# Forward Guidance and Credible Monetary Policy

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**Abstract**

The effectiveness of forward guidance depends on the credibility of the central bank. This paper sheds light on the credible forward guidance that a central bank can offer by solving for the whole set of sustainable sequential equilibrium (SSE) in a canonical New Keynesian model with occasionally binding zero lower bound (ZLB). The best credible forward guidance provides a better account of the Fed's policy practice than those under Ramsey equilibrium and Markov perfect equilibrium, including less aggressiveness in entering the ZLB, much longer duration at low or zero rate, and finally smooth and gradual exit. Quantitative results show that the Ramsey equilibrium is not generally implementable. However, central banks with credibility can boost consumption in recessions close to the degree of that provided by a commitment central bank.

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# 1 Introduction

Since the Great Recession, many major central banks, including the Federal Reserve Bank (Fed), the European Central Bank, and the Bank of England, have joined the Bank of Japan in using forward guidance to provide an economic boost.[1] In contrast to the conventional short-term interest rate policy, which explicitly sets only current rates, forward guidance is the statement and communication of the central bank's projected future path of short-term interest rates.[2] If the zero lower bound (ZLB) constrains the actions of the central bank with respect to current rates, forward guidance can be a useful instrument. In fact, an important feature of recent forward guidance practice is its promise to maintain the interest rate at, or near zero even after the economy emerges from a recession. For example, the Fed repeatedly said that it would keep short-term interest rates low for a "considerable" time after the economy emerges from the recession.[3] The prolonged stay at the ZLB increases expected inflation and stimulates current consumption.

However, the central bank also wants to raise interest rates as the economy strengthens and inflation begins to pick up, which induces the problem of time-inconsistency. This paper introduces a reputational mechanism (punishment strategies) to resolve this problem and searches for time-consistent or credible forward guidance policies. The model features a benevolent central bank that sets nominal interest rates to maximize the lifetime utility of households, firms set sticky prices à la Rotemberg (1982), and the central bank faces an occasionally binding zero lower bound constraint on the nominal interest rate, which may be activated due to large contractionary shocks. In contrast to the concepts of full commitment (Ramsey equilibrium, RE hereafter) and full discretion (Markov perfect equilibrium, MPE hereafter), this paper assumes that the central bank is playing a game with households. At the beginning of each period, the central bank sets a new policy rate. If this rate is the same as that promised in the last period, then the households believe that the central bank will behave the same as it used to. If the rate is different from what is promised, then the households will change their expectation of what the central bank is going to do and make decisions

[1] The history of the Bank of Japan using forward guidance dates back to 1999. The Fed also experimented with forward guidance in 2003 and 2004.

[2] As summarized in Issing (2014), different forms of forward guidance have been adopted by central banks.

[3] For example, the FOMC minutes of Jan 2013 stated: "To support continued progress toward maximum employment and price stability, the Committee expects that a highly accommodative stance of monetary policy will remain appropriate for a considerable time after the asset purchase program ends and the economic recovery strengthens." Officially the recession ended in 2009, nearly four years before this announcement.

about consumption, leisure, and prices in a way to punish the central bank. In equilibrium, the central bank *can* deviate but chooses not to because the current gain from deviation does not exceed the future loss due to the households' change of behavior and expectations. This type of equilibrium is a sustainable sequential equilibrium (SSE). We show that this dynamic game has infinitely many SSEs, which are defined by payoffs to the households (promised consumption and inflation) and payoffs to the central bank (welfare of the households). We then computationally characterize the entire set of SSEs and recover the strategies of interest.

Given the computed SSE set, we show that RE is generally not implementable – the corresponding payoff combinations under RE often do not lie in the set of SSEs. This conclusion is robust to a reasonable range of values for the shocks (size, frequency, and persistence) and for other model parameters, such as the Frisch labor elasticity, the elasticity of substitution among intermediate goods, and the cost of price adjustment. In other words, if such a central bank were hypothetically to re-optimize, then it would renege on its promises. In contrast, the MPE always lies within the set of SSEs and is time-consistent, naturally; however, the MPE is not a particularly good outcome. Finally, we focus on the best SSE (BSSE), which delivers the highest payoff to the central bank that can be sustained by a deviation to the worst SSE (which is not generally the MPE).[4][5] We show that the BSSE provides a better account of forward guidance practice of the Fed compared to RE and MPE.

The BSSE prescribes that forward guided policy rate should stay low or close to zero for a prolonged period even after the recession ends. In the data, the nominal federal funds rate stayed close to zero for a very long time (about 25 quarters since the NBER defined recession ended in the second quarter of 2009).[6] In the benchmark model, MPE implies that the short-term rate should exit zero immediately after the economy exits the recession. RE implies that there should be a period of zero rates after the recession ends, but the length of this period after the recession

[4] In this paper, we follow Eggertsson and Woodford (2003), Campbell et al. (2012) and many others to take forward guidance as the optimal choice of a benevolent central bank. Many papers such as Lasen and Svensson (2011) and Del Negro et al. (2012) use anticipated policy shocks to capture forward guidance in DSGE models. Others assume ad hoc and exogenous extensions of zero rates to show the idea and power of forward guidance.

[5] Campbell et al. (2012) distinguishes two types of forward guidance, "Delphic" and "Odyssean", depending on whether the central bank communication is interpreted partly as a signal of the underlying state of the economy or exclusively as a commitment about future policy. This paper is concerned with the second kind of forward guidance.

[6] Recently the literature seems to have shifted to calling the bound the ELB (effective lower bound), since it could be operating at positive or negative values; nothing important hinges on the value of the bound, only that it binds in some state of the world.

is usually quite limited. For example, the extended time span is about three to four quarters in the baseline calibration. In contrast, forward guidance under BSSE involves low rates for much longer time; given our calibration, this period of low rates is at least six quarters longer. Thus, we conclude that BSSE delivers a length of forward guidance that better fits the data.

Credible forward guidance features a smooth and gradual exit. Under MPE, the policy rate reverts to its normal level immediately after the recession ends; the extra duration of the ZLB after the recession is zero. In RE, the policy rate stays at the ZLB for an extended period, followed by a sharp increase and overshooting of its normal level.[7] If we call the liftoff of the short-term rate a normalization, then optimal policies under MPE and RE state that as long as the central bank starts to normalize, it would not be accommodative in the sense that the rate stays the same as or higher than the one if the central bank is allowed to reset. However, credible forward guidance shows that liftoff is not a signal of a changing stance of monetary policy. As the recession unfolds, the policy rate will gradually and smoothly go back to its normal level. This series of lower-than-normal rates continue to accumulate and accommodate to boost the economy. This gradual stance of normalization (not non-accommodation) is consistent with the Fed's minutes. The two prevailing arguments for gradualism in the exit of accommodative monetary policy are worries about the headwinds of the economy and the over-sensitivity of agents to interest rate changes due to the experiences of the last few years (Plosser, 2012). In contrast, our work shows that the gradualism of exit may simply be a consequence of the requirement that policy announcements need to be credible while also stimulative.

The optimal policy under BSSE involves substantially less aggressive adjustments in the nominal interest at the onset of a recession, which is consistent with the Fed's action at the beginning of the Great Recession. Under MPE and RE, the central bank immediately sets the nominal rate to zero, whereas under BSSE the central bank lowers the rate gradually over time. This result is in contrast to the conventional wisdom of "keeping the powder dry" (Williams, 2014). We then approximate the BSSE and RE outcomes using rule-based forward guidance and find that the elasticity of nominal rate with inflation in the BSSE is about 1/3 of that in the RE. Once again,

[7] This is a typical result when using a two-state Markov chain to characterize the shock; see Nakata (2018) and Levin et al. (2010). When shocks follow an AR(1) process and in general smoothly fade out, optimal policy under Ramsey also features a smooth and gradual exit. This paper emphasizes that a smooth and gradual exit is not just a result of continued headwinds on economic growth but it is also likely from the central banks' need to be credible.

the general message is that central banks react less aggressively if credibility is a concern.

A central bank with reputation can do much better than one with full discretion. In the absence of fiscal subsidy and the ZLB, the inflation bias under BSSE is only half of that under MPE. Given that lower inflation bias means higher welfare, the central bank with reputation can achieve higher welfare with access to future policy rates, which the central bank with full discretion cannot. In the presence of full fiscal subsidy and the ZLB, the output gap under BSSE is on average $-1.55\%$ during a recession, which is much smaller than $-4.55\%$ under MPE and comparable to $-1.05\%$ under RE. Therefore, a central bank with reputation has a larger toolkit and can credibly twist current and future policy rates to achieve higher lifetime utility for households.

This paper is related to the literature examining whether RE can be made time-consistent via reputation; see Rogoff (1987), Phelan and Stacchetti (2001), Chang (1998), and Ireland (1997) for early contributions, and Kurozumi (2008), Sunakawa (2015) for recent contributions. The most related work to ours is Nakata (2018), who examines whether reputation can make RE time-consistent in the presence of the ZLB. While we have essentially the same idea of using punishment to force a central bank to stick to its promises, Nakata (2018) focuses on only two equilibria – MPE and RE. He also uses MPE as the upfront determined punishment to enforce RE. In contrast, in our model the punishment (i.e., the worst SSE, which can deliver a lower payoff than the MPE) is endogenously determined and actually enforces infinitely many equilibria (the entire set of SSEs). Importantly, the set of SSEs does not include RE, which means that the latter is not implementable as in Nakata (2018); the reason is that the central bank would deviate to the BSSE, which dominates the MPE and cannot enforce the RE outcome. Nakata and Sunakawa (2018) weakens the punishment to only a finite deviation to the MPE, which also would not render the RE sustainable as in our model.[8] On the other hand, the welfare loss associated with moving from the RE to the BSSE appears not to be very large, so the fact that RE is not implementable is not as important either. Finally, we argue that the BSSE provides a better account of actual forward guidance practice in terms of a 'lower for longer' path for interest rates.

This paper is also related to Bodenstein et al. (2012), and Nakata and Schmidt (2018), both of which try to reconcile properties of actual forward guidance practice with optimal policies.

[8] Feng (2015) and Domnguez and Feng (2017) show that the Ramsey equilibrium is generally not implementable in real business cycle models with capital and labor taxes.

Bodenstein et al. (2012) show that the central bank promises longer duration at the ZLB and how it reduces output gap while not overshooting inflation target when there is imperfect credibility. In their paper, they model imperfect credibility by allowing central banks, with an exogenous probability, to discard their earlier promises from commitment and re-optimize with full discretion. In contrast, in our model, households have a full understanding of what the central bank does given the state and central banks have complete freedom to renege in any given period. In equilibrium, the central bank never deviates, so the probability of deviation is zero. Furthermore, the policy outcomes from our paper do not lie in between the conventional extremes of optimal policy under commitment and under discretion, as assumed in Bodenstein et al. (2012) and subsequently shown in Fujiwara et al. (2016). Finally, Nakata and Schmidt (2018) show how a gradual exit of policy rates from the bound improves welfare in the presence of ZLB. While they explicitly assume interest-rate smoothness in the central banks' objective function, we show that gradualism is an intrinsic feature of the optimal policy of credible central banks.

This paper is related to theoretical papers exploring the effectiveness of forward guidance. For example, Levin et al. (2010) shows that forward guidance is effective in offsetting natural rate shocks of moderate size and persistence but not of larger and persistent shocks under commitment. Cole (2015) shows that the assumption of rational expectations overstates the effects of forward guidance relative to adaptive learning. Angeletos and Lian (2018) show that agents without common knowledge attenuate the effects of forward guidance. McKay et al. (2016) show that when agents face uninsurable income risk and borrowing constraints, a precautionary savings effect tempers their responses to changes in future interest rates, which makes forward guidance substantially less able to stimulate the economy. Hagedorn et al. (2018) uses a similar setup to McKay et al. (2016) and finds that the effect of forward guidance is negligible if the model matches the income and wealth distributions of the data and has nominal government bonds. Kiley (2016) shows that effects of forward guidance are negligible with information stickiness rather than price stickiness as in standard New Keynesian models. Our paper echoes Krugman (1998), Eggertsson and Woodford (2003), Campbell (2013), and especially Bodenstein et al. (2012) by showing that the power of forward guidance relies on the central bank's credibility. In particular, in contrast to the literature, it is the shrunken policy space from requiring central banks to be credible that leads to less powerful forward guidance. Although less powerful than that under commitment, credible forward guidance is much more powerful than that under discretion – reputation has value.

Finally, the method for computing the equilibrium payoff set closely follows Feng (2015), Phelan and Stacchetti (2001), and Chang (1998). The key is to find variables that summarize the history of the game. The equilibrium payoff set is found by repeatedly applying a monotonic set-valued operator, as in Fernández-Villaverde and Tsyvinski (2002) and Sleet and Yeltekin (2000). To deal with aggregate shocks, we apply Feng (2015)'s method to represent the equilibrium set on a computer. This paper is the first one to calculate the whole SSE set with aggregate shocks and ZLB in a New Keynesian model.

The rest of the paper is structured as follows. Section 2 presents the model. Section 3 introduces MPE and RE while Section 4 presents SSE and the strategies to solve the model. Sections 5 and 6 are devoted to results, without and with the ZLB, respectively. Section 7 is the conclusion.

## 2 The Model and Competitive Equilibrium

This section presents a standard New Keynesian model and gives the definition of competitive equilibrium. The economy is populated by a continuum of households and firms and a monetary authority. At the beginning of each period $t$, the economy is hit by a discount factor shock, $\beta_t$.[9] A higher $\beta_t$ means that the agents are more patient and desire to save more and consume less and, hence, there is weak aggregate demand. We assume that $\beta_t$ follows a two-state Markov chain process, characterized by the transition matrix

$$\boldsymbol{P} = \begin{bmatrix} p_{NN} & 1 - p_{NN} \\ 1 - p_{RR} & p_{RR} \end{bmatrix}$$

and two states $\{\beta^N, \beta^R\}$, where $N$ and $R$ mean normal and recession states, respectively. A normal state $\beta^N$ is when $\beta_t$ is low, while a recession state $\beta^R$ is when $\beta_t$ is high. $1 - p_{NN}$ is the probability of the economy switching from normal state to recession and $p_{RR}$ is the probability of the economy remaining in a recession.

