Homework #1
Mathematical Methods II
Spring 2009
Due: In class Monday, March 30

1. Dynamic programming involves iterating on the Bellman operator

$$(Tw)(x) = \max_{a \in \Gamma(x)} \{r(x,a) + \beta w(t(x,a))\}$$

to convergence. Consider the alternative operator

$$(T_\pi w)(x) = r(x, \pi(x)) + \beta w(t(x, \pi(x))),$$

for some specified policy function $\pi(x)$. Assume that $\beta < 1$ and that r is bounded and continuous.

(a) Prove that T_π is a strict contraction mapping on the space of continuous and bounded functions for any π.

(b) In practice we cannot carry out the Bellman iterations exactly – we must carry out iterations on an approximation to the current w function. Consider approximations L that map functions w into a family of functions F that satisfy two conditions. First, they are **projections**: $L \circ L = L$ (that is, if $w \in F$ then $Lw = w$). Second, they are **nonexpansive**:

$$\|Lv - Lw\|_\infty \leq \|v - w\|_\infty$$

$\forall v, w$. Define

$$\begin{aligned}\left(\widehat{T}w\right)(x) &= [(L \circ T)(w)](x) \\ &= \left[L\left(\max_{a \in \Gamma(x)} \{r(x,a) + \beta w(t(x,a))\}\right)\right](x)\end{aligned}$$

to be the Bellman operator on approximated functions. Let v^* denote the true value function and v_π denote the fixed point of T_π. Show that

$$v^*(x) - v_\pi(x) \leq \frac{2}{(1-\beta)^2}\left(\beta\epsilon + \|v^* - Lv^*\|_\infty\right)$$

for some $\epsilon > 0$.

(c) Interpret the error bound condition.

2. Consider the dynamic program

$$v(x) = \max_a \left\{Qx^2 + Ra^2 + 2Wax + \beta v(x')\right\}$$

with law of motion

$$x' = Ax + Ba.$$

Assume that (Q, R, W) are such that the objective function is strictly increasing and strictly concave over the relevant ranges for (x, a). Both x and a are scalars.

(a) Prove that the value function takes the form

$$v^* (x) = Px^2$$

and the decision rule takes the form

$$a^* (x) = Fx.$$

(b) Does P exist? Is P unique? What is the sign of P?

(c) Assume that $Q = -1$, $R = -2$, and $W = 0$. Let $\beta = 0.96$, $A = 0.5$, and $B = 0.1$. Compute P and F, where

$$a^* (x) = -Fx.$$