Neoclassical Inequality*

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Abstract

In a model with a worker-capitalist dichotomy, we show that the relationship between inequality (measured as a ratio of incomes for the two types) and growth is complicated; zero growth generally lowers inequality, except under extreme parameterizations. In particular, the elasticity of substitution between capital and labor in production needs to be substantially greater than one in order for income inequality be higher with zero growth, in fact higher than nearly all estimates. If this condition is not met, factor prices adjust strongly causing the fall in the return to capital (the rise in wages) to reduce income inequality. Our results extend to models with endogenous growth.

Keywords: Inequality; Growth; Elasticity of Substitution

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1 Introduction

The effect of low growth on the distribution of income and capital has gained attention recently. Certainly the most prominent example is Thomas Piketty's *Capital in the Twenty-First Century* which documents the dynamics of wealth inequality over hundreds of years and across several countries. In addition, Piketty puts forth an economic model to account for these data. Roughly speaking, Piketty's model has two groups of households, laborers and capitalists, who derive all of their income from a single source (labor and capital, respectively); in this environment a natural measure of inequality is capital's share of national income. Under the assumption that capitalists set net saving equal to a constant fraction of net output, a decline in growth to zero leads to an explosion in inequality. This result has been found in models where wealth is accumulated exogenously through multiplicative random shocks (Piketty and Saez 2014) and in models where wealth enters directly into the utility function of households (Piketty and Zucman 2015). Krusell and Smith (2015) show, however, that under more standard assumptions about capitalist saving, the rise in inequality from low growth is less dire than Piketty predicts: with constant gross saving out of gross income, the increase in inequality is substantially smaller (but, it should be noted, still quite large).

Our goal in this paper is to characterize the relationship between growth and inequality in a macroeconomic model minimally extended to generate inequality. Our model has the following ingredients: some households, called capitalists, own claims to the productive technology while other ones, called laborers, do not; both types have an endowment of time that can be rented to firms in return for labor income and their labor productivity is time-invariant. We can analytically characterize many features of the relationship between growth and inequality (measured as the ratio of capitalist income to laborer income), and we use numerical tools to uncover the behavior of income inequality as the long run growth rate of the economy goes to zero; we consider both the short-run and long-run effects (transitions and steady states).

We find that steady states with zero growth generally have lower inequality than steady states with positive growth rates. When growth is low, capitalists discount the future at a lower rate, and thus accumulate more capital; however, this accumulation leads to an abundance of capital relative to labor and results in higher equilibrium wages, both absolutely and relative to capital's return. In terms of inequality, the movement in relative factor prices away from capital and toward labor mitigates the effects of increasing wealth. Generally, the factor price effect dominates the effect of the wealth increase. Zero growth steady states are associated with higher inequality only if the elasticity of substitution in production between capital and labor is substantially greater than one (i.e., strong substitutes in production). A high elasticity of substitution mutes the factor price effect by preventing wages from becoming "too large" relative to the return to capital.

The elasticity of substitution cannot be arbitrarily high, however. Conditional on a particular capital share in production, balanced growth places a limit on the elasticity of substitution: the higher is capital's share the lower is the upper limit on the elasticity of substitution. We find that the explosive increase in long run inequality only occurs when the elasticity of substitution is very close to its upper bound. Moreover, the values required for this explosion in inequality to obtain are starkly at odds with empirical measurements of the elasticity of substitution given estimates of capital's share of income; to be specific, if capital's share of income is 0.36 (as documented by Gollin 2002 for a large sample of countries) the required elasticity is 1.33, which is higher than most of the estimates surveyed by Chirinko (2008) (only one extreme outlier estimate is significantly higher than 1.33, and only two others are close to this value). Note that these estimates are specifically aimed at measuring the long-run elasticity of substitution, which is the relevant one for our study – short-run elasticities are likely to be even smaller.[1]

We then characterize how the inequality-growth relationship changes in the presence of redistribution via capital income taxation. We consider two cases – either tax revenue is rebated lump-sum to all households (uniform transfers) or only to laborer households (targeted transfers). Not surprisingly, we find that redistribution does reduce inequality, but it operates by shrinking the income of the laborer household by less than that of the capitalist household, rather than by increasing the laborer's income and decreasing the capitalist's. Furthermore, the minimum elasticity of substitution needed to get higher inequality at zero growth is not sensitive to the value of the capital tax, provided we remain on the upward-sloping portion of the Laffer curve.