Let $s_t = \{\beta_t\}$ be the shock at period $t$, drawn from $\boldsymbol{S} := \{\beta^R, \beta^N\}$. The history of shocks up

[9] This shock is also called an aggregate demand shock in the literature (see Nakata, 2018; Burgert and Schmidt, 2014; and many others). Eggertsson and Woodford (2003), and Christiano et al. (2011) view this shock as standing in for a wide variety of factors that alter households' propensity to save, including financial and uncertainty shocks.

to and including time $t$ is defined as $s^t = (s_0, s_1, ..., s_t)$ given an initial shock $s_0$. The probability of each of these histories is given by $\pi(s^t)$ and the conditional probability of moving from $s^t$ to $s^{t+1}$ is $\pi(s^{t+1}|s^t)$. Agents in the economy make decisions after observing shocks $s_t$ and have full information of the history $s^t$ and $\pi(s^t)$.

## 2.1 Households

The representative household maximizes its lifetime utility,

$$\sum_{t=0}^{\infty}\sum_{s^t}\left(\prod_{i=0}^{t}\beta\left(s^i\right)\right)\pi\left(s^t\right)\left\{\log\left(c\left(s^t\right)\right)-\frac{l\left(s^t\right)^{1+\chi}}{1+\chi}\right\} \tag{1}$$

subject to

$$c\left(s^t\right)+\frac{B\left(s^t\right)}{P\left(s^t\right)}=w\left(s^t\right)l\left(s^t\right)+R\left(s^{t-1}\right)\frac{B\left(s^{t-1}\right)}{P\left(s^t\right)}+\tau\left(s^t\right)+d\left(s^t\right) \tag{2}$$

where $w\left(s^t\right)$ is the real wage, $R\left(s^{t-1}\right)$ the nominal interest rate from period $t-1$ to $t$, $\tau\left(s^t\right)$ a real lump-sum transfer or tax, and $d\left(s^t\right)$ real profits of the firms in the economy. The household chooses labor to supply $(l\left(s^t\right))$, nominal bonds to buy $(B\left(s^t\right))$ and goods to consume $(c\left(s^t\right))$ to maximize its lifetime utility. The optimality conditions for the household are

$$\frac{1}{c\left(s^t\right)} = \beta_t R\left(s^t\right)\mathbb{E}_t\left[\frac{1}{c\left(s^{t+1}\right)}\frac{1}{\Pi\left(s^{t+1}\right)}\right] \tag{3}$$

$$w\left(s^t\right) = l\left(s^t\right)^{\chi}c\left(s^t\right) \tag{4}$$

where $\Pi\left(s^t\right)=P\left(s^t\right)/P\left(s^{t-1}\right)$ is the inflation from period $t-1$ to $t$.

## 2.2 Firms

The final good producer maximizes its profits by solving the following problem:

$$\max_{\{y_i(s^t)\}_{i\in I}}\left\{P\left(s^t\right)y\left(s^t\right)-\int_0^1 P_i\left(s^t\right)y_i\left(s^t\right)di : y\left(s^t\right)=\left(\int_0^1 y_i^{\frac{\epsilon-1}{\epsilon}}\left(s^t\right)di\right)^{\frac{\epsilon}{\epsilon-1}}\right\};$$

$\epsilon$ is the elasticity of substitution among intermediate goods. Given the prices of final and intermediate goods, the demand for intermediate good $i$ is given by:

$$y_i\left(s^t\right) = \left(\frac{P_i\left(s^t\right)}{P\left(s^t\right)}\right)^{-\varepsilon} y\left(s^t\right). \tag{5}$$

The typical intermediate goods producer has a linear technology with labor as its only input, that is, $y_i\left(s^t\right) = l_i\left(s^t\right)$. It maximizes discounted profits by paying a cost to adjust its price, à la Rotemberg (1982),

$$\max_{\{p_i(s^t)\}_{i\in I}} \left\{\sum_{t=0}^{\infty}\sum_{s^t}\left(\prod_{i=0}^{t}\beta(s^i)\right)\pi(s^t)\frac{\lambda(s^t)}{\lambda(s^0)}d_i(s^t)\right\}$$

subject to the demand function (5), and

$$d_i\left(s^t\right) = \frac{P_i\left(s^t\right)y_i\left(s^t\right)}{P\left(s^t\right)} - (1-\xi)\,w\left(s^t\right)l_i\left(s^t\right) - \frac{\phi}{2}\left(\frac{P_i\left(s^t\right)}{P_i\left(s^{t-1}\right)} - 1\right)^2 y\left(s^t\right).$$

The coefficient of the quadratic term $\phi$ captures how costly it is to adjust prices and $\lambda\left(s^t\right)$ is the Lagrangian multiplier for the household at date $t$. $\xi$ is a subsidy to eliminate steady state distortion due to monopolistic pricing. After solving the problem and using symmetry conditions, the behavior of the firm sector can be summarized by the Phillips curve equation

$$\left[(\epsilon-1) - (1-\xi)\,\epsilon w\left(s^t\right) + \phi\left(\Pi\left(s^t\right)-1\right)\Pi\left(s^t\right)\right]\frac{l\left(s^t\right)}{c\left(s^t\right)} = \phi\beta_t\mathbb{E}_t\left[\left(\Pi\left(s^{t+1}\right)-1\right)\Pi\left(s^{t+1}\right)\frac{l\left(s^{t+1}\right)}{c\left(s^{t+1}\right)}\right] \tag{6}$$

Equation (6) states that the marginal benefit of adjusting prices (LHS) must equate to the marginal cost (RHS).

## 2.3 The Central Bank

The central bank is benevolent and chooses the nominal interest rate $\{R_t\}_{t=0}^{\infty}$ to maximize the households' lifetime utility (1).[10] However, the central bank cannot reduce the rate below $R_t = 1$ – the net nominal interest rate is bounded below by zero. The usual argument for the ZLB is that people can hold cash, which of course pays no interest, rather than lend money out at a

[10] Appendix A.1 provides a simple case where the interest rate is implementable through the central banks' open market operations.

negative rate of return (Williams, 2013).[11] At each period after the central bank sets the rate, the households make their decisions about consumption and leisure, and the firms set their new prices. The credibility of the central bank determines the available rate instruments in its toolkit.

### 2.4 Market Clearing

The goods market must clear:

$$c\left(s^t\right) = \left(1 - \frac{\phi}{2}\left(\Pi\left(s^t\right) - 1\right)^2\right) l\left(s^t\right). \tag{7}$$

The net bond supply is assumed to be zero and therefore plays no role here.

### 2.5 Competitive Equilibrium

**Definition 2.1 (Competitive Equilibrium)** *Suppose that the economy starts with $\Upsilon\{s_0, R_0\}$, a competitive equilibrium (CE) for $\Upsilon\{s_0, R_0\}$ is characterized by a state-contingent sequence ($c(s^t)$, $l(s^t)$, $w(s^t)$, $\Pi(s^t)$) such that, for all $t \geq 1$, $s^t \in \boldsymbol{S^t}$, and $R(s^t) \geq 1$ and equations (3), (4), (6), and (7) hold.*

*Remarks*: Given the central bank's policy rates $R\left(s^t\right)$, the households and firms behave optimally subject to the resource constraint in any CE.

## 3 Full Discretion and Full Commitment

We now present different definitions of equilibrium based on the central bank's ability to commit or not.

[11] One may argue that the negative rate adopted by ECB and BoJ is a violation of the ZLB constraint. However, we argue that a general lower bound (not too much lower than zero) still exists. First, commercial banks seem to avoid charging customers for deposits, although paying for the reserve to the central bank. Second, the degree of negative rate is so far limited. Finally, as to countries like the United States, overnight rates are reduced to (or even slightly below) the rate of interest paid on overnight balances at the central bank, so that further expansions of the supply of bank reserves do not bring about any additional material reduction in the level of overnight rates, given the rate of interest paid on reserves.

### 3.1 Ramsey Equilibrium (RE)

**Definition 3.1** *A Ramsey equilibrium is where the central bank at time* 0 *instructs all of the policies of future depending on the possible shocks to maximize the lifetime utility of the household (1). Namely,*

$$\max_{\{c(s^t),l(s^t),w(s^t),\Pi(s^t),R(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t}\left(\prod_{i=0}^{t}\beta(s^i)\right)\pi(s^t)\left\{\log c(s^t)-\frac{l(s^t)^{1+\chi}}{1+\chi}\right\}$$

*subject to (3), (4), (6) and (7) and* $R(s^t)\geq 1$ *for* $\forall t$.

The central bank in the RE has full commitment, which means that, by some means, it would not or is not allowed to change behavior chosen at date 0. We denote the maximized lifetime utility of the Ramsey central bank at period 0 as $V^{RAM}(s^0)$ and discounted utility at period $t$ as $V^{RAM}(s^t)$. By definition, the Ramsey equilibrium delivers the highest lifetime utility at time 0. However, it does not guarantee that for any given period $t>1$, the discounted utility $V^{RAM}(s^t)$ exceeds what could be achieved if the central bank is given the chance to re-optimize. In this case, the time-inconsistency comes from the fact that the central bank may have a strong incentive to close saving and inflation gaps that were promised in earlier periods to boost the economy due to the zero lower bound constraint.

To solve the Ramsey equilibrium of this model, we follow Marcet and Marimon (2011), and Adam and Billi (2006) and first write the Ramsey problem recursively with the introduction of two Lagrange multipliers for the two Euler equations of households and firms. A time iteration method is then applied on Karush-Kuhn-Tucker conditions to get the policy functions, with a simple transformation of the ZLB constraint following Dong (2012).[12] Linear interpolation is used to approximate values that are not on the pre-assigned grids.

[12]The idea of the transformation follows Garcia and Zangwill (1981). The key is to introduce an auxiliary variable $\mu_t$ and rewrite the complementary slackness condition, $\lambda(R-1)=0$ as $\lambda=\max\{0,\mu\}^2$ and $R_t=1+\max\{-\mu_t,0\}^2$, where $\lambda$ is the Lagrangian of the ZLB constraint. The essence of this transformation is to make any inequalities become equalities and the system differentiable everywhere so that we can apply a nonlinear optimization solver to the problem.

### 3.2 Markov Perfect Equilibria (MPE)

**Definition 3.2** *An MPE is the case where the central bank at any time $t$ maximizes the household's lifetime utility (1) from that period onwards by choosing consumption $c(s_t)$, labor $l(s_t)$, wage $w(s_t)$, inflation $\Pi(s_t)$ and interest rate $R(s_t)$ subject to conditions (3), (4), (6), and (7), and non-negativity of the nominal interest rate, taking as given the behavior of the future central bank and households' expectations.*

The central bank in MPE has full discretion since it can always re-optimize. Let $V^{MPE}(s)$ denote the discounted lifetime utility with state $s$ and primes variables of next period. The Bellman equation is:

$$V^{MPE}(s) = \max_{c,l,w,\Pi,R} \left\{ \log(c) - \frac{l^{1+\chi}}{1+\chi} + \beta \mathbb{E}[V^{MPE}(s')|s] \right\}$$

subject to

$$\frac{1}{c} = R\beta\mathbb{E}\left[\frac{1}{c'\Pi'}\right], w = l^{\chi}c, c = \left[1 - \frac{\phi}{2}(\Pi - 1)^2\right] l,$$
$$[(\epsilon - 1) - (1-\xi)\epsilon w + \phi(\Pi - 1)\Pi] \frac{l}{c} = \phi\beta\mathbb{E}\left[(\Pi' - 1)\Pi'\frac{l'}{c'}\right], and\ R \geq 1.$$

An MPE is characterized by a sequence of time-invariant value function and policy functions of consumption, labor, wage, inflation, and interest rate, that is, $\{c(s), l(s), w(s), \Pi(s), R(s), V^{MPE}(s)\}$. Given that the discretionary central bank re-optimizes every period, MPE is time-consistent. Value function iteration is then used to solve this equilibrium; see Bodenstein et al. (2012).

## 4 Sequential Sustainable Equilibria (SSE)

The game is played between households and the central bank.[13] Denote by $\Gamma(s_0)$ the game where the economy starts with $s_0$. The public history of the game is $\zeta^t = \{\zeta_0, \zeta_1, ..., \zeta_t\}$, where $\zeta_t = (c_t, l_t, w_t, \Pi_t, R_t, s_t)$. Let $\sigma_H$ be the strategy of households and $\sigma_B$ that of the central bank. Both are measurable functions. Strategy $\sigma_B$ maps the publicly observed history $\zeta^{t-1}$ and the current shock $s_t$ into an interest rate for date-event $s^t$, namely, $R(s^t) = \sigma_B(\zeta^{t-1}, s_t)$. Similarly, strategy

[13] The firms are owned by households. To simplify, we assume that individual households coordinate in their actions when needed, following Phelan and Stacchetti (2001).

$\sigma_H$ specifies $c\left(s^t\right)$, $w\left(s^t\right)$, $l\left(s^t\right)$, $\Pi\left(s^t\right)$ as functions of the expanded history $\left(\zeta^{t-1}, s_t, R\left(s^t\right)\right)$; that is, $\left(c\left(s^t\right), w\left(s^t\right), l\left(s^t\right), \Pi\left(s^t\right)\right) = \sigma_H\left(\zeta^{t-1}, s_t, R(s^t)\right)$. We use $\sum\left(s_0\right) = \sum_H\left(s_0\right) \times \sum_B\left(s_0\right)$ to denote the set of all symmetric strategy profiles for $\Gamma(s_0)$, where $\sum_H\left(s_0\right)$ represents the set of strategies for households, and $\sum_B\left(s_0\right)$ the set of strategies for the central banks. The value of a strategy $\sigma = (\sigma_H, \sigma_B)$ for the central bank is defined as

$$\Phi_B\left(s_0, \sigma\right) = \sum_{t=0}^{\infty}\sum_{s^t}\left(\prod_{i=0}^{t}\beta\left(s^i\right)\right)\pi\left(s^t\right)\left\{\log\left(c\left(s^t\right)\right) - \frac{l\left(s^t\right)^{1+\chi}}{1+\chi}\right\} \tag{8}$$

**Definition 4.1** *A strategy profile $\sigma$ of the game $\Gamma(s_0)$ is an **SSE** if for any $t \geq 0$ and history $\zeta^{t-1}$:*

1. *$\Phi_B(s_t, \sigma|_{\zeta^{t-1}}) \geq \Phi_B(s_t, (\sigma_H|_{\zeta^{t-1}}, \gamma))$ for any strategy $\gamma$ in $\sum_B(s_t)$ for the central bank;*

2. *$\{c(s^j), l(s^j), w(s^j), \Pi(s^j)\}_{j=t}^{\infty}$ is a CE for $\Gamma\{s_t, R_{s^t}\}$, where $R_{s^t} := \{R(s^j)\}_{j=t}^{\infty}$, $R(s^t) \in \sigma_B(\zeta^{t-1}, s_t)$, and $(c(s^t), l(s^t), w(s^t), \Pi(s^t)) \in \sigma_H(\zeta^{t-1}, s_t, R(s^t))$.*

In line with Phelan and Stacchetti (2001), $\sigma|_{\zeta^{t-1}}$ denotes the strategy profile in an SSE with history $\zeta^{t-1}$, and $\left(\sigma_H|_{\zeta^{t-1}}, \gamma\right)$ the strategy profile in which the household plays a SSE strategy under history $\zeta^{t-1}$ while the central bank plays an alternative strategy $\gamma$. The first condition above says that the continuation payoff for the central bank's strategy $\sigma_B$ is better than that from any deviation to a different strategy. The second condition requires that the household always responds to a central bank strategy with decisions that induce a competitive equilibrium.

