# 1 Dynamic Programming

These notes are intended to be a very brief introduction to the tools of dynamic programming. Several mathematical theorems – the Contraction Mapping Theorem (also called the Banach Fixed Point Theorem), the Theorem of the Maximum (or Berge's Maximum Theorem), and Blackwell's Sufficiency Conditions – are referenced but may not be proven or even necessarily stated explicitly. The omission of these theorems is merely a result of time considerations; the notes are intended to be brief enough to cover in one two-hour lecture.

Suppose we wish to solve the following problem:

$$\max_{\{a_t,x_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t r(x_t, a_t) \tag{1}$$

subject to

$$a_t \in \Gamma(x_t) \tag{2}$$

$$x_{t+1} = t(x_t, a_t) \tag{3}$$

$$x_0 \text{ given.} \tag{4}$$

The horizon length, $T$, can be infinite. The reason we need a new method to solve this problem – rather than simply taking Kuhn-Tucker conditions – is that such a problem will have a huge number of decision variables if the horizon is large. If it is in fact infinite we cannot write down enough equations to completely solve our model. Note: the problem would become even more severe in the presence of uncertainty, especially uncertainty with a continuous support. We must attack this problem in a new manner –the method of **dynamic programming**. This approach rewrites the problem into one which has only one choice rather than an infinite number; we begin by discussing the components of a dynamic program and how to set them up.

A deterministic stationary discounted dynamic programming problem consists of five basic objects. These objects are

- The **state space** $X$ (a state $x$ is a set of variables sufficient to tell a household everything needed to make decisions, such as prices and income);
- The **feasible control space** $A = \cup_{x \in X} \Gamma(x)$ and the **feasible control correspondence** $\Gamma : X \to 2^A$ (these are the choices $a$ that are permissible at a given point in time when state $x$ is realized – a budget set is the simplest example of a feasible control correspondence);
- The **return function** $r : X \times A \to \mathcal{R}$ (this is the reward in current terms for taking actions $a$ in state $x$);
- The **transition function** $t : X \times A \to X$ (this determines tomorrow's state $x' = t(x, a)$ when action $a$ is taken in current state $x$);
- The **discount factor** $\beta$.

At this point, an example is instructive – we will choose the deterministic optimal growth model of Cass-Koopmans, which extended the famous Solow model to permit elastic savings rates. In this model, output is produced using capital only – the production technology is given by $f(k_t)$. The representative household or planner chooses sequences of consumption $\{c_t\}_{t=0}^T$ and capital $\{k_{t+1}\}_{t=0}^T$ to maximize lifetime utility

$$\sum_{t=0}^{T} \beta^t u(c_t) \tag{5}$$

subject to the budget constraints

$$c_t + k_{t+1} \leq f(k_t) \tag{6}$$

and a given initial capital stock $k_0$. As a simple exercise, we will formulate all five components of our dynamic programming problem.

1. The state space: Current capital $k_t$ is the only thing the planner needs to know – thus, the state vector $(k_t) \in K = \mathcal{R}_+$ (capital must be nonnegative);

2. The feasible control space: The controls here are $(c_t, k_{t+1})$; they must lie in the set
$$B(k_t) = \{(c_t, k_{t+1}) | c_t + k_{t+1} \leq f(k_t)\}; \tag{7}$$
that is, we cannot use on consumption plus savings more than we create in production; the feasible control correspondence is simply $B(k_t)$;

3. The return function: this is simply the period utility function $u(c_t)$;

4. The transition function: tomorrow's state is the control $k_{t+1}$, so the transition function is trivial;

5. The discount factor: simply $\beta$.

So far we haven't done anything particularly useful. However, I will show that this approach has value by solving a series of problems with progressively longer horizons, showing that the solutions display simple patterns. Using these patterns we will rewrite our problem recursively; that is, we will write it in a way that only depends on the current state and only has a choice for the current control.

Now we will proceed forward by solving this problem for $T = 0$ (a static problem). Setting lifetime utility to zero after death (we are free to normalize death utility in any way we see fit) this problem becomes

$$v_0(k_0) = \max_{k_1} \{u(f(k_0) - k_1)\}. \tag{8}$$

We will impose the condition that $k_1 \geq 0$; that is, capital cannot be negative in the final period. This restriction is necessary for there to exist a solution if $u$

is increasing. Furthermore, we will specialize to the following functional forms because it makes the algebraic solution possible:

$$f(k) = Ak^{\alpha} \tag{9}$$

$$u(c) = \log(c). \tag{10}$$

The solution to the above problem is obviously

$$k_1 = 0 \tag{11}$$

$$c_0 = Ak_0^{\alpha}. \tag{12}$$

The solution is trivial; the planner tells the household to eat everything and then go off to die. The value function is therefore

$$v_0(k_0) = \log(Ak_0^{\alpha}) = \log(A) + \alpha \log(k_0). \tag{13}$$

Lifetime utility depends on the existing stock of capital; endowing an economy with more capital will generate more utility for consumers in this static world.

Now, examine the problem for $T = 1$. This problem is

$$v_1(k_0) = \max_{k_1, k_2} \{\log(Ak_0^{\alpha} - k_1) + \beta \log(Ak_1^{\alpha} - k_2)\} \tag{14}$$

subject to nonnegativity on $k_2$ (but not necessarily on $k_1$). Clearly we wish to set $k_2 = 0$. The first-order condition for $k_1$ is

$$\frac{1}{Ak_0^{\alpha} - k_1} = \frac{\beta A \alpha k_1^{\alpha-1}}{Ak_1^{\alpha}} \Rightarrow k_1 = \frac{\alpha\beta}{1+\alpha\beta} Ak_0^{\alpha}. \tag{15}$$

The value of making these decisions is

$$\begin{aligned} v_1(k_0) &= \log\left(Ak_0^{\alpha} - \frac{\alpha\beta}{1+\alpha\beta}Ak_0^{\alpha}\right) + \beta \log\left(A\left(\frac{\alpha\beta}{1+\alpha\beta}Ak_0^{\alpha}\right)^{\alpha}\right) \\ &= \log\left(\frac{1}{1+\alpha\beta}A\right) + \alpha\log(k_0) + \beta\log\left(A\left(\frac{\alpha\beta}{1+\alpha\beta}A\right)^{\alpha}\right) + \alpha^2\beta\log(k_0). \end{aligned}$$