Our experiments crucially assume that the growth rate is an exogenous parameter; we do not allow for feedback that goes from interest rates to growth rates. While this assumption has the virtue of not requiring us to take a stand on why growth falls (it is simply a decline in the growth

[1] Rognlie (2015) notes that net elasticities are smaller than gross elasticities, so that extreme values for the gross elasticity are even less plausible.

rate of exogenous labor-augmenting technology), it is obviously limiting – if interest rates and growth rates are jointly determined, they may not behave in the same manner. With this point in mind, we extend the model to include a production externality, as in Romer (1986), that renders the social production function constant returns to scale in capital. Once again, we find that zero growth only produces extreme inequality when the elasticity of substitution between capital and labor is well above 1. The key is that the tight link between returns and growth – namely that falling growth must come with falling returns – still obtains in endogenous growth models with two types of agents.[2]

2 Model

The model economy is populated by two groups, called capitalists and workers, who are situated in dynasties that live forever and value the utility of descendants; the size of the two groups are $\mu \in (0, 1)$ and $1 - \mu$. Both groups have identical isoelastic preferences over consumption and leisure (non-work time). Our main assumption is that there is no mobility across groups – at some point in the infinite past, some dynasties were lucky enough to get granted access to a productive asset called capital, and some were not, and that situation has persisted.

Both groups supply labor elastically and have constant labor productivities. We normalize the productivity of laborers to 1 and denote by e the labor productivity of capitalists. Throughout the paper, we maintain that $e \geq 1$, and generally will assume $e = 1$. One effective unit of labor earns w units of wage as compensation, and capital pays a gross return $1 + r$.

[2] Other types of growth models reduce to similar social production functions; see Hammond and Rodríguez-Clare (1993). Pure AK-style models would have no labor income for laborer households and would therefore not be suitable for studying the question at hand.

2.1 Household Problems

2.1.1 Laborers

Given the current stock of capital in the economy, the laborer's problem is a static choice of how many hours, l, to supply at market wage w:

$$V(K) = \max_{l \in [0,1)} \left\{ \frac{\left[wl(1-l)^{\theta}\right]^{1-\sigma}}{1-\sigma} + \beta(1+g)^{1-\sigma} V(K') \right\}. \tag{1}$$

The solution is

$$l^* = \frac{1}{1+\theta} \tag{2}$$

$$x^* = wl^* \tag{3}$$

where x^* is the laborer's consumption.

2.1.2 Capitalists

A capitalist chooses consumption, hours, and savings to solve the dynamic program

$$v(k, K) = \max_{k', h, c} \left\{ \frac{\left[c(1-h)^{\theta}\right]^{1-\sigma}}{1-\sigma} + \beta(1+g)^{1-\sigma} v(k', K') \right\} \tag{4}$$

subject to

$$c + (1+g)k' \leq whe + (1+r)k \tag{5}$$

$$k' \geq 0, c \geq 0, h \in [0,1).$$

Since the first two boundary conditions will never bind, we ignore them from now on. Taking the first-order conditions and applying the envelope condition produces three conditions:

$$\left[c(1-h)^{\theta}\right]^{-\sigma}(1-h) = \beta(1+g)^{-\sigma}\left[c'(1-h')^{\theta}\right]^{-\sigma}(1-h')(1+r') \tag{6}$$

$$\left[c(1-h)^{\theta}\right]^{-\sigma}\left[we(1-h)^{\theta} - \theta c(1-h)^{\theta-1}\right] \leq 0 \tag{7}$$

$$c + (1+g)k' = whe + (1+r)k; \tag{8}$$

the second condition holds with equality if $h > 0$.

Capitalists choose consumption c, work effort h, and capital holdings k' to maximize lifetime utility; we have already incorporated growth in labor productivity g in the usual method to ensure the (normalized) wealth of the capitalist remains bounded over time (see King, Plosser, and Rebelo 1988 for details on how this normalization is done). We require that the inverse of the intertemporal elasticity of substitution satisfies $\sigma > 0$.[3]

We can obtain the aggregate capital stock and labor input by summing over all individuals.

$$K = \mu k \tag{9}$$

$$N = \mu h e + (1-\mu)\, l. \tag{10}$$

Note the asymmetry – capitalists supply all the capital, but labor is (at least in principle) supplied by both types; note also that aggregate labor input is in terms of "effective" units of labor (hours weighted by productivity) and that both types' hours are perfect substitutes in the production of effective hours.