## 4.1 Recursive Formulation of SSE

Following Kydland and Prescott (1980) and Feng (2015), define

$$m_1\left(s^t\right) = \frac{1}{c\left(s^t\right)\Pi\left(s^t\right)} \tag{9}$$

$$m_2\left(s^t\right) = \phi\left(\Pi\left(s^t\right) - 1\right)\Pi\left(s^t\right)\frac{l\left(s^t\right)}{c\left(s^t\right)}, \tag{10}$$

which represent the marginal utility of saving one unit of consumption goods and the marginal cost of changing relative prices, respectively.[14] It is important to emphasize that each individual household is atomistic, so when they make a decision they take $m_1(s^t)$ and $m_2(s^t)$ as given. Using the market clearing condition, equation (10) can be used to calculate $\Pi$ as a function of $m_2$:

$$\Pi = \frac{\phi\,(1+m_2) + \sqrt{2\phi m_2^2 + 4\phi m_2 + \phi^2}}{\phi\,(2+m_2)}.$$

With (9), consumption is given by

$$c = \frac{1}{m_1 \Pi}.$$

Therefore, $m_1$ and $m_2$ are also referred to as expected or promised consumption and inflation.[15] For any $s^t$, $m_1\left(s^{t+1}\right)$ and an arbitrary specified interest rate $R$, households solve the following problem:

$$\max_{c(s^t),l(s^t),B(s^t)} \left\{ \log\left(c\left(s^t\right)\right) - \frac{l\left(s^t\right)^{1+\chi}}{1+\chi} + \beta_t \mathbb{E}_t\left[ m_1\left(s^{t+1}\right) \frac{B\left(s^t\right)}{P\left(s^t\right)} R \right] \right\} \tag{11}$$

subject to the budget constraint (2). By construction, the recursive problem is equivalent to the sequential problem provided that the transversality condition is satisfied:

$$\lim_{t\to\infty} \left\{ \sum_{t=0}^{\infty} E_t \left( \prod_{i=0}^{t} \beta\left(s^i\right) \right) m_1\left(s^t\right) \frac{B\left(s^t\right)}{P\left(s^t\right)} R \right\} = 0. \tag{12}$$

This is shown in the following proposition, which is an extension of the results in Feng (2015), and Phelan and Stacchetti (2001).

**Proposition 4.2** *Assume $R \in [1, \bar{R}]$ and $l \in [0, \bar{l}]$, where $\bar{R} < +\infty$ and $\bar{l} < +\infty$. Given the functional forms of preference and production function, the recursive and sequential problems are equivalent.*

**Proof.** See Appendix A.5. ■

The firms also solve a recursive problem given $m_2\left(s^{t+1}\right)$.[16] Now, we define a static CE using

[14] Chang (1998) and Phelan and Stacchetti (2001) show that equilibria can be characterized in terms of their value to the government and their marginal value of private variables.

[15] The other root is $\Pi = \frac{\phi(1+m_2) - \sqrt{2\phi m_2^2 + 4\phi m_2 + \phi^2}}{\phi(2+m_2)}$, the limit of which is $1 - \sqrt{2/\phi}$ as $m_2$ goes to $+\infty$ and $1 + \sqrt{2/\phi}$ as $m_2$ goes to $-\infty$. Since this root never visits the range of $[1 - \sqrt{2/\phi}, 1 + \sqrt{2/\phi}]$ (see the proof of Proposition 4.2 for why this condition is important) and violates the Friedman rule for reasonable values of $\phi$, we ignore this case.

[16] See Appendix A.2.

the two variables introduced.

**Definition 4.3** *Let $\Upsilon\{s, R, \{m_1^+, m_2^+\}\}$ be the static economy in which the current shock is $s$, the current interest rate set by the central bank is $R$, and agents have expectations about the future summarized in $\{m_1^+, m_2^+\}$. $(c, l, w, \Pi)$ is a CE for $\Upsilon\{s, R, \{m_1^+, m_2^+\}\}$ if and only if the following conditions are satisfied: (1) Households satisfy the temporal equilibrium conditions $c^{-1}(s) = R\beta\mathbb{E}\{m_1^+\}$ and $w(s) = l(s)^\chi c(s)$; (2) Firms satisfy the temporal equilibrium condition $[(\epsilon - 1) - (1 - \xi)\epsilon\frac{w(s)}{A} + \phi\left(\Pi(s) - 1\right)\Pi(s)]\frac{l(s)}{c(s)} = \beta\mathbb{E}m_2^+$; and (3) Goods market clears at each node $c(s) = (1 - \frac{\phi}{2}(\Pi(s) - 1)^2)l(s)$.*

We denote this equilibrium as $(c, l, w, \Pi) \in \mathbf{CE}^S\left\{s, R, m_1^+, m_2^+\right\}$. The following lemma allows us to think of the original economy as a sequence of static economies with endogenously-changing state variables and exogenous stochastic shocks.

**Lemma 4.4** *Given a feasible interest rate policy $R = \{R_t\}_{t=0}^\infty$, suppose that the sequence $\{c(s^t)$, $l(s^t)$, $w(s^t)$, $\Pi(s^t)\}_{t=0}^\infty$ is such that for each $t$,*

$$\{c(s^t), l(s^t), w(s^t), \Pi(s^t)\} \in \mathbf{CE}^S\{s_t, R_t, \{m_1(s^{t+1}), m_2(s^{t+1})\}\}$$

*where*

$$m_1(s^{t+1}) = \frac{1}{c(s^{t+1})\Pi(s^{t+1})} \tag{13}$$

$$m_2(s^{t+1}) = \phi(\Pi(s^{t+1}) - 1)\Pi(s^{t+1})\frac{l(s^{t+1})}{c(s^{t+1})} \tag{14}$$

*then $\{c(s^t), l(s^t), w(s^t), \Pi(s^t)\}_{t=0}^\infty$ constitutes a competitive equilibrium for $\Upsilon\{s_0, R_0\}$.*

The lemma says that the promised marginal value of investment in bonds and the promised cost for adjusting prices summarize the expectations of households. Let $h$ denote the equilibrium continuation payoff of the central bank $\Phi_B$ defined by (8). The equilibria of the economy can be characterized by

$$V\left(s\right) := \{(m_1, m_2, h) \mid \sigma \text{ is a SSE for } \Gamma\left(s\right)\}$$

which is a mapping from the values of the states $s$ into set of possible payoffs associated with a strategy profile $\sigma$ that constitutes a SSE.

### 4.2 Credible Plans

To recursively characterize $V(s)$, we introduce two definitions.

**Definition 4.5 (Consistency)** *Let $W : S \rightarrow R^3$ denote the set of all equilibrium payoffs. A vector $\psi = (R, c, l, w, \Pi, \{m_1^+, m_2^+\})$ is consistent wrt $W$ at $s$ if*

$$(c, l, w, \Pi) \in CE^S(s, R, \{m_1^+, m_2^+\})$$

*for $(m_1(s,\psi),\ m_2(s,\psi),\ h(s,\psi)) \in W(s)$, and $(m_1^+, m_2^+, h^+) \in W(s^+)$, where the values of $m_1$, $m_2$ and $h$ are given by*

$$m_1(s,\psi) = \frac{1}{c\Pi},\ m_2(s,\psi) = \phi(\Pi - 1)\Pi\frac{l}{c},\ h(s,\psi) = log(c) - \frac{l^{1+\chi}}{1+\chi} + \beta E h^+.$$

**Definition 4.6 (Admissibility)** *The vector $\psi$ is admissible wrt $W$ if it is consistent wrt $W$ at $s$ and*

$$h(s,\psi) \geq h(s,\psi')$$

*for any other consistent $\psi'$.*

Consistency guarantees that the vector $\psi$ delivers an allocation that is optimal for households and feasible. In addition, it requires that the promised continuation values $(m_1^+, m_2^+, h^+)$ belong to the same equilibrium set as $(m_1, m_2, h)$. Admissibility says that the interest rate set by the central bank is optimal and it has no incentive to deviate. That is, the central bank cannot increase its payoff by setting a different interest rate $R'$. A credible plan thus is the strategy of households and central banks instructed by an admissible $\psi$.

With these two definitions, we define an operator $\mathbb{B}$ with its fixed point being the set of equilibrium values $V$ as follows:

For a given set of equilibrium values $W$,

$$\mathbb{B}(W)(s) = \{(m_1, m_2, h) |\ \psi \text{ admissible wrt } W \text{ at } s\}$$

The interpretation of the operator and the constraints is as follows. $\mathbb{B}(\cdot)$ is the convex hull of the payoffs $(m_1, m_2, h)$ such that there are associated values of consumption, labor supply, wage, inflation and government policy rates and next period payoffs that belong to the value correspondence $\mathbb{W}$ for every possible realization of the shock compatible with the current state and that satisfy certain conditions.

Following Phelan and Stacchetti (2001) and Abreu et al. (1986, 1990), the operator $\mathbb{B}$ has the following properties:

**Proposition 4.7** *The operator* $\mathbb{B}$ *has the following properties:*

1. *If* $W \subseteq \mathbb{B}(W)$*, then* $\mathbb{B}(W) \subseteq V$*;*
2. $V$ *is compact and the largest set of equilibrium values* $W$ *such that* $W = \mathbb{B}(W)$*;*
3. $\mathbb{B}(.)$ *is monotone and preserves compactness;*
4. *If we define* $W_{n+1} = \mathbb{B}(W_n)$ *for all* $n \geq 0$*, and the equilibrium value correspondence* $V \subset W_0$*, then* $lim_{n\to\infty} W = V$*.*

***Proof.*** *See Appendix A.5* ■

With results of Proposition 4.7, the equilibrium value correspondence $\mathbb{B}$ is calculated numerically as follows:

$$\mathbb{B}(W)(s)=\left\{(m_1,m_2,h)|\exists R,(c,l,w,\Pi),\ and\ (m_1^+,m_2^+,h^+)\in W(s)\text{ for all }s^+\succ s\right\}\text{ such that}$$

$$m_1=\frac{1}{c\Pi} \tag{15}$$

$$m_2=\phi\left(\Pi-1\right)\Pi\frac{l}{c} \tag{16}$$

$$h=\log\left(c\right)-\frac{l^{1+\chi}}{1+\chi}+\beta\mathbb{E}h^+ \tag{17}$$

$$(m_1,m_2,h)\in W(s) \tag{18}$$

$$h\geq[u(c',l')+\beta\mathbb{E}h^{+'}|(m_1^{+'},m_2^{+'},h^{+'})],\forall(m_1^{+'},m_2^{+'},h^{+'})\in W(s^+) \tag{19}$$

$$1/c=R\beta\mathbb{E}\{m_1^+\} \tag{20}$$

$$w=l^\chi c \tag{21}$$

$$c=(1-\frac{\phi}{2}\left(\Pi-1\right)^2)l \tag{22}$$

$$[(\epsilon-1)-(1-\xi)\epsilon w+\phi\left(\Pi-1\right)\Pi]\frac{l}{c}=\beta\mathbb{E}m_2^+ \tag{23}$$

$$R\geq 1 \tag{24}$$

where $s^+\succ s$ denotes all possible shocks that follow $s$. Constraints (15) to (18) are called "regeneration constraints", while (19) is an "incentive constraint" saying that the strategy corresponding to $(m_1,m_2,h)$ generates a higher payoff to the central bank than any other possible ones. Constraints (20) to (24) are necessary to ensure that continuation of a sustainable plan after any deviation is consistent with a CE.

Following Feng (2015) and Chang (1998), we replace (19) with the following condition,

$$h\geq\tilde{h}\left(s\right) \tag{25}$$

where $\tilde{h}\left(s\right)$ is the worst possible payoff for the central bank if it announces an unexpected interest rate $R'$. $\tilde{h}(s)$ is defined as

$$\tilde{h}(s)=\max_R\left\{\min_{c,l,w,\Pi,(m_1^+,m_2^+,h^+)\in W(s^+)}\left\{\log\left(c\right)-\frac{l^{1+\chi}}{1+\chi}+\beta\mathbb{E}h^+\right\}\right\}$$

such that

$$(c,l,w,\Pi)\in CE^S\{s,R,\Pi,\{m_1^+,m_2^+\},\forall s^+\succ s\}.$$

The idea of replacing (19) with (25) is that: (1) if the central bank deviates from the claimed policy $R$, households will collectively impose the worst punishment for any alternative $R'$ (here is where our coordination assumption has bite); (2) given the response of households, the central bank picks the policy rate that delivers the highest payoff as long as it decides to deviate. Condition (25) is equivalent to (19) in the sense of leading to the same fixed point $V$ by applying the operator $\mathbb{B}$.