Once again the value is increasing in the initial capital stock $k_0$; if we give people more capital they will be better off. Notice that

$$\begin{aligned} v_1(k_0) &= \log(A) + \alpha\log(k_0) + \log\left(\frac{1}{1+\alpha\beta}\right) + \beta\log\left(A\left(\frac{\alpha\beta}{1+\alpha\beta}A\right)^{\alpha}\right) + \alpha^2\beta\log(k_0) \\ &= v_0(k_0) + \log\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta\log\left(\frac{\alpha\beta}{1+\alpha\beta}\right) + (1+\alpha)\beta\log(A) + \alpha^2\beta\log(k_0); \end{aligned}$$

that is, the value function for the two-period case is the value function for the static case plus some extra terms. That is,

$$v_1(k_0) = \max_{k_1} \{\log(Ak_0^{\alpha} - k_1) + \beta v_0(k_1)\}.$$

Finally, let's examine the case $T = 2$; this problem is given by

$$v_2(k_0) = \max_{k_1, k_2, k_3} \left\{ \log(Ak_0^\alpha - k_1) + \beta \log(Ak_1^\alpha - k_2) + \beta^2 \log(Ak_2^\alpha - k_3) \right\} \quad (16)$$

subject to nonnegativity on $k_3$. Once again, we set $k_3 = 0$. The first-order conditions for $k_1$ and $k_2$ are given by

$$\frac{1}{Ak_0^\alpha - k_1} = \frac{\alpha\beta A k_1^{\alpha-1}}{Ak_1^\alpha - k_2} \quad (17)$$

$$\frac{\beta}{Ak_1^\alpha - k_2} = \frac{\alpha\beta^2 A k_2^{\alpha-1}}{Ak_2^\alpha}. \quad (18)$$

The solution to these equations is

$$k_1 = \frac{\alpha\beta + (\alpha\beta)^2}{1 + \alpha\beta + (\alpha\beta)^2} A k_0^\alpha \quad (19)$$

$$k_2 = \frac{\alpha\beta}{1 + \alpha\beta} A k_1^\alpha. \quad (20)$$

The value function for this problem is a big mess

$$\begin{aligned}
v_2(k_0) &= \log\left(\left(\frac{1}{1 + \alpha\beta + (\alpha\beta)^2}\right) A k_0^\alpha\right) + \beta \log\left(\left(\frac{1}{1+\alpha\beta}\right)\left(\frac{\alpha\beta + (\alpha\beta)^2}{1 + \alpha\beta + (\alpha\beta)^2}\right)^\alpha A^{1+\alpha} k_0^{\alpha^2}\right) + \\
&\quad \beta^2 \log\left(\left(\frac{\alpha\beta}{1+\alpha\beta}\right)^\alpha \left(\frac{\alpha\beta + (\alpha\beta)^2}{1 + \alpha\beta + (\alpha\beta)^2}\right)^{\alpha^2} A^{1+\alpha+\alpha^2} k_0^{\alpha^3}\right) \\
&= \log(A) + \alpha \log(k_0) + \log\left(\frac{1}{1 + \alpha\beta + (\alpha\beta)^2}\right) + \beta \log\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta \log\left(\frac{\alpha\beta + (\alpha\beta)^2}{1 + \alpha\beta + (\alpha\beta)^2}\right) + \\
&\quad (1+\alpha)\beta \log(A) + \alpha^2\beta \log(k_0) + \alpha\beta^2 \log\left(\frac{\alpha\beta}{1+\alpha\beta}\right) + \alpha^2\beta^2 \log\left(\frac{\alpha\beta + (\alpha\beta)^2}{1 + \alpha\beta + (\alpha\beta)^2}\right) + \\
&\quad \left(1 + \alpha + \alpha^2\right)\beta^2 \log(A) + \alpha^3\beta^2 \log(k_0) \\
&= v_1(k_0) + \log\left(\frac{1}{1 + \alpha\beta + (\alpha\beta)^2}\right) + \alpha\beta \log\left(\frac{\alpha\beta + (\alpha\beta)^2}{1 + \alpha\beta + (\alpha\beta)^2}\right) + \alpha^2\beta^2 \log\left(\frac{\alpha\beta + (\alpha\beta)^2}{1 + \alpha\beta + (\alpha\beta)^2}\right) + \\
&\quad \left(1 + \alpha + \alpha^2\right)\beta^2 \log(A) + \alpha^3\beta^2 \log(k_0)
\end{aligned}$$

and it satisfies

$$v_2(k_0) = \max_{k_1} \left\{ \log(Ak_0^\alpha - k_1) + \beta v_1(k_1) \right\}.$$

The general solution to a problem with horizon $T$ is

$$\begin{aligned}
k_1 &= \frac{\alpha\beta + \cdots (\alpha\beta)^T}{1 + \alpha\beta + \cdots (\alpha\beta)^T} A k_0^\alpha \\
&= \alpha\beta \left[\frac{1 - (\alpha\beta)^T}{1 - (\alpha\beta)^{T+1}}\right] A k_0^\alpha.
\end{aligned}$$

Letting $T \to \infty$ we have the decision rule

$$k_1 = \alpha \beta A k_0^{\alpha}. \tag{21}$$

Note: here we are using the partial sum formula

$$\sum_{t=0}^{T-1} p^t = \frac{1-p^T}{1-p}$$

if $|p| < 1$.

Some things are important to note here. One, examining the decision rules from the problems with horizons of 2 and 3 periods, we see that the only thing that matters is the current capital stock $k_t$. Another important observation is that the decision rules depend not on the current period $t$ but on the number of periods before the end $T - t$; that is, a household makes the same decisions $n$ periods from death no matter how long they have been alive, conditional on current capital. We have then what is known as a **recursive problem**; the state of the world is given by $k_t$ and it is sufficient to determine current behavior. Furthermore, we may drop the time subscripts in the infinite horizon case, as the consumer will always be infinitely far from death:

$$k_{t+1} = \alpha \beta A k_t^{\alpha} \tag{22}$$

or

$$k' = \alpha \beta A k^{\alpha} \tag{23}$$

where primes denote next period values.

What about the values of these problems? It turns out that the value function (function of existing capital) also converges in the infinite horizon case. Although messy, you can show that it converges to a function of the form

$$v(k) = a + b \log(k) \tag{24}$$

where

$$a = \frac{1}{1-\beta}\left[\log(A(1-\alpha\beta)) + \frac{\alpha\beta}{1-\alpha\beta}\log(A\alpha\beta)\right] \tag{25}$$

and

$$b = \frac{\alpha}{1-\alpha\beta}. \tag{26}$$

Note that we have dropped the subscripts on the value function as well; since consumers decisions only depend on the current capital stock and the time until death, only the current capital stock matters.

