \right).$$

Your program would then guess P_0 (arbitrary $n \times n$ matrix), set

$$\begin{aligned} F &= -\left(R + R^T + \beta B^T \left(P_0 + P_0^T \right) B \right)^{-1} \left(W^T + S + \beta B^T P_0^T A + \beta B^T P_0 A \right) \\ P_1 &= Q + F^T R F + W F + F^T + \beta \left(A^T + F^T B^T \right) P \left(A + BF \right), \end{aligned}$$

check whether $||P_1 - P_0|| < \epsilon$, terminate if yes, if no set $P_0 = P_1$ and compute a new P_1.

2. Let a firm choose a sequence of labor demands $\{n_t\}_{t=0}^{\infty}$ to maximize the net present value of its stream of profits:

$$\max_{\{n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \left[f_0 n_t - \frac{1}{2} n_t^2 - w n_t - \frac{d}{2} (n_t - n_{t-1})^2 \right]$$

where n_{-1} is given. In the above problem, $f_0 n_t - \frac{1}{2} n_t^2$ is the production of n_t workers, w is their wage, and parameters f_0 and d are positive. The last term captures adjustment costs in the hiring and firing of workers. r is the interest rate at which firms discount the future and is positive.

   (a) Write down the Bellman equation for this model. What are the matrices Q, R, W, A, S, and B? What is β?

The Bellman equation is

$$\left[\begin{array}{cc} 1 & n_{-1} \end{array}\right] P \left[\begin{array}{c} 1 \\ n_{-1} \end{array}\right] = \max_n \left\{ f_0 n - \frac{1}{2}n^2 - wn - \frac{d}{2}\left(n - n_{-1}\right)^2 + \frac{1}{1+r}\left[\begin{array}{cc} 1 & n \end{array}\right] P \left[\begin{array}{c} 1 \\ n \end{array}\right] \right\}.$$

Thus, the matrices are

$$\begin{aligned} Q &= \left[\begin{array}{cc} 0 & 0 \\ 0 & -\frac{d}{2} \end{array}\right] \\ R &= \left[-\frac{1}{2} - \frac{d}{2}\right] \\ S &= \frac{1}{2}\left[\begin{array}{cc} f_0 - w & d \end{array}\right] \\ W &= \frac{1}{2}\left[\begin{array}{c} f_0 - w \\ d \end{array}\right] \\ A &= \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] \\ B &= \left[\begin{array}{c} 0 \\ 1 \end{array}\right]. \end{aligned}$$

Checking equality yields

$$\begin{aligned} & \left[\begin{array}{cc} 1 & n_{-1} \end{array}\right] \left[\begin{array}{cc} 0 & 0 \\ 0 & -\frac{d}{2} \end{array}\right] \left[\begin{array}{c} 1 \\ n_{-1} \end{array}\right] + n\left[-\frac{1}{2} - \frac{d}{2}\right] n + n\frac{1}{2}\left[\begin{array}{cc} f_0 - w & d \end{array}\right] \left[\begin{array}{c} 1 \\ n_{-1} \end{array}\right] + \left[\begin{array}{cc} 1 & n_{-1} \end{array}\right] \\ = & -\frac{d}{2}n_{-1}^2 - \frac{1}{2}n^2 - \frac{d}{2}n^2 + \frac{1}{2}nf_0 - \frac{1}{2}nw + \frac{d}{2}nn_{-1} + \frac{f_0}{2}n - \frac{w}{2}n + \frac{d}{2}n_{-1}n \\ = & f_0 n - \frac{1}{2}n^2 - wn - \frac{d}{2}\left(n - n_{-1}\right)^2. \end{aligned}$$

Finally, $\beta = \frac{1}{1+r}$.

(b) Find the steady state level of employment n. How does this value depend on the parameters r, w, f_0, and d?

The first-order condition for the consumer is

$$f_0 - n - w - dn + dn_{-1} + (1+r)^{-1}v'(n) = 0.$$

The envelope condition for this problem is

$$v'(n) = dn - dn_{-1}$$

which yields an Euler equation

$$(f_0 - w) - (1+d)n + dn_{-1} + (1+r)^{-1}d(n_{+1} - n) = 0.$$

In the steady state we then have

$$n = f_0 - w.$$

The steady-state level of n does not depend on r or d; it depends positively on f_0 and negatively on w.

(c) Use your computer program to compute P given values for the parameters. Set $w = 1$ and $f_0 = 2$. Solve the model for the decision rule n for each of the nine points

$$(r, d) \in \{\infty, 1, 0.1\} \times \{0, 1, 5\}.$$

Verify that the steady state level of employment agrees with your answer in the previous part for each parameter value. How do changes in r and d affect the coefficient on n_{-1} in your computed decision rules?

The table of decision rules can be written down as

$r \backslash d$	0	1	5
∞	1	$0.5000 + 0.5000n_{t-1}$	$0.1667 + 0.8333n_{t-1}$
1	1	$0.5616 + 0.4384n_{t-1}$	$0.2434 + 0.7566n_{t-1}$
0.1	1	$0.6083 + 0.3917n_{t-1}$	$0.3356 + 0.6644n_{t-1}$

.

In each case the steady state level of employment is 1, which coincides with the analytical expression. When $d = 0$ r does not alter the coefficient on n_{t-1}. When $d > 0$, increases in d increase the coefficient on n_{t-1} and so do increases in r.