Following Feng (2015), the whole equilibrium set can be characterized by the upper and lower boundaries of $W(s)$, which are:[17]

$$\bar{h}(s, m_1, m_2) = \sup_h \{h | (m_1, m_2, h) \in W(s)\} \tag{26}$$

$$\underline{h}(s, m_1, m_2) = \inf_h \{h | (m_1, m_2, h) \in W(s)\} \tag{27}$$

As Phelan and Stacchetti (2001) observe, the lowest value in $W(s)$ yields the value of the worst punishment for the central bank $\tilde{h}(s) = \min_{m_1, m_2} \underline{h}(s, m_1, m_2)$, which corresponds to the equilibrium in which the central bank decides to deviate, that is, WSSE. The highest value $\max_{m_1, m_2} \bar{h}(s, m_1, m_2)$ corresponds to equilibrium in which the central bank obtains the maximum payoff, that is, the BSSE. Since $W(s, m_1, m_2)$ is convex-valued and this paper focuses on the boundaries of the equilibrium value correspondence, we then define the outer approximation of $W$ as follows:

$$\hat{W}(s) = \{(m_1, m_2, h) | h \in [\underline{h}(s, m_1, m_2), \bar{h}(s, m_1, m_2)]\}$$

**Proposition 4.8** *For all $(m_1, m_2, h) \in V(s)$,*

$$\begin{aligned}
\bar{h}(s, m_1, m_2) &= \underset{R}{max} \, \{u(c, l) + \beta \mathbb{E} \bar{h}(s', m_1', m_2')\} \\
\underline{h}(s, m_1, m_2) &= \underset{R}{max} \, \{u(c, l) + \beta \mathbb{E} \underline{h}(s', m_1', m_2')\} \\
\tilde{h}(s) &= \underset{m_1, m_2}{min} \, \underline{h}(s, m_1, m_2)
\end{aligned}$$

*subject to the constraint $(c, l, w, \Pi) \in \boldsymbol{CE}^S\{s, R, m_1', m_2'\}$ for all $s' \succ s\}$, where $\bar{h}(s, m_1, m_2)$ and $\underline{h}(s, m_1, m_2)$ are sup and inf of $V(s)$ for a given $(m_1, m_2)$ as defined by (26) and (27).*

***Proof.*** *See Appendix A.5* ■

[17] Assumptions 1 and 2 in Feng (2015) are easy to check to hold in the current setup.

Proposition 4.8 provides guidelines for computing boundaries of the equilibrium set $V(s)$ and its approximation $\hat{W}$. We define a new operator $\mathbb{F}$ to implement the proposition.

**Definition 4.9 (The operator $\mathbb{F}$)** *For any convex-valued correspondence $\hat{W}$,*

$$\mathbb{F}(\hat{W})(s) = \{(m_1, m_2, h) | h \in [\underline{h}^1, \bar{h}^1]\}$$

*where*

$$\begin{aligned}
\bar{h}^1 &= \max_R \{u(c,l) + \beta \mathbb{E} \bar{h}^0\} \\
\underline{h}^1 &= \max \{\max_R u(c,l) + \beta \mathbb{E} \underline{h}^0, \tilde{h}^0\} \\
\tilde{h}^0 &= \max_R \{ \min_{c,w,l,\Pi,\{m_1', m_2'\}} u(c,l) + \beta \mathbb{E} \underline{h}^0 \}
\end{aligned}$$

*such that the vector $(R, c, w, l, \Pi, (m_1', m_2', h'))$ is admissible with respect to $\hat{W}$ at $s$. To implement $\mathbb{F}$ on a computer, define $\underline{h}(s, m_1, m_2) = -\infty$ and $\bar{h}(s, m_1, m_2) = +\infty$ if no such vector exists.*

The following theorem shows that this operator has good convergence properties and repeated application of this operator generates a sequence of sets that converge to the equilibrium value correspondence $V$. The details of the algorithm are in Appendix A.3.

**Theorem 4.10** *Let $\hat{W}_0$ be a convex-valued correspondence such that $\hat{W}_0 \supset V$. Let $\hat{W}_n = \mathbb{F}(\hat{W}_{n-1})$. Then $\lim_{n \to \infty} \hat{W}_n = V$.*

**Proof.** See Appendix A.5. ■

### 4.3 Recovering Strategies

This subsection shows how to find a strategy that supports the BSSE in the deterministic case. The procedure here can be generalized to find strategies supporting any point (pair of payoffs) belonging to the equilibrium value correspondence in both deterministic and stochastic cases. As we have mentioned, in general many paths support a given payoff.

- Step 1: At $t = 0$, find the highest possible value of $h_0 = \sup\{h | (m_{1,0}, m_{2,0}, h_0) \in W^*(s)\}$ and its corresponding $(m_{1,0}, m_{2,0})$. Then search for the central bank's interest rate policy that supports $(m_{1,0}, m_{2,0}, h_0)$; that is, pick $R_0$ such that

$$u(c_0, l_0) + \beta h_1 = h_0$$

where $h_1 = \bar{h}(m_{1,1}, m_{2,1})$, $m_{1,1} = \frac{u_{c,0}}{R_0 \beta}$, $m_{2,1} = [(\epsilon - 1) - (1 - \xi)\epsilon w_0 + \phi\,(\Pi_0 - 1)\,\Pi_0]\, l_0/c_0/\beta$, and $(m_{1,1}, m_{2,1}, h_1) \in W^*(s)$. Given $(R_0, m_{1,0}, m_{2,0})$, values of $(c_0, l_0, w_0, \Pi_0)$ can be calculated via definitions of $m_1$ and $m_2$, optimality condition of leisure, and market clearing condition as:

$$\begin{aligned}
\Pi_0 &= \frac{\phi(1 + m_{2,0}) + \sqrt{2\phi m_{2,0}^2 + 4\phi m_{2,0} + \phi^2}}{\phi(2 + m_{2,0})}, \\
c_0 &= \frac{1}{m_{1,0}\Pi_0} \\
l_0 &= c_0(1 - \frac{\phi}{2}(\Pi_0 - 1)^2)^{-1} \\
w_0 &= l_0^{\chi} c_0.
\end{aligned}$$

Therefore, the above problem is well-defined in terms of $(R_0, m_{1,0}, m_{2,0}, h_0)$.

- Step 2: $t = 1$, $m_{1,1}$, $m_{2,1}$, $h_1$ are given by the solution in step 1. Now search for the central bank's policy $R_1$ such that

$$u(c_1, l_1) + \beta h_2 = h_1$$

as in step 1.

- Step 3: Repeat step 2 for $t = 2, ...T$, for $T$ sufficiently large.

# 5 Optimal Monetary Policies Without the ZLB

In this section, we first calibrate the model. We then present the solution for the deterministic case and the stochastic case with and without the ZLB. For each case, the solutions are compared with those from RE and MPE. This serves as a first step to understand the role of credibility in conducting monetary policy.

### 5.1 Calibration

The parameterization here mostly follows the literature. The Frisch elasticity is set to 1. The elasticity of substitution among intermediate goods is set to 6. The price adjustment cost $\phi$ is set to 60 to deliver a slope of 1/6 for the Phillips curve. There are four parameters to calibrate for the Markov process of the discount factor $\beta$. The value of $\beta^N$ is set to 0.994 to yield an average 2.5 percent real annual rate during normal times and $\beta^R$ to 1.011 to have a natural rate of $-4.5$ percent for recession times. Setting $p_{RR} = 2/3$ implies an average of three quarters duration of the economy at recession states as estimated by Kulish et al. (2014). Williams (2013) reports that expectations from financial markets consistently showed the federal funds rate lifting off from zero within just a few quarters from 2009 to mid-2011. $1-p_{NN}$ is the frequency of the economy entering the recession state when the current state is normal and set to be 0.0068 to make the likelihood of being in a recession about one quarter every 12 years.[18] These two values are within the ranges used by Nakata (2018) and are summarized in Table 1.

Table 1: Baseline Parameterization

| Parameters | Explanations | Values | Targets/Sources |
|---|---|---|---|
| $\chi$ | Frisch elasticity of labor | 1 | Hall (2009a,b) |
| $\epsilon$ | elasticity among inter. goods | 6 | Fernández-Villaverde et al. (2015) |
| $\phi$ | price adjustment cost | 60 | slope of Phillips curve= 1/6 |
| $\beta^R$ | recession shock of preference | 1.011 | -4.5% natural rate |
| $\beta^N$ | normal shock of preference | 0.994 | 2.5% natural rate |
| $p_{RR}$ | prob. from recession to recession | 2/3 | aver. duration of 3Qs at recession |
| $1 - p_{NN}$ | prob. from normal to recession | 0.0068 | 1 quarter out of 12 years |

### 5.2 The Deterministic Case

#### 5.2.1 Without the ZLB

In the deterministic case, the economy is always in the normal state. Figure 1 shows the equilibrium set of SSE for the deterministic case. The $x$- and $y$-axes are the two auxiliary quantities introduced above, $m_1$ and $m_2$. The $z$-axis is the payoff to the central bank, $h$. A few straightforward notes and

[18] The unconditional probability of a state being recession is $\frac{1-P_{NN}}{2-P_{RR}-P_{NN}} = 0.02$.

observations follow. First, any point $(m_1, m_2, h)$ on and in the mound-shape area is an equilibrium. Second, given $(m_1, m_2)$, there are infinitely many payoffs $h$ to the central bank ranging from $\underline{h}(m_1, m_2)$ to $\bar{h}(m_1, m_2)$. We focus mainly on the BSSE, which is in the set of $\bar{h}(m_1, m_2)$ and on the surface of the mound. Third, there are in general infinitely many combinations of $(m_1, m_2)$ that lead to a certain level of payoff $\bar{h}$. This implies that the central bank can have multiple (and possibly infinitely many) choices of $(m_1, m_2)$ and, hence, nominal interest rates $R$ to induce a particular equilibrium path. Fourth, the shape of the state space implies that there is a trade-off between $m_1$ and $m_2$. To support a certain payoff, any decrease in $m_1$ must be accompanied by an increase in $m_2$. Since a higher $m_2$ implies higher inflation, and a lower $m_1$ implies lower marginal return of saving, this trade-off means that to achieve the same level of payoff, households have a trade-off between higher inflation and a lower return on saving.[19] Finally, given the level of $m_1$, there are, in general, two $m_2$ generating same level of $\bar{h}(s, m_1, m_2)$. Taking the example of $m_1 = 1$, a same level of $\bar{h}$ can be supported by both deflation (low $m_2$) and inflation (high $m_2$).

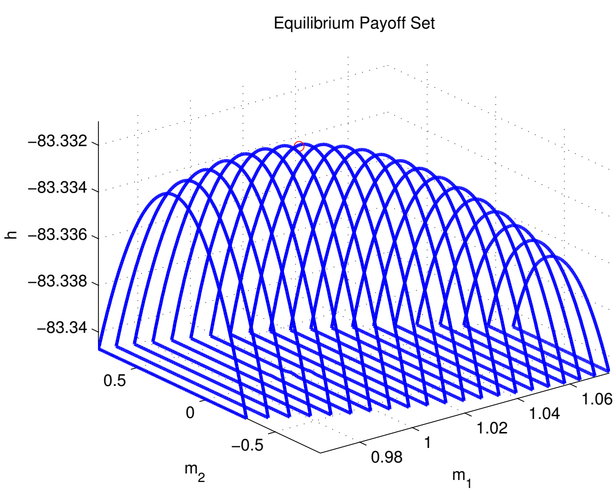



Figure 1: Equilibrium Payoff Set


Figure 2: Cross Section when $m_1 = 1$

The BSSE corresponds to that highest payoff $h$, as denoted by a red circle at the top of the mound in Figure 1. Our first observation is that the BSSE features $(m_1, m_2) = (1, 0)$, of which $m_2 = 0$ means zero inflation.[20] This is also shown in Figure 2, which is a cross-section of the

[19] The return on saving of course also depends on inflation. However, higher inflation also leads to higher adjustment cost and thus more resource loss due to price adjustment.

[20] Since we have limited grids for $m_1$ and $m_2$ and use linear interpolation for values not on grids, computational errors arise when comparing to get the maximum $\bar{h}$.

equilibrium set as a function of $m_2$ when fixing $m_1 = 1$. The maximum $h$ is achieved when $m_2 = 0$. The second observation is that only the BSSE is a steady state that can be supported, whereas the other equilibria are not. These two observations imply a steady state of prices and allocations that coincides with MPE and RE; that is, $\Pi = 1$, $c = l = 1$, $w = 1$, and $R = 1/\beta$.

#### 5.2.2 With the ZLB

The ZLB restricts the central bank from using negative rates to enforce credible plans. To simplify the discussion of this point, we use promised consumption and inflation instead of $(m_1, m_2)$ from now on. Figure 3 shows the supported state space of $(c, \Pi)$, which is the bottom plane of the whole set in Figure 1 with appropriate transformation. The area inside the solid line is the space with the ZLB and the area inside the dashed line is the space without the ZLB. It is clear to see that the supported state space shrinks due to the ZLB. Given promised $c$, the lower bound of promised $\Pi$ that can be supported by credible plans shifts up. Figure 4, which is a cross section of the equilibrium set, shows the same shrink to $\Pi$ when fixing $c = 1$. In other words, households will not believe a low promised inflation that can be achieved in the absence of the ZLB if they know that the central bank is handcuffed by the ZLB. In other words, the ZLB prevents the central bank from reducing nominal rates to have lower real rates when promised inflation is small. Mathematically, the ZLB renders the central bank unable to close the gap between the real rate $R/\Pi$ and the natural rate $1/\beta$. However, the dots with coordinates $(1, 1)$ in Figures 3 and 4 are always within the $(c, \Pi)$ plane, which means that the BSSE can always be supported.

#### 5.2.3 Inflation Bias and Target

Compared to discretionary central banks with only the current period policy instrument $R_t$, central banks with credibility at period $t$ have an enlarged toolkit $\{R_t, R_{t+1}, R_{t+2}, ...\}$, which gives them more freedom to achieve targets. To show the power of this larger toolkit, this subsection deviates temporarily and discusses inflation bias and targets.