Solving the problem this way is not very fast when we know the form of the value function. Note that the above value functions imply

$$v_1(k) = \max_{k_1} \left\{\log(c_0) + \beta v_0(k_1)\right\}. \tag{27}$$

What this says is that we can solve our problem turning our utility into the sum of two parts – what we get today and what we get in the future, assuming we make the proper choices tomorrow; we then only need to worry about making the proper choice today. With an infinite horizon we have

$$v(k_t) = \max_{k_{t+1}} \left\{ \log(Ak_t^\alpha - k_{t+1}) + \beta v(k_{t+1}) \right\}. \tag{28}$$

$v(k_t)$ is the lifetime utility from having $k_t$ units of capital. This equation – the celebrated **Bellman equation** named after inventor/discoverer Richard Bellman – gives us a convenient method for solving the problem: if we could somehow know the form of the value function (and that no other function satisfied this equation) we could simply insert it into the above problem, maximize, and be done. If this sounds too good to be true, well it almost is; knowing the form of the value function is generally impossible. For the above case, we could insert a guess of the form

$$v(k) = a + b\log(k) \tag{29}$$

into the Bellman equation and take derivatives:

$$\frac{1}{Ak_t^\alpha - k_{t+1}} = \frac{\beta b}{k_{t+1}}. \tag{30}$$

The solution to this is

$$k_{t+1} = \frac{\beta b}{1+\beta b} Ak_t^\alpha. \tag{31}$$

The only problem is that we don't know $b$. But if we insert our solution into the Bellman equation we get

$$a + b\log(k_t) = \log\left(\frac{1}{1+\beta b} Ak_t^\alpha\right) + \beta a + \beta b \log\left(\frac{\beta b}{1+\beta b} Ak_t^\alpha\right) \tag{32}$$

where the max disappears because it is embedded within the choice for $k_{t+1}$. The important thing here is to note that this must hold for every value of $k_t$; this implies that all the coefficients for the constant terms and $\log(k_t)$ must be the same on each side of the equation. For $\log(k_t)$ this requires

$$b = \alpha + \beta b \alpha \tag{33}$$

or

$$b = \frac{\alpha}{1-\alpha\beta} \tag{34}$$

which is exactly what we got before. Matching up the constants gives us

$$a = \log\left(\frac{A}{1+\beta b}\right) + \beta a + \beta b \log\left(\frac{\beta b A}{1+\beta b}\right) \tag{35}$$

or

$$a = \frac{1}{1-\beta}\left[\log\left(\frac{A}{1+\beta b}\right) + \beta b \log\left(\frac{\beta b A}{1+\beta b}\right)\right]. \tag{36}$$

Inserting our solution for $b$ yields

$$\begin{aligned}
a &= \frac{1}{1-\beta}\left[\log\left(\frac{A}{1+\beta b}\right)+\beta b\log\left(\frac{\beta bA}{1+\beta b}\right)\right] \\
&= \frac{1}{1-\beta}\left[\log\left(\frac{A}{1+\beta\frac{\alpha}{1-\alpha\beta}}\right)+\beta\frac{\alpha}{1-\alpha\beta}\log\left(\frac{\beta\frac{\alpha}{1-\alpha\beta}A}{1+\beta\frac{\alpha}{1-\alpha\beta}}\right)\right] \\
&= \frac{1}{1-\beta}\left[\log((1-\alpha\beta)A)+\frac{\alpha\beta}{1-\alpha\beta}\log\left(\alpha\beta A\right)\right].
\end{aligned}$$

We have completely solved the consumer's problem now; with the given solution for $b$ optimal capital accumulation is given by

$$k_{t+1} = \alpha\beta A k_t^{\alpha}, \tag{37}$$

exactly what we got before. The only problem with this method is that there are a very small number of economic problems where we know the form of the value function; of these, some are impossible to solve for the coefficients analytically and others are simply not very interesting from an economic standpoint as they involve odd choices for parameters.

What if we don't know the form of the value function (either because we don't know it or it simply does not exist in closed-form)? Go back to the general Bellman equation

$$v_t(k_t) = \max_{k_{t+1}}\{u(f(k_t)-k_{t+1})+\beta v_{t+1}(k_{t+1})\}. \tag{38}$$

In the above problem we know that the value function is constant over time but we don't know its form. Can we still use this equation? In fact, we can and will. The key is the recursive nature of the value function. Imagine if we guessed that the value in period $T+1$ was zero, as we did solving the problem the first way. Then the Bellman equation would imply that

$$v_T(k_T) = u(f(k_T)). \tag{39}$$

But now we know by Bellman recursion that

$$v_{T-1}(k_{T-1}) = \max_{k_T}\{u(f(k_{T-1})-k_T)+\beta v_T(k_T)\}. \tag{40}$$

That is, we update our guess $v_T$ by replacing it with $v_{T-1}$ after solving for $k_T$ as a function of $k_{T-1}$. That is,

$$v_{T-1}\left(k_{T-1}\right) = u\left(f\left(k_{T-1}\right)-k_T\left(k_{T-1}\right)\right)+\beta v_T\left(k_T\left(k_{T-1}\right)\right)$$

Then, according to the Bellman equation we must now have that

$$v_{T-2}(k_{T-2}) = \max_{k_{T-1}}\left\{u(f(k_{T-2})-k_{T-1})+\beta v_{T-1}(k_{T-1})\right\} \tag{41}$$

and so on. If we let $T \to \infty$ we would hope to eventually get close to the value function of the infinite horizon problem

$$v(k) = \max_{k'} \{u(f(k) - k') + \beta v(k')\} \tag{42}$$

under certain conditions. What we will attempt to do in the next sections is prove that this algorithm will converge to the true value function, that this value function exists, and that we can say something about how it "looks." In effect, we will be searching for the "certain conditions" that make this the appropriate way to proceed.

Take the Bellman equation

$$v(x) = \max_{a}\{r(x, a) + \beta v(x') : a \in \Gamma(x), x' = t(x, a)\}. \tag{43}$$

The function $v(x)$ is the unknown of this equation. We know that inserting a function into the right-hand-side for $v(\cdot)$ and performing the maximization gives us a new function for the left-hand-side; furthermore, these functions are not necessarily the same. Let our generic guess be given by $w$ so as not to confuse it with the true value function $v$, which may not exist and which we certainly do not know. We can view the Bellman equation as mapping functions into functions, a functional operator. Calling this thing $T$, we have the operator $T$ takes a function $w : X \to \mathcal{R}$ and "turns it into" a function $Tw : X \to \mathcal{R}$ via the process

$$(Tw)(x) = \max_{a}\{r(x, a) + \beta w(x') : a \in \Gamma(x), x' = t(x, a)\}. \tag{44}$$

Lest this all seem very strange, I can point out that differentiation and integration perform the exact same tasks. For example, let $D$ be the differentiation operator. Then $Dx^2 = 2x$, a new function which is related to the old one via the operator $D$. The Bellman operator works exactly the same way. We know that, if the true value function exists, it satisfies Equation 1. That is, if we feed the Bellman operator $v$ we get $v$ back; $(Tv)(x) = v(x)$. In other words, the value function is a fixed point of the $T$ operator in the space of functions.

For example, one fixed point for the differentiation operator is the zero function:

$$D0 = 0. \tag{45}$$

Another one is $e^x$:

$$De^x = e^x. \tag{46}$$

If we could somehow prove that the Bellman operator had a fixed point, we would definitely know that the value function existed. If we could further prove that it only had one fixed point, then we would know that the value function was that fixed point; otherwise, if there were more than one we would need to find the best fixed point. That is our first task.

Finally, one point of terminology. Assuming that we have enough structure that solutions to the Bellman equation exist, there will be (at least) one action for each value of the state that is optimal. Denote this mapping the **policy function**:

$$a^* = \pi(x). \tag{47}$$

Policy functions are like demand functions – they tell you what to do in the event that the state becomes $x$; note that, in the infinite horizon case, policy functions are invariant (they are used in every period).

We will need the notion of a contraction mapping. I assume that you have already been exposed to the ideas of a Banach space (a complete normed vector space) and a uniformly continuous mapping.

**Definition 1** *A mapping $T$ from a metric space $(\Xi, \rho)$ into itself is a* ***strict contraction map*** *if $\exists \theta \in (0,1)$ such that $\rho(Tf, Tg) \leq \theta\rho(f,g)\ \forall f, g \in \Xi$.*

Basically, a contraction brings objects closer together with a modulus $\theta$. First, an important property of strict contraction maps.

**Theorem 2** *If $T : (\Xi, \rho) \to (\Xi, \rho)$ is a strict contraction map then $T$ is uniformly continuous.*

**Proof.** If $T$ is a contraction, then for some $\theta \in (0,1)$ we have

$$\frac{|Tx - Ty|}{|x - y|} \leq \theta < 1$$

$\forall x, y \in S$. Let $\delta \equiv \frac{\epsilon}{\theta}$; then for any $\epsilon > 0$, if $|x - y| < \delta$ we have

$$|Tx - Ty| \leq \theta\,|x - y| < \theta\delta = \epsilon.$$

Thus, $T$ is uniformly continuous. ■

We now prove the key theorem in this section.

**Theorem 3** *(**Contraction Mapping Theorem**) A strict contraction map on a Banach space has a unique fixed point. Furthermore, the sequence $\{f, Tf, T^2f, ...\}$ converges to that unique fixed point.*

**Proof.** Let $(\Xi, \rho)$ be a complete metric space and $T : (\Xi, \rho) \to (\Xi, \rho)$ be a strict contraction map. For any $f \in \Xi$ define $f^n = T^n f$. Since $T$ is a strict contraction map, there is $\theta < 1$ such that if $n \geq m$ we have

$$\rho(f^n, f^m) \leq \theta^m \rho(f^{n-m}, f). \tag{48}$$

This result is obtained by using the contraction property $m$ times. Using the triangle inequality for metrics we must have

$$\rho(f^{n-m}, f) \leq \rho(f^{n-m}, f^{n-m-1}) + \cdots + \rho(f^1, f). \tag{49}$$

We also know that

$$\rho(f^{n-m}, f^{n-m-1}) \leq \theta^{n-m-1}\rho(f^1, f) \tag{50}$$

and

$$\rho(f^{n-m-1}, f^{n-m-2}) \leq \theta^{n-m-2}\rho(f^1, f) \tag{51}$$

and so on. Putting these together we get that

$$\begin{aligned}\rho(f^{n-m}, f) &\leq \rho(f^{n-m}, f^{n-m-1}) + \rho(f^{n-m-1}, f^{n-m-2}) + \cdots \rho(f^1, f) \\ &\leq \rho(f^1, f) \times (\theta^{n-m-1} + \theta^{n-m-2} + \cdots \theta + 1).\end{aligned}$$

We know that

$$\theta^{n-m-1} + \theta^{n-m-2} + \cdots \theta + 1 = \frac{1-\theta^{n-m}}{1-\theta} \leq \frac{1}{1-\theta}. \tag{52}$$

Combining all our inequalities yields

$$\rho(f^n, f^m) \leq \frac{\theta^m}{1-\theta}\rho(f^1, f). \tag{53}$$

The case of $m \geq n$ is similar. Therefore, we must have

$$\rho(f^m, f^n) \to 0 \tag{54}$$

or $\{f^n\}$ is Cauchy. Since $\Xi$ is complete, this sequence has a limit point $f^* \in \Xi$. We simply need to show that $f^*$ is a fixed point of $T$. With strict contraction maps being uniformly continuous, it follows that

$$f^* = \lim_{n\to\infty} T^n f = T\left(\lim_{n\to\infty} T^{n-1} f\right) = Tf^*. \tag{55}$$

Thus we have a fixed point. This fixed point must be unique, since if $g^*$ were another fixed point we must have

$$\begin{aligned}\rho(f^*, g^*) &= \rho(Tf^*, Tg^*) \\ &\leq \theta\rho(f^*, g^*).\end{aligned}$$

With $\theta < 1$ we must have $f^* = g^*$. ∎

The CMT proves that a sequence generated by a contraction map converges to a limit point independent of the initial condition for that sequence. A weaker condition can be applied to monotone sequences – they converge whenever they are bounded.

Our goal is to find conditions under which the Bellman operator is a strict contraction on a Banach space. We then have achieved a large part of our goal; we will have proven that the value function exists, is the unique solution to the Bellman equation, and that iterating on the Bellman operator will converge to the value function from any initial guess. All that would be left would be to examine the behavior of that limiting object and ensure that we have solved our initial problem; that is, prove that any solution to the Bellman equation solves the sequential version of the household problem. For some problems, we will not be able to prove that the Bellman operator is a contraction; in that case, if we can prove that it is monotone we will know that it possesses at least one fixed point. For this class of problems, we know that iterating from something strictly higher than the value function will converge to the best fixed point.