Many central banks with implicit or explicit inflation targeting have a positive inflation target in practice. In the United States, for example, this target is 2 percent annually. One justification of this positive target is the inflation bias due to a discretionary central bank's correction for monopolistic pricing distortion (see, for example, Gertler et al., 1999); in the case of RE with full

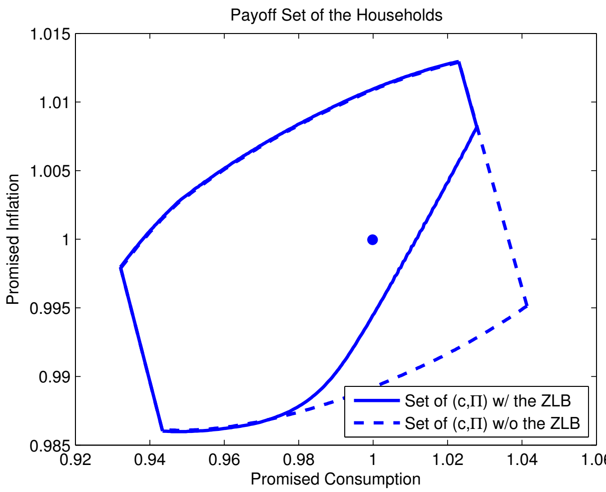



Figure 3: The Sets of Promised Payoffs $(c, \Pi)$

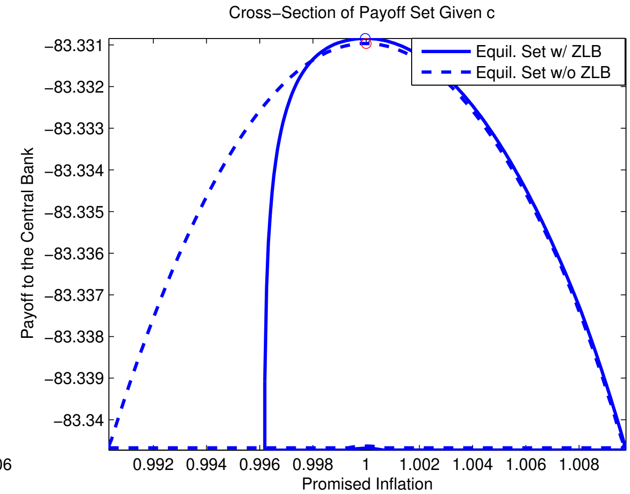



Figure 4: Cross Section when $c = 1$

commitment the inflation bias is always zero, as shown in Nakata (2018) and Dong (2012).[21] This bias comes at the cost of lower consumption, which leads to lower lifetime utility. A central bank with credibility improves outcomes by lowering inflation.

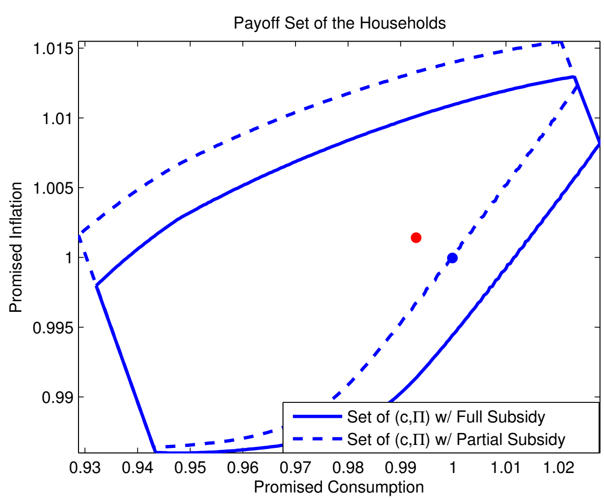



Figure 5: Comparison of SSE sets with full (solid line) and partial (dashed line) subsidy. The blue (red) dot is BSSE under full (partial) subsidy.

In the benchmark, the parameter that governs the subsidy $\xi$ is set to $1/\epsilon$ to completely eliminate the steady state distortion due to monopolistic pricing (full subsidy). Figure 5 displays the

[21]The logic of this argument is the desire for discretionary central banks to reduce unemployment if fiscal policy fails; see, for example, Kydland and Prescott (1980), Barro and Gordon (1983), Ruge-Murcia (2004), and Kim and Ruge-Murcia (2009). Another argument for inflation bias is the fear of plunging into a deep recession or liquidity trap; see for example, Cukierman (2002), Coibion et al. (2012), and Yellen (2012).

equilibrium set if $\xi$ is reduced to 88 percent of the benchmark level (partial subsidy). The first observation is that, not surprisingly, having only a partial subsidy moves the whole SSE set to where inflation is in general higher given any promised consumption of the same level. Households know that the central bank has an incentive to use higher inflation to correct the monopoly distortion and will not trust the promised consumption and inflation bundles from the case of a full subsidy. Second, the BSSE also shifts to higher inflation, but with a bias that is smaller than that under MPE. The BSSE with a partial subsidy has $m_2 = 0.0821$, which corresponds to an annual inflation rate of 0.52 percent in the steady state. In contrast, the implied annual rate in the MPE is 0.94 perecent. In other words, a central bank with credibility is able to have a smaller target than one with discretion; a central bank with a high inflation target is one that lacks credibility. Lastly, Table 2 shows the implied inflation bias and target for different fiscal subsidies. If the fiscal subsidy is smaller and the central bank needs to correct more of the pricing distortion, there is a larger gap between the inflation biases of MPE and BSSE. These results will also hold in the stochastic economies.

Table 2: Fiscal Subsidy and Implied Inflation Targets (%)

| Fiscal Subsidy | MPE | SSE |
|---|---|---|
| 100% | 0 | 0 |
| 88% | 0.94 | 0.52 |
| 76% | 1.85 | 1.12 |
| 64% | 2.74 | 1.60 |

### 5.3 The Stochastic Case Without the ZLB

The equilibrium set for the stochastic case without the ZLB is a combination of the set for $\beta^N$ as in subsection 5.2.1, and a similar one for $\beta^R$. Note that the set for $\beta^R \geq 1$ cannot exist independently because it violates transversality. Given that there are no distortions or ZLB constraints, the BSSE features an efficient allocation for any period $t$ and state $s$, that is, $c_t = l_t = \Pi_t = w_t = 1$. The associated policy rate and promised values are $R_t = 1/\beta_t$ and $(m_{1t}, m_{2t}) = (1, 0)$ for all $t$. In this case, the BSSE is identical to the RE and the unique MPE.

# 6 Forward Guidance and Credible Monetary Policy

We now move to a discussion of forward guidance. The relevant message from the results above is that when the nominal rate is flexible enough (not constrained by the ZLB) and there is a full fiscal subsidy, the optimizing central bank always has a zero inflation bias and delivers the efficient consumption level in each period ($c_{t+1} = c_t = 1$ for any $t$). We define any deviations from zero inflation as an inflation gap and call any discrepancy between the real rate ($R_t/E\Pi_{t+1}$) and the natural rate ($1/\beta_t$) a shadow rate gap. The inflation gap measures the intratemporal distortion between consumption and leisure – excessive inflation creates resource losses that lead to lower consumption and more work effort. The shadow rate gap measures the intertemporal distortion between consumption today and tomorrow – excessively high interest rates lead to low consumption today. The goal of the central bank is to use the current period policy instrument $R_t$ or the enlarged toolkit $\{R_t, R_{t+1}, ...\}$ to close these two gaps.

In the absence of the ZLB, the central bank can simultaneously close both of these two gaps to zero with only $R_t$ (the divine coincidence). In the presence of the ZLB, however, the central bank may not be able to close both gaps, even with the enlarged toolkit. In this case, there is a trade-off between these two gaps, which in turn will create a time-inconsistency problem.

## 6.1 The Time-inconsistency Problem

To illustrate the issue of time-inconsistency, we assume a particular realization of shocks. Suppose that the recession shock $\beta^R$ hits the economy at period $t$ and is followed by normal shocks thereafter. The natural rate at period $t$ is $1/\beta^R$, which is smaller than 1. The real interest rate $R_t/E\Pi_{t+1}$ should decrease to close the gap. In the case of the MPE, the central bank today cannot affect inflation tomorrow (in the absence of endogenous states, the MPE bank is powerless to affect the future in any manner), and therefore the shadow rate gap will not be closed even at the minimum value of $R_t = 1$. Consequently, the real rate is higher than the natural rate. Therefore, the households save more and consume less, leading to weak aggregate demand and deflation, opening the inflation gap.

In the RE, the central bank has another way to close the shadow rate gap when $R_t$ is bounded below by 1. The central bank can raise expected inflation by promising to have zero nominal rates tomorrow, no matter what happens in period $t+1$. By doing so, aggregate demand in the current period will increase, in turn reducing the inflation gap in period $t$. The result is increased consumption and a less severe deflation.

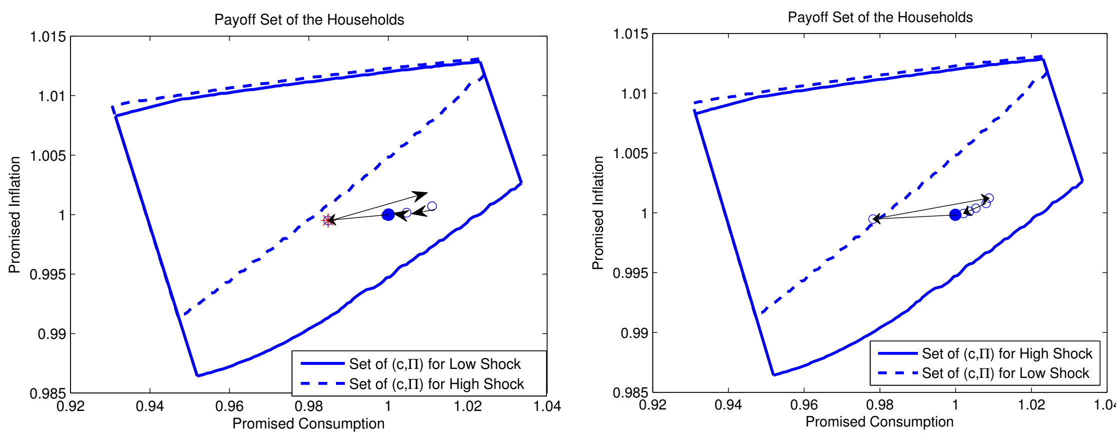
The time-inconsistency arises in period $t+1$. With the economy having returned to the normal state, the natural rate will be $1/\beta^N$. If the central bank follows its promise of having $R=1$, then it will then have a shadow rate gap of $1/E\Pi_{t+2} - 1/\beta^N$. To simplify, suppose that the central bank need only promise one period and is able to stick to a policy supporting the efficient allocations after period $t+2$, so that $E\Pi_{t+2} = 1$. The shadow rate gap will be $1 - 1/\beta^N < 0$, which leads to excessive aggregate demand and a positive inflation gap. These two gaps are costly to the central bank and they will renege and close them by setting $R = 1/\beta^N$ and $\Pi = 1$. Clearly, this response is not consistent with what the central bank promised in period $t$.

The key to the time-inconsistency problem is the temptation to close the two gaps without cost when the economy reverts back to its normal state. While the RE simply excludes the ability to deviate, an SSE introduces a finite punishment that may outweigh the temptation. Given the worst punishment that the households can enforce, the content of the statement becomes key. The private actors in the economy consider two aspects of the central bank announcement regarding future interest rates: the level of the interest rate and the length of time it will be unusually low. If the promised rate is too low or the duration of low rates is too long, then the forward guidance will not be credible.

## 6.2 RE is Not Implementable

Before examining the optimal policy in the BSSE, it is useful to first investigate whether the RE is implementable or not. The strategy is to calculate the corresponding promised values to the private sector and continuation payoff to the central bank for the Ramsey case. We then check if the triplet $(c, \Pi, h)$ from the RE dynamics lies within the equilibrium set of SSEs. For this discussion, we assume that shocks materialize in the following manner: a recession shock $\beta^R$ hits the economy only at period 1, preceded and followed by normal shocks. We first check the implementablility of the RE by looking at the promised values $(c, \Pi)$. The associated values and the sustainable sets under SSE are drawn in Figure 6(a). The area framed by a solid line is the sustainable equilibrium set for the normal shock state and the one by a dashed line is the set for the recession shock state.

The filled circle is the promised consumption and inflation bundle before the recession shock hits the economy. The star is the bundle at the time of the shock. Unfilled small circles are the promised values after the shock, which revert to the position of the filled circle following a clockwise orbit, as shown by the arrows. All of the circles lie in the sustainable set for the normal state. However, the star, the promised value in the recession state in the RE, lies outside the sustainable set for that state, the dashed line area – the promised consumption and inflation bundle under the RE is not available. In contrast, the promised values under BSSE lie in the sustainable set both in and out of the recession state as shown in Figure 6(b).




(a) RE (b) BSSE

Figure 6: Dynamics of promised consumption and inflation under RE and BSSE. The area in the solid line is the supported payoff plane with normal shock, while the dashed line area the set with recession shock. The circles are the calculated consumption and inflation after the one-time recession shock.

A second way to check the implementablility is to see if the maximized lifetime utility from RE is always within the payoff range of the central bank under SSE. Mathematically, we need to check if $V^{RAM}(s^t) \in [\underline{h}(s, m_1, m_2), \bar{h}(s, m_1, m_2)]$ for any $(s, m_1, m_2)$. If there exists any $V^{RAM}(s^t)$ that do not belong to $[\underline{h}(s, m_1, m_2), \bar{h}(s, m_1, m_2)]$, then we say RE is not implementable.

Figure 7 draws the lifetime utility of RE for a group of given realizations of shocks. The dashed lines represent value sequences after the recession shock hits for 1, 2, 3, 4, 5 and 10 periods. The solid line connects the values of first after-shock period (the dots). The red line is the payoff for the BSSE. The first observation is that there are after-shock periods (the dots) that are higher than the payoff under BSSE. In the time of periods 2, 3, and 4, which are the first periods after the recession ends, the payoff to the central bank under RE is higher than that under BSSE. The second observation is that the payoffs under RE are always higher than those under BSSE if recession has ended more than two periods ago. For example, the payoff in period 7 is higher than that under BSSE after a 5-period recession. These two observations also show that RE is not implementable.

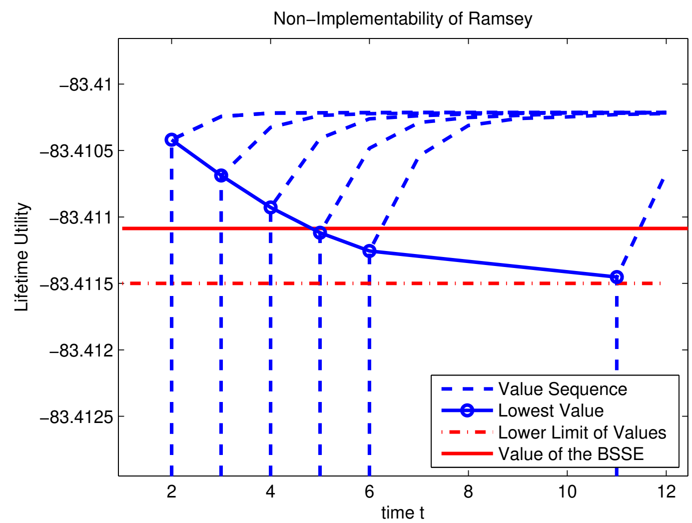
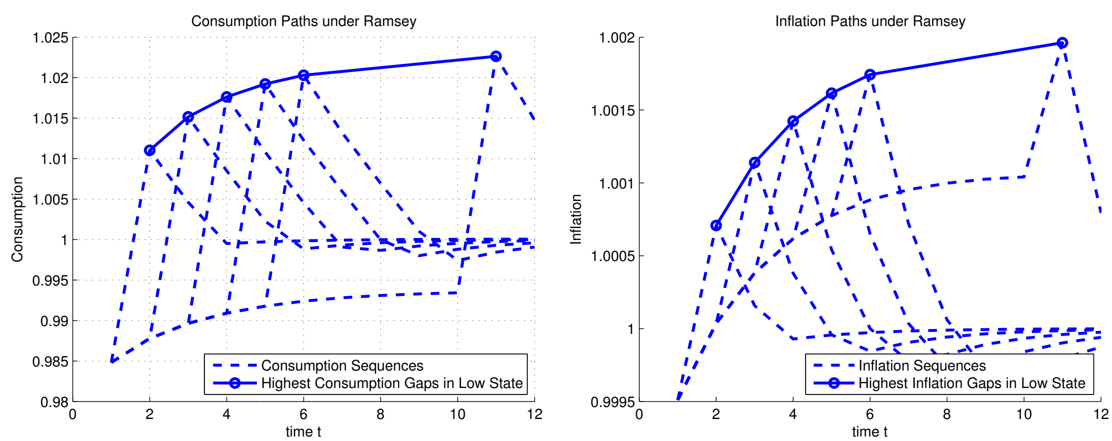



Figure 7: The non-implementability of RE: Compare payoffs to the central bank. The dashed lines represent value sequences for realizations of 1, 2, 3, 4, 5, and 10 periods of recession shocks. The solid line connects the values of first after-shock period (the dots). The red solid line is the payoff of BSSE.




(a) Consumption

(b) Inflation

Figure 8: Consumption and inflation paths after recession ends. Dashed lines represent consumption and inflation paths since recession shock first hits the economy and lasts for 1, 2, 3, 4, 5, and 10 quarters, respectively. The solid line connects the highest consumption gaps associated with each realization of shocks.

Three points are worth discussing here. First, if the committed central bank is hypothetically allowed to re-optimize (and this re-optimization is assumed by households to occur with zero probability before it happens), then it will always deviate from the promised values in the first period after the recession disappears. For a given realization of shocks, Figure 7 shows that the first after-shock period delivers the lowest payoff compared to the hypothetical re-optimization level, which is the horizontal dashed line. This result can also be seen in the shadow rate and inflation gaps. Figure 8(a) shows the shadow rate gaps compared to the efficient level ($c = 1$). It is clear that the gaps right after the recession shocks are the largest. For example, if the recession shock hits the economy for only one period, then the shadow rate gap is largest in the second period, decreasing quickly to nearly zero by period 4. The same is true for inflation gaps, as shown in Figure 8(b).

Second, in recessions of longer duration, the gaps between the commitment payoff and hypothetical re-optimization after the recession ends will be larger. This result means that there is a stronger temptation for the central bank to deviate. One can also see this from consumption and inflation gaps in Figures 8(a) and 8(b). Therefore, if the central bank can get a higher continuation payoff by deviating when a recession shock hits the economy for the first time, then it will surely deviate if the recession continues. However, the reverse is not true – the central bank may be able to commit in short recessions but not protracted ones.

Finally, there is a maximum limit of utility loss for the central bank to commit and not to re-optimize as the duration of recession increases, shown by the decreasing solid line in Figure 7 and the increasing solid lines in Figures 8(a) and 8(b). In other words, the gain for the central bank to deviate is limited. This opens the possibility of making a central bank to commit to increasing frequency or persistence of recessions, for example, as in Nakata (2018). However, after experimenting over a wide range of key parameter values (the recession shock size $\beta^R$, the recession persistence $p^{RR}$, the recession frequency $1 - p^{NN}$, and the price adjustment cost $\phi$), we find no evidence that the RE is implementable. This result is in stark contrast to Nakata (2018), who finds that the RE is implementable given the parameterization in this paper. While we have essentially the same idea of using punishment to enforce a central bank to stick to its promises, Nakata (2018) uses MPE as the only permissible deviation; in contrast, we allow deviation to any SSE, and find that the central bank would deviate from the RE to the BSSE. Since the BSSE is better than the

MPE, it cannot sustain as large a set of equilibria, including RE.

### 6.3 Credible Forward Guidance

In this subsection, we will answer the question raised at the beginning: what is the best credible forward guidance? Or, in other words, how does the policy rate behave if a central bank plays the BSSE strategy? We also show that forward guidance under BSSE provides a better account of the Fed's policy after 2008 than either RE or MPE.

Our strategy for construcing the policy outcomes follows Subsection 4.3. Since there exist infinitely many policy paths that support the same equilibrium, we have refined the strategy as described in Appendix A.4 to find the policy paths of interests. First, we only pick strategies that support the upper boundary of central bank's payoff set; that is, $\bar{h}(s, m_1, m_2)$. Second, central banks may have unmodeled preferences over what level of policy rate to have in different states. For example, one central bank may prefer to have high rates during recessions and low rates during normal times while another central bank may prefer the opposite. We back out all of the possible combinations of maximum high and minimum low rates at normal and recession states to guarantee that we have both upper and lower bounds. Finally, to account for the monetary practice of the Fed since 2008, we focus on the path that features the lowest interest rates when a recession starts and continues to have low rates after the recession ends.

Figure 9 shows the range of policy rates for a specific realization of shocks, where we assume that at period 1, the economy is hit by a recession shock and this shock persists for six periods. The length of recession is to match the NBER definition of the recent financial recession (2008Q1-2009Q2). The dashed line is the upper bound of the nominal rates and the dash-dotted one is the lower bound for all of the paths we find. Note that neither the upper or lower bound necessarily corresponds to a given path; that is, it is not an equilibrium to follow the lower bound. Our first observation is that the ZLB barely binds in recession times in the BSSE. The only chance that it will bind happens during the period when the high shock disappears. This result is mainly due to the size of $\beta^R$. A higher $\beta^R$ can lead the central bank to set the nominal rate to zero during the recession.[22] Second, the upper and lower bounds are not smooth, because these bounds are the supremum of rates at a given period for different selected paths. Finally, the specific path that

[22] However, this result does not change the conclusions we make about forward guidance.

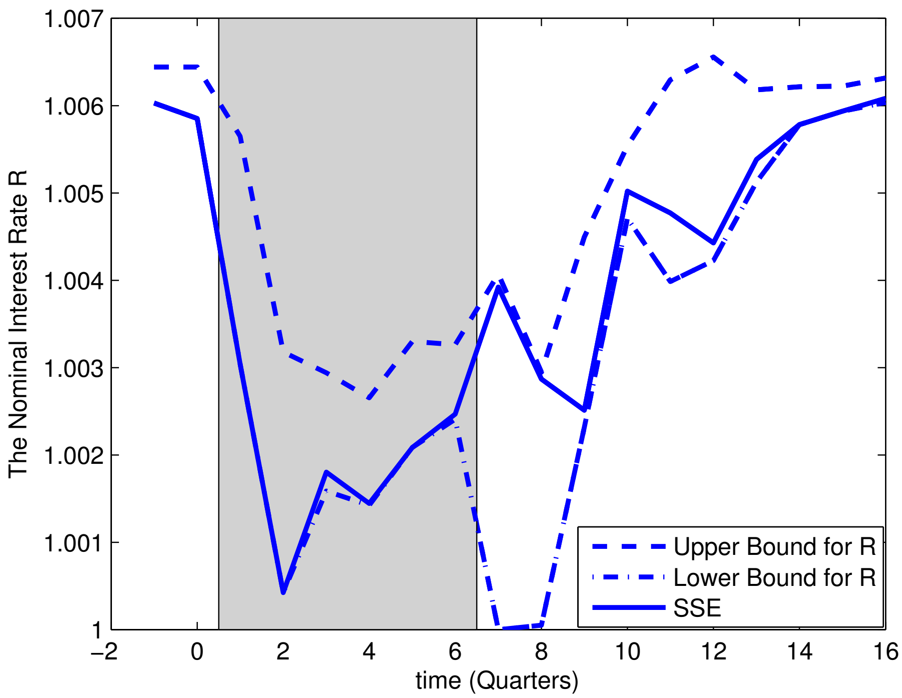



Figure 9: The upper and lower bounds and a specific path of policy rates for BSSE. The recession shock starts to hit the economy at time 1 and stays for another five quarters.

features lowest nominal rates when recession starts is represented by the blue solid line. Consistent with the bounds, this path does not touch the ZLB when recession starts. As the recession unfolds, it gradually moves back to its normal level.[23]

The properties of the specific policy path under BSSE are clearer when compared to that in data (Figure 10) and those under RE and MPE (Figures 11(a) and 11(b)). First, when the recession starts at period 1, the central bank under RE and MPE reacts aggressively to cut the rate to zero; this result also holds for many models with optimized Taylor Rules (see, for example, Reifschneider and Willams, 2000 and Fernández-Villaverde et al., 2015). These models suggest not to "keep your powder dry" by waiting in case the future is even worse. In contrast, the central bank under BSSE does not react so aggressively and only sets the interest rate at about half that in normal times (1.2 percent compared to 2.5 percent). We see the same pattern in the data – it took the Fed several quarters to get to a zero rate after the recession began in the first quarter of 2008. While many observers take this as a result of time lag of monetary policy, the policy here indicates that cautious

[23]The non-smoothness of this policy path is due to computational inaccuracies and has no economic content. At substantial computational cost we can reduce the irregularities in the path but that delivers no additional insights.

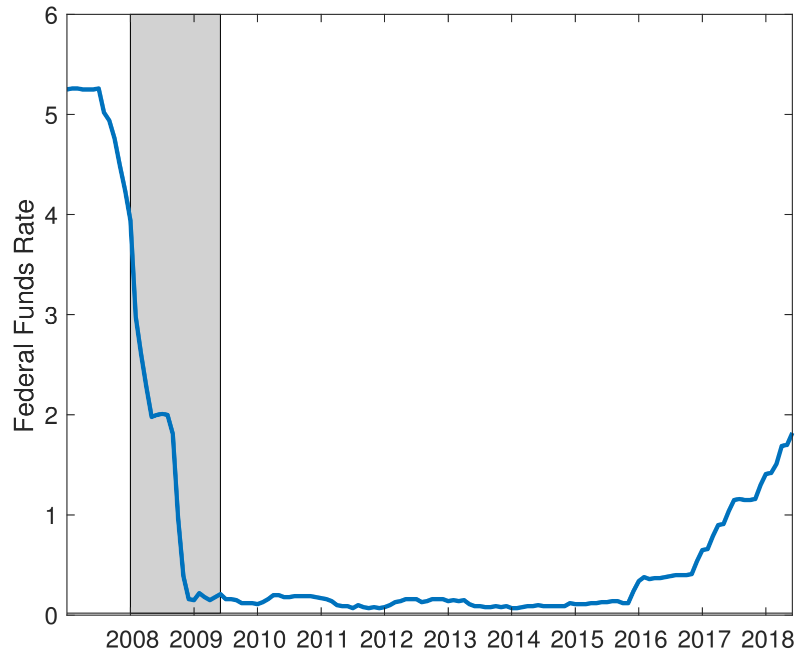
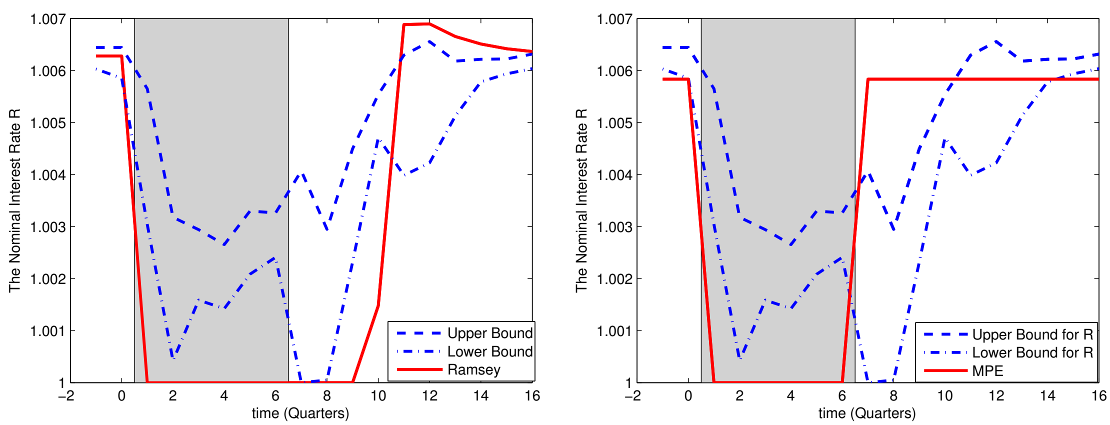



Figure 10: Federal funds rate from 2007Q1-2018Q2; The shaded area is the NBER definition of the Great Recession of 2008Q1-2009Q2. (Source: FRED)




(a) RE

(b) MPE

Figure 11: Dynamics of policy rates under RE and MPE

and sluggish response to a recession may come arise from credibility concerns.[24]

Second, under RE, although the recession ends in period 6, the nominal rate stays low another

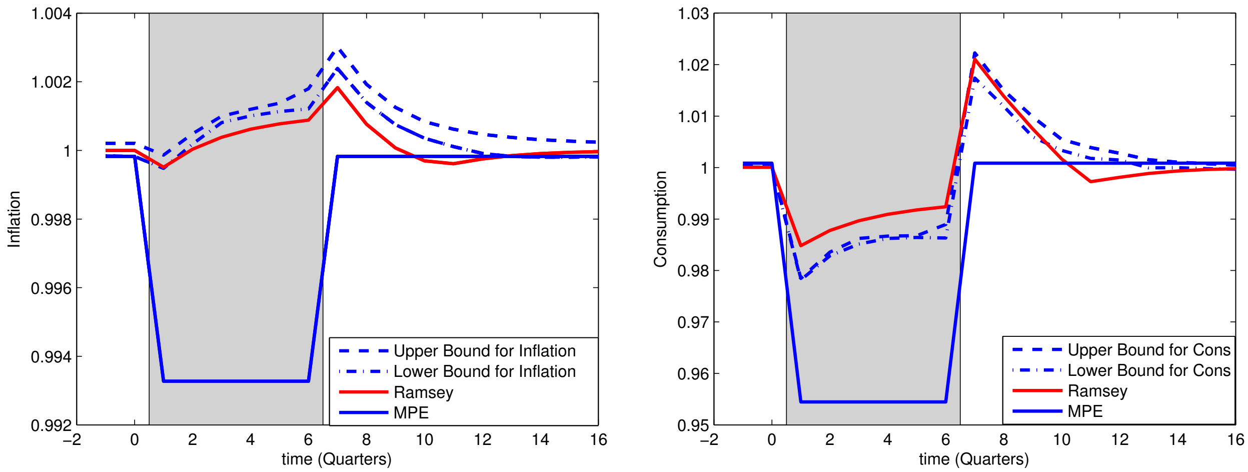
[24]Policymakers are generally dismissive of the idea that interest rates determination must occur with a substantial lag (the Open Market Committee could easily meet within a few days); the effects of course might take longer (Friedman's famous long and variable delays).

four quarters (three quarters at zero and one quarter close to zero) even after the economy emerges from recession. The nominal rate then jumps to a higher-than-normal level to contain inflation. Under BSSE, the nominal rate stays low but is non-zero for at least eight quarters after the recession ends. Furthermore, the nominal rate increases as the probability of exiting recession becomes larger. Why is this so? If the central bank as a player sets the nominal interest rate too low from the rate at normal times, then it is understood by households that the central bank has large current gains to a deviation. Consequently, the households will not trust the central bank and the equilibrium falls apart. The credible policy path must be relatively smooth in all states and sloped upward to reduce the incentive for the central bank to renege.