We now examine the conditions under which the Bellman operator will be a contraction. Suppose $\phi$ is a function in the space of continuous functions $\Xi$ that satisfies $\phi(x) > 0\ \forall x \in X$. Let's call the following the "$\phi$-norm" of a function:

$$\|f\|_\phi = \sup_x \left\{ \frac{|f(x)|}{\phi(x)} : x \in X \right\}. \tag{56}$$

That is, deflate the value of $f$ by the positive function $\phi$; formally, $||\cdot||_\phi$ is a member of the class of **weighted norms**. Denote by $\Xi_\phi$ the subset of $\Xi$ for which this norm is finite. This subset equipped with the $\phi$-norm $\left(\Xi_\phi, ||\cdot||_\phi\right)$ is a Banach space. We can prove the following sufficiency conditions for a contraction mapping easily:

**Theorem 4** ***(Blackwell-Boyd Sufficiency Conditions)*** *Let $T$ be a mapping from $\left(\Xi_\phi, \|\cdot\|_\phi\right) \to \left(\Xi_\phi, ||\cdot||_\phi\right)$ such that (a) $T$ is* ***monotone*** *– $f, g \in \Xi$ and $f \geq g$ implies $Tf \geq Tg$ – and (b) $T$* ***discounts constant functions*** *– $\forall f \in \Xi$ and any constant function $A$, $T(f + A\phi) \leq Tf + \theta A\phi$ for some $\theta < 1$. Then $T$ is a strict contraction map.*

We now show that the Bellman operator is a contraction provided we make some reasonable assumptions.

**Theorem 5** *Assume the one-period return function $r$ and the transition function $t$ are continuous on $X \times A$; the feasible actions correspondence $\Gamma$ is continuous and compact-valued; and there is a function $\phi \in \Xi$ with $\phi(x) > 0$ such that (i) there exists an $M$ with*

$$\max_{a \in \Gamma(x)} \{r(x, a)\} \leq M\phi(x) \tag{57}$$

*and (ii) there exists $\theta < 1$ such that*

$$\beta \max_{a \in \Gamma(x)} \{\phi(t(x, a))\} \leq \theta\phi(x) \tag{58}$$

*$\forall x \in X$. Under these conditions the Bellman operator is a strict contraction map.*

Because both $r$ and $w \in \Xi_\phi$ are continuous functions and $\Gamma(x)$ is compact-valued and continuous, the maximization is well-defined for each $x$ (by application of the maximum theorem). It is then easy to see that $T : \Xi_\phi \to \Xi_\phi$; that is, if a function $w$ is bounded by $\phi$ then $Tw$ will also be bounded by $\phi$, which follows from (i) of the condition (which also incidentally proves that $T0 \in \Xi_\phi$). Let $v(x) \geq w(x)\ \forall x \in X$. Then

$$\begin{aligned}
(Tv)(x) &= \max_{a \in \Gamma(x)} \{r(x, a) + \beta v(t(x, a))\} \\
&\geq \max_{a \in \Gamma(x)} \{r(x, a) + \beta w(t(x, a))\} \\
&= (Tw)(x).
\end{aligned}$$

Monotonicity is proven. Let $A$ be a constant function and let $w \in \Xi_\phi$. We have

$$\begin{aligned}(T(w+A\phi))(x) &= \max_{a\in\Gamma(x)}\{r(x,a)+\beta w(t(x,a))+\beta A\phi(t(x,a))\} \\ &\leq \max_{a\in\Gamma(x)}\{r(x,a)+\beta w(t(x,a))\}+\beta A\max_{a\in\Gamma(x)}\{\phi(t(x,a))\} \\ &\leq (Tw)(x)+\theta A\phi(x).\end{aligned}$$

Discounting constant functions is proven if (ii) holds. We have shown that the Bellman operator is a contraction under the stated assumptions, which are almost entirely assumptions about the primitives of the model (except for the existence of $\phi$, but we will turn to this in a moment).

We need only make one further step to show that there exist optimal paths that solve the sequence problem; after all, these are really the objects we are after. It may be the case that the value function is unattainable, that the max should be replaced by a sup. We are not interested in these types of problems – they do not solve the sequence problem we defined at the beginning of this section. To this end, note that the value function must satisfy

$$v(x) \geq \sum_{t=0}^{\tau}\beta^t r(x_t,a_t)+\beta^\tau v(x_{\tau+1}) \tag{59}$$

for any $\tau \geq 0$ and the optimal sequence of actions. To match up with our previous problem in sequence form, we need only show that the "size" of the last term goes to zero as $\tau \to \infty$. But we know that

$$|v(x_{\tau+1})| = \frac{|v(x_{\tau+1})|}{\phi(x_{\tau+1})}\frac{\phi(x_{\tau+1})}{\phi(x_\tau)}\cdots\frac{\phi(x_1)}{\phi(x)}\phi(x) \tag{60}$$

since $\phi$ is always positive. Furthermore, we have

$$\frac{|v(x_{\tau+1})|}{\phi(x_{\tau+1})} \leq \|v\|_\phi < \infty \tag{61}$$

because $v$ is $\phi$-bounded. We also have, from our condition, that

$$\frac{\phi(x_{s+1})}{\phi(x_s)} = \frac{\phi(t(x_s,a_s))}{\phi(x_s)} \leq \frac{\theta}{\beta}. \tag{62}$$

Thus, we have

$$|v(x_{\tau+1})| \leq \left(\frac{\theta}{\beta}\right)^\tau \|v\|_\phi\,\phi(x) \tag{63}$$

or

$$\beta^\tau|v(x_{\tau+1})| \leq \theta^\tau\|v\|_\phi\,\phi(x). \tag{64}$$

The right hand side converges to zero because $\theta < 1$; therefore we have

$$\beta^\tau|v(x_{\tau+1})| \to 0. \tag{65}$$

We have thus proven that the value function exists; that it can be obtained by iterating on the Bellman operator; and that optimal paths generated by the policy function solve the sequence problem:

$$\{x_t^{seq}\} = \{x_0, \pi(x_0), \pi(\pi(x_0)), ...\}.$$

This last property implies that once we are on an optimal path it is optimal to remain there, so that "plans" announced at time $t$ for time $t+k$ are carried out at $t+k$. This is called the **Principle of Optimality**.

Returning to our example, we will use the one-sector growth model to demonstrate how to employ this machinery. Let $u : \mathcal{R}_+ \to \mathcal{R}$ be increasing, continuous, and bounded below; since $u$ is bounded below we can set $u(0) = 0$ without loss of generality. Let $f : \mathcal{R}_+ \to \mathcal{R}$ be continuous and increasing with $f(0) \geq 0$. All of the continuity requirements for our condition are met. We need only find $\phi$, the function which bounds the value function.