Third, a credible central bank keeps monetary policy accommodative for a substantially longer time. Compared to zero length under MPE and the one year length under RE in which the policy rate is lower than normal times, the central bank in the BSSE has a lower than normal policy rate for more than eight quarters, in order to compensate for its inability to have an aggressively low rate. This property of the BSSE matches the data along two dimensions. First, the Fed's accommodation period was very long – the US did not liftoff from the ZLB until 24 quarters after the recession ended. Second, the liftoff has been gradual and smooth. Some researchers have argued that this gradualism is intended to avoid a weak recovery and the resulting downturn uncertainty (Plosser, 2012); we provide an alternative view that gradualism comes hand-in-hand with credibility. Finally, normalization does not mean non-accommodation. As long as the rate is lower than what should be (which will last some time), the monetary policy stance is accommodative. This result is consistent with Nakata and Schmidt (2018), who show that gradualism helps to boost the economy during liquidity traps.

We now turn in issues of welfare: how effective is the central bank at using the enlarged toolkit to close the inflation and shadow policy gaps? Figure 12(a) shows that credible central banks have higher and more persistent inflation gaps compared to the RE. It takes at least three quarters longer after a recession for inflation to first revert to its normal level under BSSE than under RE. The shadow rate gap is also larger under BSSE because higher inflation is associated with (even) higher nominal rates. As a result, the BSSE delivers welfare lower than RE (which was already noted because RE is not implementable).

Finally, compared to a central bank with full discretion, the central bank with credibility is




(a) RE

(b) MPE

Figure 12: Dynamics of inflation and consumption

more effective at boosting consumption during recessions. Given the baseline parameterization, the BSSE central bank has an average shadow rate gap of only $-1.55$ percent compared to $-4.55$ percent under MPE and $-1.05$ percent under RE. It is appropriate to emphasize here that although our discussions so far have focused on the specific policy path that we have chosen, there are many other paths that also support the BSSE. For example, the central bank can set a high rate during a recession and normal times to achieve the same payoff; however, this path would look strange to a practitioner.

### 6.4 Rule-based Forward Guidance

In practice central banks implement forward guidance according to an implicit or explicit policy rule, rather than a complicated state-contingent rule, presumably because they are easier both to actually use and to communicate to the private sector. Here, we consider how the various different equilibrium rules can be approximated by "Taylor-like" rules that are functions of inflation, output gaps, and lagged rates only. We simulate the economy under BSSE, RE, and MPE $1 \times 10^6$ times and approximate a Taylor-style rule for the central bank; that is, what Taylor Rule looks most like the equilibrium rule? Under BSSE with the specific policy rate path picked previously, we obtain

$$r_t = 1.84 + 0.18 r_{t-1} - 1.94\pi_t + 0.025 y_t + 2.22\pi_t^2 - 0.02 y_t^2 + \epsilon_t,\ R^2 = 0.75,$$

where $\pi_t = \Pi_t - 1$ is the net inflation rate and $y_t$ is the percentage deviation of output from the steady state (the squared terms, while nonstandard, improve the fit of the regression substantially).[25] In contrast, under RE we obtain

$$r_t = 1.19 + 0.52 r_{t-1} - 2.16\pi_t + 0.08 y_t + 7.27\pi_t^2 - 0.06 y_t^2 + \epsilon_t,\ R^2 = 0.92.$$

First, the average interest rate under BSSE is close to that under RE; the ranking of these numbers can switch if we choose a different interest rate path. Second, the interest rate responds to inflation negatively under both RE and BSSE. Positive inflation is a sign of recession, as shown in Figure 12(a). Whenever inflation is high, the interest rate should decrease to boost the economy. This is different from what is usually encompassed in the Taylor principle, which states that the nominal rate should increase more than one-to-one with inflation to contain the latter. Third, the interest rate reacts more aggressively to inflation and output gaps under RE than under BSSE. Under RE, the elasticity of nominal rates with respect to inflation is about three times as big as that under BSSE. Similarly, the reaction of the nominal rate to the output gap under RE is about three times as big as that under BSSE. In other words, if the central bank has full commitment, then it can and should act more aggressively to adjust its policy rate in response to inflation and output gaps, closing output and inflation gaps as quickly as possible.

These rules relate interest rate to contemporaneous inflation and output. Suppose now the central bank announces forward guided rate for the future only based on current inflation and output, then what will the rate look like? To shed light on this issue, we run the following regressions:

$$r_{t+j} = \beta_0 + \beta_{\pi,j}\pi_t + \beta_{y,j} y_t + \epsilon_{t+j}$$

where $r_{t+j}$ is the policy rate $j$ periods forward. It should be noted that the rule-based forward guidance here is an approximation of the optimal polices and it has a flexible form that reflects the dynamic change of forward guidance over the time horizon. However, the rule-based forward guidance in the literature is either non-optimal or has exogenous parameters (see, for example, Katagiri, 2016). Figure 13 plots $\beta_\pi$ as a function of $j$. First, the average horizon of forward

[25] A positive squared inflation term implies that the central bank raises rates more strongly when inflation is far from the steady state.

guidance is about 10 to 12 quarters. The coefficients beyond this time horizon are basically zero, which means that, on average, the central bank does not use forward guidance beyond this time horizon to tackle the recession.[26] Second, as in the instantaneous response of the current policy rate to inflation, the forward-guided policy rate goes down when inflation is high. The logic is again that positive inflation signals recession and the central bank should continue to keep interest rates low to boost the economy. However, when that date is further away, the degree of adjustment to inflation today is smaller. Third, the forward-guided policy is, in general, much less sensitive to inflation under BSSE than under RE. Given the same time horizon, the interest rate elasticity to inflation in the BSSE is about 1/3 to 1/2 of that under RE.

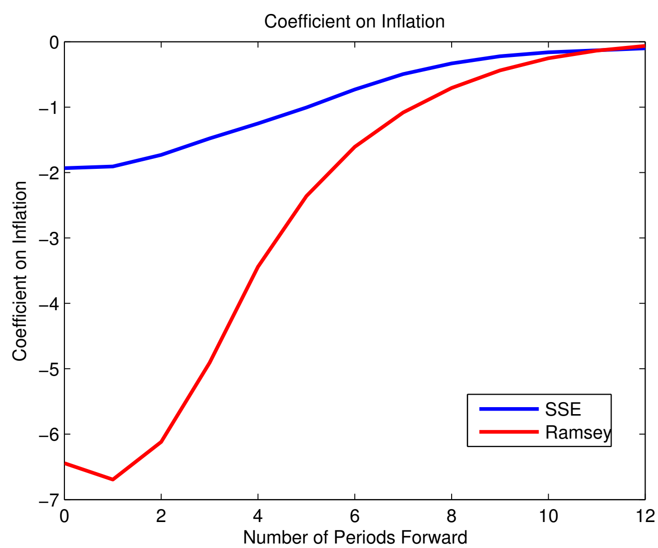



Figure 13: The Elasticity of Forward Guided Rates to Inflation

The reason for the difference of the rules is the same as explained earlier – the aggressive policy is not credible. This issue should be raised in any discussion regarding the use of aggressive monetary responses in order to ensure determinacy – to what extent is the Taylor principle undermined by the requirement of commitment?

Under MPE, because the output gap and inflation gap comove perfectly, we cannot run the same regression (we could if we add shocks that induce some opposing movements, such as markup shocks, but these come at significant computational cost). Instead, we get the rule only based on

[26]The threshold for the time span of forward guidance depends on the parameter values that we have picked. However, given the experiments that we did with a wide range of possible parameter values, forward guidance is largely an issue within horizons of less than three years.

inflation or output as follows:

$$r_t = 2.40 + 0.8841\pi_t + \epsilon_t, \ R^2 = 1,$$

$$r_t = 2.35 + 0.1355y_t + \epsilon_t, \ R^2 = 1.$$

In contrast with the rules under RE and BSSE, the Taylor-like rule under MPE is very simple: if inflation is negative (output is negative at the same time), then decrease the nominal rate appropriately.[27] However, note that the credible rule would violate the Taylor principle, since the coefficient is less than 1 on inflation.

## 7 Conclusion

This paper has explored the optimal and credible policy in a standard New Keynesian model by characterizing the entire set of sequential sustainable equilibria in the presence of a zero lower bound. In contrast to Ramsey and Markov-Perfect equilibria, the best sustainable sequential equilibrium provides a better account of the Fed's policy practice during and after the recent recession: (1) forward guidance based on BSSE states that the interest rate should stay low but non-zero for a prolonged time even after the economy recovers; (2) normalization does not mean non-accommodation; the monetary policy can be accommodative even when the rate is lifting off from the ZLB; and (3) simulated rule-based forward guidance suggests less aggressiveness in response to inflation when credibility is a requirement. Finally, the quantitative results show that RE is not generally implementable given the sound range of parameter values.

The model that we have used is very stylized. As a result, there are several limitations of the model that are worth discussing. First, we abstract from endogenous state variables, such as capital and price dispersion, in order to ease the computation of SSEs. Endogenous state variables provide discretionary central banks an additional channel to mitigate liquidity traps and they likely lead to gradual adjustment of policy rate under both RE and MPE; our expectation is that the BSSE will also have more gradual adjustment. Second, again for computation purposes, we use a two-state Markov chain instead of an AR(1) process to characterize shocks. As a result, the

[27] The reason that $R^2 = 1$ for these two regressions here is that MPE only has two states with two values for inflation and output. Thus, the regression with two parameters captures 100% of the variation in the data.

gaps are large when the economy recovers from recession under RE, which increases the benefit of deviations and the set of parameters for which the RE cannot be implemented. Third, in order to bring the ZLB into play, we study only discount factor (natural rate) shocks; these shocks lead to unusual-looking Taylor rules and the lessons may not extend unaltered to cost-push or supply shocks. A more complete exploration of the issue of forward guidance should therefore focus on extending our model to include these additional ingredients.

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# A Appendix

## A.1 The Model with Implementable Policy Rates

In Section 2, we presented a standard New Keynesian model that features private bonds. And at equilibrium, the net position of bonds is always zero. This assumption has been criticized for ignoring the issue of the implementability of policy rates, i.e., the central bank in this model has no way to realize the policy target it wants. Here we present a model similar to that in Section 2, but with government bonds and hence central banks' implementation of policy rates through open market operations.

The representative household acts to maximize same lifetime utility as in (1) subject to, however, a different budget constraint,

$$c(s^t) + \frac{B^g(s^t)}{P(s^t)} = w(s^t)l(s^t) + R(s^{t-1})\frac{B^g(s^{t-1})}{P(s^t)} + \tau(s^t) + d(s^t) + \Delta m(s^t) \tag{28}$$

where $R(s^{t-1})$ is now interpreted as the nominal interest rate paid to government bonds $B^g(s^t)$. We now interpret $\frac{B^g(s^t)}{P(s^t)} - R(s^{t-1})\frac{B^g(s^{t-1})}{P(s^t)}$ as the change of households' monetary demand and $\Delta m(s^t)$ the real change of monetary supply. At each period, the demand and supply of money much be equalized, that is

$$\Delta m(s^t) = \frac{B^g(s^t)}{P(s^t)} - R(s^{t-1})\frac{B^g(s^{t-1})}{P(s^t)} \tag{29}$$

Therefore, any policy target $R_t$ can be implemented by changing $\Delta m(s^t)$ given the households' bond holdings ($B^g(s^{t-1})$ and $B^g(s^t)$) and price ($P(s^t)$). This change of budget constraint does not change any optimality conditions or feasibility conditions. Note that though we introduce an extra state variable $B^g(s^t)$, it is irrelevant for the dynamics of the economy.