Since both $u$ and $f$ are bounded below by 0, let's choose the function

$$\phi(k) = 1 + u(f(k)). \tag{66}$$

$\phi$ is increasing, continuous, and strictly positive. (i) is satisfied since

$$\max_{c \in \Gamma(k)} \{u(c)\} \leq M\phi(k) \tag{67}$$

because $c \leq f(k)$. Requirement (ii) will then be satisfied if

$$\beta \sup_{k \geq 0} \left\{ \frac{\phi(f(k))}{\phi(k)} \right\} < 1. \tag{68}$$

We can check this for certain parametric economies. For example, take the following functional forms:

$$u(c) = \frac{c^\gamma}{\gamma} \tag{69}$$

for $\gamma \in (0, 1)$ and

$$f(k) = Ak \tag{70}$$

with $A \geq 1$. We check the condition for (ii):

$$\begin{aligned} \beta \sup_{k \geq 0} \left\{ \frac{1 + [(f \circ f)(k)]^\gamma / \gamma}{1 + [f(k)]^\gamma / \gamma} \right\} &= \beta \sup_{k \geq 0} \left\{ \frac{1 + \left(A^2 k\right)^\gamma / \gamma}{1 + (Ak)^\gamma / \gamma} \right\} \\ &= \beta A^\gamma. \end{aligned}$$

(To prove that the supremum is $\beta A^\gamma$ note that both the numerator and denominator are strictly increasing functions of $k$ and thus tend to $\infty$. Then use l'Hôpital's rule.) Therefore we have the condition that optimal paths exist only if $\beta A^\gamma < 1$.

For some problems this machinery is more elaborate than necessary. For example, if $u(f(k))$ is bounded both above and below – if either function is bounded above and below this condition will hold – then we can take $\phi(k) = 1$ for all $k$. Thus, all we need is $\beta < 1$ to ensure existence. We can also take $\phi$ to be constant if $f$ obeys the following "maximum sustainable stock" condition:

**Definition 6** *$f$ obeys the **maximum sustainable stock condition** if there exists $\overline{k} > 0$ such that $f(k) < k$ for all $k > \overline{k}$.*

This condition in effect bounds the function $u(f(k))$ by placing a strict upper bound on the capital stocks we need consider: $k \in [0, L]$ where $L = \max\{k_0, \overline{k}\}$. If we do not start with capital above $\overline{k}$ we will never observe it, and if we do start with a capital stock that high we will be forced to run it down. The function $f(k) = Ak^{\alpha}$ is one which satisfies the maximum sustainable stock condition. Certain important classes of return functions – quadratic return functions – are always bounded above and below; $\phi$ can be taken to be a constant in this case as well. When the production function is linear there may not exist a maximum capital stock, so unless the utility function is bounded we require a nontrivial $\phi$ to prove existence. To handle the $\gamma \leq 0$ cases (which includes the log case $\gamma = 0$) – where the utility function is unbounded even when $k$ is restricted to a compact interval $[0, L]$ – requires some additional tools. Those interested can consult Alvarez and Stokey (1998) "Dynamic Programming with Homogeneous Functions" for the details.

# 2 Properties of the Value Function

Now we have the value function. By the Theorem of the Maximum, we have that $v(k)$ is continuous under the assumptions given above. Also by the Theorem of the Maximum, the policy correspondence $a^* = \pi(k)$ is upper hemicontinuous and continuous if single-valued. What else can we prove about $v$ and $\pi$? It depends on the problem. As a concrete example, we will once again examine the optimal growth model. Let us assume that $u(c)$ and $f(k)$ are both increasing, concave, and continuously-differentiable. It is natural to assume that the value function will inherit these properties, but will it?

If $w : X \to \mathcal{R}$ is weakly increasing, then so is

$$(Tw)(k) = \max_{k'} \{u(f(k) - k') + \beta w(k')\}$$

because if $k'$ is feasible from $k$ it is feasible from any $\overline{k} > k$ and the residual consumption cannot decrease the RHS because $u$ and $f$ are increasing. Thus, the Bellman operator maps weakly increasing functions into weakly increasing functions. Since we can initialize our iterations with a weakly increasing function, the value function itself must be weakly increasing provided this property is preserved in the limit. Since weakly increasing functions are defined by a weak inequality,

$$g(x) \geq g(y)$$

if $x \geq y$, then the set of these functions is closed and therefore preserved in the limit. Thus, the value function is weakly increasing. This approach cannot guarantee that $v$ is strictly increasing, since that set is not closed.

Similarly, note that if $w : X \to \mathcal{R}$ is concave, then so is

$$(Tw)(k) = \max_{k'}\{u(f(k) - k') + \beta w(k')\}. \tag{71}$$

To see this, take any $k$, $\overline{k} \geq 0$ and note that the set of feasible $k'$ is convex, given $k$. Then

$$(Tw)(\alpha k+(1-\alpha)\overline{k}) \geq u(f(\alpha k+(1-\alpha)\overline{k})-\alpha k'-(1-\alpha)\overline{k}')+\beta w(\alpha k'+(1-\alpha)\overline{k}'). \tag{72}$$

Using the fact that both $u$ and $w$ are concave, we have

$$u(f(\alpha k+(1-\alpha)\overline{k})-\alpha k'-(1-\alpha)\overline{k}')+\beta w(\alpha k'+(1-\alpha)\overline{k}') \geq \alpha(Tw)(k)+(1-\alpha)(Tw)(\overline{k}). \tag{73}$$

That is, we know that $(Tw)(k)$ is concave; evidently, the Bellman operator takes concave functions into concave functions. Since the limit of any sequence of functions is the value function, all we need to prove is that the limit of a sequence of concave functions must be concave because we can start our iterations with a concave function. But note that concave functions are defined by a weak inequality:

$$g(\alpha x + (1-\alpha)y) \geq \alpha g(x) + (1-\alpha)g(y) \tag{74}$$

if $\alpha \in [0,1]$. Thus, the set defining it must be closed, so the value function is concave. This proof will not establish that the value function is strictly concave since that is not a closed set.

Finally we consider whether $v$ is differentiable. The approach using the Bellman operator would fail – the limit of a sequence of differentiable functions may not be differentiable. Directly applying the envelope theorem is impossible because the function whose differentiability is in question appears on the RHS of the expression. Instead we approach the problem indirectly. Let $\pi$ denote the optimal decision rule for tomorrow's capital. Now fix $k$ at some value $k^*$. If we can prove that the value function is differentiable here, we can claim differentiability everywhere on the interior of the constraint set. Consider the function

$$w(k) = u(f(k) - \pi(k^*)) + \beta v(\pi(k^*)). \tag{75}$$

Essentially, $w$ is the value of pretending one has $k^*$ capital today instead of $k$ but behaving optimally from this point onward. It should be clear that $w(k) \leq v(k)$ and that $w(k^*) = v(k^*)$. Since $u$ and $f$ are differentiable, $w$ is also differentiable with

$$w'(k) = u'(f(k) - \pi(k^*))f'(k). \tag{76}$$

The following lemma gives us our key result:

**Lemma 7** ***(Benveniste-Scheinkman Lemma)*** *Let $v$ be a real-valued, concave function defined on a convex set $D \subset \mathcal{R}^n$. If $w \in C^1$ is a concave function on a neighborhood $N$ of $x_0 \in D$ such that $w(x_0) = v(x_0)$ and $w(x) \leq v(x)$ $\forall x \in N$ then $v \in C^1$ at $x_0$.*

Essentially, this lemma states that if we can find a function that is everywhere below $v$, agrees with $v$ at $k^*$, and is continuously-differentiable at $k^*$, then $v$ will be continuously-differentiable at $k^*$ as well. $w(k)$ is that function, so the value function is differentiable.

We may want to rule out boundary solutions in our models – for example, in an economy with only Robinson Crusoe it will not make much sense if we examine solutions where all the existing capital is consumed. We therefore examine conditions – known as *Inada conditions* – which guarantee an interior solution to our problem. First, let the utility function be such that $u'(c) \to \infty$ as $c \to 0$. We further assume that the utility function obeys $u'(c) \to 0$ as $c \to \infty$; that is, marginal utility vanishes for large enough consumption. We also assume the production function obeys the same conditions for all of its marginal products. It is easy to see that consumption will never be zero – the marginal utility would be infinite and thus an increase of $\epsilon$ would definitely increase utility for any finite cost of consumption. Similarly, next period's capital will also never be zero – the return on capital $f'(k)$ would exceed any cost. We therefore have interior decisions for both consumption and savings. The upper boundary conditions ensure that both remain bounded by driving the return below any positive cost.

As a simple homework, suppose that $f(k)$ is linear and $u(c)$ is homogeneous of degree $\gamma$. Can you prove that $v$ is also homogeneous of degree $\gamma$ without explicitly calculating it?

# 3 Euler Equations

One of the problems with dynamic programming is that the form of the value function will typically be unknown. We can get around this problem using tools from variational calculus – the optimality conditions for such problems are known as Euler equations. Euler equations are useful for analyzing the dynamics of our system and often are easier to use for intuitive purposes.

Let's take the Bellman equation for the growth model:

$$v(k) = \max_{k'} \left\{ u(f(k) - k') + \beta v(k') \right\}. \tag{77}$$

Assuming enough structure for interior differentiable solutions, the first-order condition is

$$u'(f(k) - k') = \beta v'(k'). \tag{78}$$

This does not seem like much of an improvement; while we don't need to know the value function we need to know its derivative. Let's assume that the above equation is satisfied by the policy function

$$k' = \pi(k). \tag{79}$$

Inserting this into the Bellman equation gives us

$$v(k) = u(f(k) - \pi(k)) + \beta v(\pi(k)). \tag{80}$$

Since this must hold at every feasible $k$, we can differentiate (at least if we have an interior optimum), yielding

$$v'(k) = u'(f(k) - \pi(k))(f'(k) - \pi'(k)) + \beta v'(\pi(k))\pi'(k). \tag{81}$$

Collecting terms involving $\pi'(k)$ we have

$$v'(k) = u'(f(k) - \pi(k))f'(k) - [u'(f(k) - \pi(k)) - \beta v'(\pi(k))]\,\pi'(k). \qquad (82)$$

From the first-order condition the term in the square brackets is zero, so we have

$$v'(k) = u'(f(k) - \pi(k))f'(k). \qquad (83)$$

This "envelope condition" must also hold at every $k$; in particular, it holds at $k' = \pi(k)$. This implies

$$v'(k') = u'(f(k') - \pi(k'))f'(k'). \qquad (84)$$

Inserting this term into the first-order condition we reach the Euler equation

$$u'(f(k) - \pi(k)) = \beta u'(f(k') - \pi(k'))f'(k') \qquad (85)$$

or

$$u'(f(k) - \pi(k)) = \beta u'(f(\pi(k)) - \pi(\pi(k)))f'(\pi(k)). \qquad (86)$$

We can now explore how $k' = \pi(k)$ and $c = f(k) - \pi(k)$ depend on $k$. It seems natural that both would be increasing in $k_t$, but can we prove it? Assume as before that $k_t < \overline{k}_t$; can we conclude that

$$\pi(k_t) < \pi\left(\overline{k}_t\right)$$

and

$$f(k_t) - \pi(k_t) < f\left(\overline{k}_t\right) - \pi\left(\overline{k}_t\right)?$$

Under the assumption that $\pi$ is differentiable (and even if it is not, we can do this in finite differences) the first-order condition for optimality

$$u'(f(k) - \pi(k)) = \beta v'(\pi(k))$$

can be differentiated to obtain

$$\pi'(k) = \frac{u''(f(k) - \pi(k))\,f'(k)}{u''(f(k) - \pi(k)) + \beta v''(\pi(k))}.$$

Since $u''$ and $v''$ are negative, this expression is positive. Therefore, savings is increasing in capital. The same steps imply that

$$f'(k) - \pi'(k) = \frac{\beta v''(\pi(k))\,\pi'(k)}{u''(f(k) - \pi(k))} > 0,$$

so that consumption is also increasing in capital.

We could solve the Euler equation for the policy function $\pi(k)$ (at least in principle) if we were certain that it constituted a necessary and sufficient condition for the problem at hand. Unfortunately, it does not – the problem being that we have discarded information about the value function. To more clearly see this, let's write this in terms of sequences:

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}). \tag{87}$$

This is a second-order difference equation in the capital stock. As we have seen before, second-order difference equations require two boundary conditions, and we have only one: the initial capital stock $k_0$. At time 1, the Euler equation is

$$u'(f(k_0) - k_1) = \beta u'(f(k_1) - k_2) f'(k_1). \tag{88}$$

If we somehow knew $k_1$ we could use the above equation to solve for $k_2$ and move forward. But we are supposed to learn what $k_1$ is from the model; this is not an appropriate condition to impose. We will therefore look for a "terminal condition" on the capital stock – something that happens at the end of the time horizon that will pin down the path $\{k_t\}$ uniquely. Remembering back to our finite horizon case, one of the necessary conditions for optimality was

$$k_{T+1} = 0. \tag{89}$$

That is, the household set the capital stock to zero in the last period of life. Something analogous will hold here. Suppose that $u$ and $f$ are both concave – we have already shown that this is sufficient for $v$ to be concave as well. Concave functions satisfy the following inequality:

$$g(z^*) + g'(z^*)(z - z^*) \geq g(z) \tag{90}$$

for every set of points $(z, z^*)$. If we can assume that $u$ and $f$ are bounded below we know that the value function will also be bounded below – we can therefore freely set $v(0) = 0$. With concavity we have

$$v(k_{t+1}) + v'(k_{t+1})(0 - k_{t+1}) \geq v(0) \tag{91}$$

or

$$v(k_{t+1}) \geq v'(k_{t+1}) k_{t+1} \geq 0. \tag{92}$$

(The last inequality comes from the envelope condition.) Since we know that $v'(k) = f'(k) u'(f(k) - k')$ we have

$$\beta^{t+1} v(k_{t+1}) \geq \beta^{t+1} k_{t+1} f'(k_{t+1}) u'(f(k_{t+1}) - k_{t+2}) \geq 0. \tag{93}$$