## A.2 Firm's Dynamic Problem

This subsection shows that given the promised values about adjusting prices, the firm solves the same recursive problem as without it. The dynamic problem of an individual firm can be written as follows:

$$V(P_{it-1}) = \lambda_t d_{it} + \beta_t E_t V(P_{it})$$

subject to constraint (5) and the definition of dividend $d_{it}$. $\lambda_t$ is the Lagrangian multiplier of households' budget constraint. Solving the problem gives the following Euler equation:

$$\left[(\epsilon-1)(\frac{P_{it}}{P_t})^{-\epsilon} - (1-\xi)\epsilon w_t + \phi\left(\frac{P_{it}}{P_{it-1}} - 1\right)\frac{P_t}{P_{it-1}}\right] y_t\lambda_t = \beta_t E_t\left[\left(\frac{P_{it+1}}{P_{it}} - 1\right)\frac{P_{it+1}P_t}{P_{it}^2}\right]\phi y_{t+1}\lambda_{t+1}$$

Imposing symmetry gives the equation (6) in the text:

$$\left[(\epsilon-1) - (1-\xi)\epsilon w_t + \phi\left(\Pi_t - 1\right)\Pi_t\right] y_t\lambda_t = \beta_t E_t\left[\left(\Pi_{t+1} - 1\right)\Pi_{t+1}\right]\phi y_{t+1}\lambda_{t+1}$$

Now let $m_2^+ = [(\Pi_{t+1} - 1)\,\Pi_{t+1}]\,\phi y_{t+1}\lambda_{t+1}$ be the marginal value (payoff) of adjusting price relative to aggregate price. Then the firm's problems becomes:

$$V(P_{it-1}) = \lambda_t d_{it} + \beta_t E_t m_2^+ P_{it}/P_t$$

Solving and imposing symmetry gives exactly the same solution as the original recursive problem.

### A.3 Numerical implementation of the operator $\mathbb{F}$

Let $\boldsymbol{S}\times\boldsymbol{M_1}\times\boldsymbol{M_2}\times\boldsymbol{H}$ denote the space of all equilibrium state vectors and associated payoffs to the central bank $(s, m_1, m_2, h)$. $\boldsymbol{W} : \boldsymbol{S} \to \boldsymbol{M_1}\times\boldsymbol{M_2}\times\boldsymbol{H}$ is a correspondence from $\boldsymbol{S}$ to $\boldsymbol{M_1}\times\boldsymbol{M_2}\times\boldsymbol{H}$.

With an initial guess $\boldsymbol{W}^0(s) = \{(m_1(s), m_2(s), h(s))\}$ and a pre-determined tolerance level $\epsilon$, the algorithm goes as follows:

- Step 1: For $\forall s \in \boldsymbol{S}$, find $\boldsymbol{\Omega}(s) := \{(m_1, m_2, h)|(m_1, m_2, h) \in \boldsymbol{W}^0(s), \exists R \in [\underline{R}, \bar{R}]$ and $(m_1', m_2', h') \in \boldsymbol{W}^0(s')$ such that :

$$\begin{aligned}
h &= u(c,l) + \beta\mathbb{E}h' \geq \tilde{h}^0(s) \\
Em_1' &= \frac{1}{c}\frac{1}{R\beta} \\
Em_2' &= \frac{1}{\beta}\left[(\epsilon-1) - (1-\xi)\epsilon w + \phi\,(\Pi-1)\,\Pi\right]\frac{l}{c} \\
m_1 &= \frac{1}{c\Pi} \\
m_2 &= \phi(\Pi-1)\Pi\frac{l}{c} \\
w &= l^{\chi}c \\
c &= (1-\frac{\phi}{2}(\Pi-1)^2)l
\end{aligned}$$

where

$$\begin{aligned}
\bar{h}^0(s, m_1(s), m_2(s)) &= \underset{h}{max}\{h|(m_1, m_2, h) \in \boldsymbol{W}^0(s)\} \\
\underline{h}^0(s, m_1(s), m_2(s)) &= \underset{h}{min}\{h|(m_1, m_2, h) \in \boldsymbol{W}^0(s)\} \\
\tilde{h}^0(s) &= \underset{(m_1,m_2)}{min} \underline{h}^0(s, m_1, m_2)
\end{aligned}$$

- Step 2: For $\forall s \in \boldsymbol{S}$, and $\boldsymbol{\Omega}(s)$, denote $\boldsymbol{\Omega}^M(s, h) := \{(m_1, m_2)|(m_1, m_2, h) \in \boldsymbol{\Omega}(s), h = h(s, m_1, m_2)\}$, and define

$$\begin{aligned}
\bar{h}^1(s, m_1, m_2) &= \underset{R}{max} \underset{\substack{c,l,\Pi,w, \\ (m_1', m_2', h') \in \boldsymbol{W}^0(s')}}{max} u(c, l) + \beta \mathbb{E} \bar{h}^0(s', m_1', m_2') \\
\underline{h}^1(s, m_1, m_2) &= max\{\underset{R}{max} \underset{\substack{c,l,\Pi,w, \\ (m_1', m_2', h') \in \boldsymbol{W}^0(s')}}{min} u(c, l) + \beta \mathbb{E} \underline{h}^0(s', m_1', m_2'), \tilde{h}^0(s)\}
\end{aligned}$$

for all $(m_1, m_2) \in \boldsymbol{\Omega}^M(s, h)$. Otherwise, set

$$\begin{aligned}
\bar{h}^1(s, m_1, m_2) &= +\infty \\
\underline{h}^1(s, m_1, m_2) &= -\infty
\end{aligned}$$

Further, let

$$\tilde{h}^1(s) = \underset{(m_1,m_2) \in \boldsymbol{\Omega}^M(s,h)}{min} \underline{h}^1(s, m_1, m_2)$$

- Step 3: Define $\boldsymbol{W}^1(s) = \{(m_1, m_2, h)|(m_1, m_2) \in \boldsymbol{\Omega}^M(s, h), h \in [min\{\bar{h}^0(s, m1, m2), \underline{h}^1(s, m_1, m_2)\}, max\{\underline{h}^0(s, m_1, m_2), \bar{h}^1(s, m_1, m_2)\}]\}$

- Step 4: Set $\boldsymbol{W}^* = \boldsymbol{W}^1$ if $\|\boldsymbol{W}^1 - \boldsymbol{W}^0\| < \epsilon$; otherwise, set $\boldsymbol{W}^0 = \boldsymbol{W}^1$ and repeat the steps above.

In the deterministic case, we set the number of grids for $m_1$ and $m_2$ to be 200, the number of points for interest rate 500 with the range $[1, 1.1]$, implying a 8 basis point change when optimizing. In the stochastic case, we reduce the number of grids for $m_1$ and $m_2$ to be 50 while keeping the grids for interest rates the same.

### A.4 Algorithm to Find All $R(m_1, m_2)$

Give $m_2$, $\Pi$ is fixed. With $m_1$, $c$, $l$, and $w$ are also determined. Expected payoffs to the firms are also prefixed as $Em_2' = [(\epsilon - 1) - \epsilon(1-\xi)w - \phi(\Pi - 1)\Pi]\frac{l}{c\beta}$. Expected payoffs to households, however, depend on $R$ as $Em_1' = \frac{1}{\beta c}\frac{1}{R}$. Since the best SSE can be supported by many $(m_1, m_2)$, there may exist different $(m_1', m_2')$ that lead to the same level payoff to the central bank. To find all such payoffs and hence interest rate, we find out all $(m_{11}', m_{21}', m_{12}', m_{22}')$ that satisfy the following two equations:

$$\omega\bar{h}(m_{11}', m_{21}') + (1-\omega)\bar{h}(m_{12}', m_{22}') = E\bar{h}(m_1', m_2')(m_1, m_2) \tag{30}$$

$$\omega m_{21}' + (1-\omega)m_{22}' = Em_2'(m_1, m_2) \tag{31}$$

where $\omega$ is the probability of the economy in state 1 next period given the state today. Note that the right hand sides of the two equations above are functions of $(m_1, m_2)$ and are fixed for the current problem.

The algorithm goes as follows:

- Step 1: Pick $(m_1, m_2)$ and hence $Em_2'(m_1, m_2)$ and $E\bar{h}(m_1', m_2')(m_1, m_2)$. Solve for $m_{21}' = (Em_2'(m_1, m_2) - (1-\omega)m_{22}')/\omega$.

- Step 2: Pick $I$ and $J$ grids for $Em_1'$ and $m_{11}'$ within their range. Given the $i^{th}$ and $j^{th}$ grids, calculate the corresponding $m_{11}' = (Em_1' - (1-\omega)m_{12}')/\omega$. Substituting $m_{11}'$ and $m_{21}'$ back to (30), we define a new function:

$$g = \omega\bar{h}((Em_1' - (1-\omega)m_{12}')/\omega, (Em_2'(m_1, m_2) - (1-\omega)m_{22}')/\omega) + (1-\omega)\bar{h}(m_{12}', m_{22}') - E\bar{h}(m_1', m_2')(m_1, m_2)$$

- Step 3: Fixing $m_{11}'$, $g$ is a function of $m_{22}'$. Use root finding solver to solve for the roots of $g = 0$. (There are at most 2 roots because $h$ is convex along the dimension of $m_2$.)

- Step 4: Keep the roots if any. Go to step 3 if $j \leq J$. Go to step 2 if $j > J$. Go to step 1 if $i > I$.

## A.5 Proofs

**Proposition 4.2.** To simplify the exposition, we abstract away from uncertainty. In the text, we have shown that the sequential and recursive problems lead to the same Euler equation. The only thing left is to show that the transversality condition holds.

Let $b_t = B_t/P_t$. To prevent Ponzi schemes, we assume that there are debt limits for $b_t$, that is, $b_t \in [\underline{b}, \bar{b}]$, where $\underline{b_t} > -\infty$ and $\bar{b_t} < \infty$. Since $R \leq \bar{R} < \infty$, it is sufficient to show that $m_t \leq \bar{m} < \infty$. From the goods market clearing condition, the upper (lower) bound of inflation is $\bar{\Pi} = 1 + \sqrt{2/\phi}$ ($\underline{\Pi} = 1 - \sqrt{2/\phi}$). If inflation is out of this range, then all the goods produced will be used for compensating price adjustment.

Given the preference of the household, we have $lim_{c\to 0} u_c(c,l) = \infty$ and $lim_{l\to 0} u_l(c,l) = 0$, which means that the household will be better off by spending a strictly positive amount of time $\underline{l} > 0$ in working so that he can obtain some income to finance a positive amount of consumption. From the first-order condition $w = cl^{\chi}$, we have $c = wl^{-\chi}$. So what is left to show is there is a lower limit for the wage $w$. From the firms' first-order condition (6), it is easy to see that $w = \frac{(1-\epsilon)-(1+\beta)\phi\Pi(\Pi-1)}{\epsilon(1-\xi)}$ is lower bounded. Therefore, there exists upper bound $\bar{m}$. Finally, $lim_{t\to\infty} \beta^t m_t b_t R_t \leq lim_{t\to\infty} \beta^t \bar{m}\bar{b}\bar{R} = 0$. ■

**Lemma 4.4.** The proof follows closely Phelan and Stacchetti (2001). ■

**Proposition 4.7.** The proof follows closely Phelan and Stacchetti (2001). ■

**Proposition 4.8.** The proof here follows Feng (2015). By definition, $\bar{h}(s, m_1, m_2)$ is the maximum value of $h$ given $(s, m_1, m_2)$, which is:

$$
\begin{aligned}
\bar{h}(s, m_1, m_2) &= \underset{R}{max} \underset{\{m_1', m_2', h'\}}{max} \left[u(c,l) + \mathbb{E}\{h(s', m_1', m_2')\}\right] \\
&= \underset{R}{max} \left[ u(c,l) + \underset{\{m_1', m_2', h'\}}{max} \mathbb{E}\{h(s', m_1', m_2')\} \right] \\
&= \underset{R}{max} \left[u(c,l) + \mathbb{E}\{\bar{h}(s', m_1', m_2')\}\right]
\end{aligned}
$$

where the first equality follows the definition of $\bar{h}(s, m_1, m_2)$, the second equality follows the fact that the instant utility only depends on promised values $(m_1, m_2)$, and the last equality uses the definition of $\bar{h}$.

A similar argument applies to $\underline{h}(s, m_1, m_2)$. A few comments go as follows. First, $\underline{h}(s, m_1, m_2) =$

$\underset{R}{max} \underset{\{m_1', m_2', h'\}}{min} u(c,l) + \beta \mathbb{E} h'$. Second, it should be noted that the value of $u(c,l) + \beta \mathbb{E} \underline{h}'$ at given $R$ might be smaller than $\tilde{h}(s)$, which says that the incentive constraint is not satisfied when the government has the lowest continuation value. When this happens, the government needs a higher continuation value so that the incentive constraint is satisfied. However, the corresponding payoff for the present government cannot be higher than $h(s)$. This is because only the minimization operates when $R$ is given. There always exists $h' \in [\underline{h}(s, m_1, m_2), \bar{h}(s, m_1, m_2)]$ that satisfies the incentive constraint when the worst continuation value violates the incentive constraint. Otherwise, $(m_1, m_2)$ would not belong to the equilibrium value correspondence. Note that $\tilde{h}(s)$ is the payoff of the worst SSE and must lie in the lower boundary of $h(s, m_1, m_2)$. Because it is the worst of all, it must be equal to $min_{\{m_1, m_2\}} \underline{h}(s, m_1, m_2)$. ■

**Theorem 4.10.** The proof follows Feng (2015) and first shows that the sequence of $\hat{W}_n$ is decreasing and $\hat{W}_n \ni \hat{W}_{n+1}$. Since $W_n$ is convex-valued, it is sufficient to show that the upper boundary is decreasing and the lower one increasing. The upper boundary is decreasing because of the fact that $\bar{h}^1(s, m_1, m_2)$ is defined as $max_R u(c,l) + \beta \mathbb{E} \bar{h}^0(s', m_1', m_2')$ such that $\psi = (R, c, l, w, \Pi, \{m_1', m_2', h'\})$ is admissible wrt $\hat{W}^0$ at $s$. The admissibility of the vector $\psi$ implies that $(m_1, m_2, \bar{h}^1(s, m_1, m_2)) \in \hat{W}_0(s)$. Therefore, $\bar{h}^1(s, m_1, m_2) \leq max\{h | (m_1, m_2, h) \in \hat{W}^0(s)\} = \bar{h}^0(s, m_1, m_2)$. Similarly, we can show that the lower boundary is increasing, i.e., $\underline{h}^1(s, m_1, m_2) \geq \underline{h}^0(s, m_1, m_2)$. The same argument thus holds for $\hat{W}_n(s)$. Since the sequence is decreasing, it has a limit $\hat{W}_\infty$. Proposition 4.7 implies that $\mathbb{F}(\boldsymbol{V}) = \boldsymbol{V}$. By a simple limit argument, we have $\lim_{n \to \infty} \hat{W}_\infty = V$. ■