Furthermore, along an optimal path we have already shown that

$$\beta^{t+1} v(k_{t+1}) \to 0. \tag{94}$$

Therefore, we establish that it is necessary for an optimal path to satisfy

$$\beta^{t+1} k_{t+1} f'(k_{t+1}) u'(f(k_{t+1}) - k_{t+2}) \to 0. \tag{95}$$

Making one final substitution from the Euler equation we get

$$\beta^{t+1} k_{t+1} u'(f(k_t) - k_{t+1}) \to 0. \tag{96}$$

This is the **transversality condition** – it is the limit of analogous finite horizon results that require capital to be nonnegative in the final period of life.

What is the interpretation of the Euler equation in the growth model? The term $u'(c_t)$ is the cost of reducing current consumption by $\epsilon$ and saving it instead. The term $\beta u'(c_{t+1}) f'(k_{t+1})$ is the benefit in consumption tomorrow if the consumer eats the extra income from the additional $\epsilon$ of saving. Reverse the two interpretations if we consider increasing current consumption by $\epsilon$. If we are on an optimal interior path neither deviation can increase utility because they are always feasible; hence the two terms must be equal. Since we can rewrite the Euler equation in terms of multiple periods deviations of more than 1 period cannot be optimal either:

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1}) f'(k_{t+1}) \\ &= \beta \left[\beta u'(c_{t+2}) f'(k_{t+2})\right] f'(k_{t+1}) \\ &= \beta^2 u'(c_{t+2}) f'(k_{t+2}) f'(k_{t+1}). \end{aligned}$$

The transversality condition rules out a permanent deviation; it cannot be better to reduce current consumption by $\epsilon$ and never consume it because the value of the extra capital that is accumulated is converging to zero fast enough.

A loose end: we have proven that the Euler equations and the TVC are necessary conditions, but not that they are sufficient. Suppose $\{k_t\}_{t=0}^{\infty}$ is a path from $k_0$ that satisfies both, with associated path $\{c_t\}_{t=0}^{\infty}$. Since $u$ is concave, we have

$$u(c_t) + u'(c_t)(c - c_t) \geq u(c)$$

$\forall c \geq 0$. Rearranging, we obtain

$$u(c_t) - u(c) \geq u'(c_t)(c_t - c).$$

Thus, for any feasible path $\{\widetilde{c}_t\}_{t=0}^{\infty}$ from $k_0$ we have

$$\sum_{t=0}^{T} \beta^t \left(u(c_t) - u(\widetilde{c}_t)\right) \geq \sum_{t=0}^{T} \beta^t u'(c_t)(c_t - \widetilde{c}_t)$$

$\forall T$. Also,

$$\begin{aligned} c_t - \widetilde{c}_t &= f(k_t) - k_{t+1} - \left(f\left(\widetilde{k}_t\right) - \widetilde{k}_{t+1}\right) \\ &= \left(f(k_t) - f\left(\widetilde{k}_t\right)\right) - \left(k_{t+1} - \widetilde{k}_{t+1}\right). \end{aligned}$$

Since $f$ is concave, we have

$$f(k_t) - f\left(\widetilde{k}_t\right) \geq f'(k_t)\left(k_t - \widetilde{k}_t\right).$$

Therefore,

$$c_t - \widetilde{c}_t \geq f'(k_t)\left(k_t - \widetilde{k}_t\right) - \left(k_{t+1} - \widetilde{k}_{t+1}\right).$$

That is, we have

$$\sum_{t=0}^{T} \beta^t u'(c_t)(c_t - \widetilde{c}_t) \geq \sum_{t=0}^{T} \beta^t u'(c_t)\left(f'(k_t)\left(k_t - \widetilde{k}_t\right) - \left(k_{t+1} - \widetilde{k}_{t+1}\right)\right).$$

The first term of the RHS is

$$u'(c_0)\left(f'(k_0)(k_0 - k_0) - \left(k_1 - \widetilde{k}_1\right)\right) = -u'(c_0)\left(k_1 - \widetilde{k}_1\right).$$

The second term is

$$u'(c_1)\left(f'(k_1)\left(k_1 - \widetilde{k}_1\right) - \left(k_2 - \widetilde{k}_2\right)\right) = \beta u'(c_1) f'(k_1)\left(k_1 - \widetilde{k}_1\right) - \beta u'(c_1)\left(k_2 - \widetilde{k}_2\right).$$

Adding the two together yields

$$(\beta u'(c_1) f'(k_1) - u'(c_0))\left(k_1 - \widetilde{k}_1\right) - \beta u'(c_1)\left(k_2 - \widetilde{k}_2\right)$$

but the first term is zero by the Euler equation, so we're left with only

$$-\beta u'(c_1)\left(k_2 - \widetilde{k}_2\right).$$

Adding the third term to the first two leaves us with only

$$-\beta^2 u'(c_2)\left(k_3 - \widetilde{k}_3\right)$$

and so on. For arbitrary $T$, we find that the sum becomes

$$-\beta^T u'(c_T)\left(k_{T+1} - \widetilde{k}_{T+1}\right).$$

Therefore, we have

$$\begin{aligned} \sum_{t=0}^{T} \beta^t (u(c_t) - u(\widetilde{c}_t)) &\geq -\beta^T u'(c_T)\left(k_{T+1} - \widetilde{k}_{T+1}\right) \\ &\geq -\beta^T u'(c_T) k_{T+1} \end{aligned}$$

because $\beta^T u'(c_{T+1}) \widetilde{k}_{T+1} \geq 0$. Now we employ the TVC, which says that the RHS of the equation above converges to 0. Thus,

$$\sum_{t=0}^{\infty} \beta^t (u(c_t) - u(\widetilde{c}_t)) \geq 0$$

or

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \geq \sum_{t=0}^{\infty} \beta^t u(\widetilde{c}_t).$$

Since the path $\{\widetilde{c}_t\}_{t=0}^{\infty}$ was arbitrary, $\{c_t\}_{t=0}^{\infty}$ must be optimal.