2.2 The Firm

The supply side of our economy consists of a single firm employing a constant returns to scale production technology (nothing would change if we had a large number of identical firms, except notation would be more tedious):

$$Y = \left(\alpha K^{\nu} + (1-\alpha)\, N^{\nu}\right)^{\frac{1}{\nu}}, \tag{11}$$

where $\alpha \in (0,1)$ is the "share" of capital in production and $\nu \leq 1$ governs the elasticity of substitution. If $\nu = 1$, capital and labor are perfectly substitutable, so that the firm will employ only the cheaper factor. If $\nu = -\infty$, capital and labor are perfect complements, and therefore will be employed in fixed ratios (given by $\frac{\alpha}{1-\alpha}$). If $\nu = 0$, we get the Cobb-Douglas case where the shares of capital and labor income in total income will be fixed at α and $1-\alpha$.[4] Profit

[3] Moll (2014) shows that the model is completely analytically tractable if $\sigma = 1$ for capitalist households. We survey briefly the literature on estimating σ later in the paper, as this parameter plays an important role in some of our results.

[4] We experimented with a form of capital-skill complementarity where capitalist hours are a complement to capital and laborer hours are a substitute to the composite of capital and capitalist hours. We found that our results are not changed; further details are available upon request.

maximization yields

$$r = \alpha\left(\alpha + (1-\alpha)\left(\frac{K}{N}\right)^{-\nu}\right)^{\frac{1-\nu}{\nu}} - \delta \tag{12}$$

$$w = (1-\alpha)\left(\alpha\left(\frac{K}{N}\right)^{\nu} + 1 - \alpha\right)^{\frac{1-\nu}{\nu}}. \tag{13}$$

Note that both the rental rate and the wage rate are related to the capital-labor ratio, but not to the levels of capital and labor.[5]

Finally, there are aggregate conditions that relate supply and demand in each of three markets – the markets for capital, labor, and "goods". First, the firm must hire all the capital and labor supplied by households (these conditions are ensured by appropriate movements in r and w). Second, the supply of goods must be sufficient to cover the consumption of capitalists, the consumption of workers, and the investment by capitalists into new capital:

$$\mu c + (1-\mu)x + \mu\left(k' - (1-\delta)k\right) = Y. \tag{14}$$

Walras's law ensures that the goods market condition will be satisfied provided both the labor and capital markets clear.

2.2.1 General Equilibrium

A recursive competitive equilibrium is a set of household functions

$$\left\{V(K), v(k,K), h(k,K), c(k,K), k'(k,K), l(k,K), x(k,K)\right\},$$

price functions $r(K)$ and $w(K)$, and aggregate labor $N(K)$ such that

1. Given pricing functions, the household functions solve the capitalist and laborer problems;

2. Given pricing functions the firm maximizes profit by demanding K and $N(K)$;

3. Markets clear:

$$K = \mu k'(K,K)$$

[5] The assumption that firms operate in perfectly competitive goods markets is not restrictive; assuming monopolistic competition does not change our results. It only changes the market clearing condition for capital by subtracting the value of profits from the savings of capitalists when determining the value of aggregate capital.

$$N(K) = \mu h(K,K)\,e + (1-\mu)\,l(K,K)$$

$$Y(K) = \mu c(K,K) + (1-\mu)\,x(K,K) + \mu(1+g)\,k'(K,K) - (1-\delta)\,\mu K.$$

2.3 Steady State

The balanced growth path is characterized by the system of equations

$$1 = \beta(1+g)^{-\sigma}(1+r) \tag{15}$$

$$h = \max\left\{\frac{we - \theta(r-g)\,k}{(1+\theta)\,we}, 0\right\} \tag{16}$$

$$c = whe + (r-g)\,k \tag{17}$$

$$l = \frac{1}{1+\theta} \tag{18}$$

$$x = w(r)\,l \tag{19}$$

$$w(r) = (1-\alpha)^{\frac{1}{\nu}}\frac{r+\delta}{\alpha}\left[\left(\frac{r+\delta}{\alpha}\right)^{\frac{\nu}{1-\nu}} - \alpha\right]^{\frac{\nu-1}{\nu}}. \tag{20}$$

The steady state Euler equation pins down r,

$$r = \frac{(1+g)^{\sigma} - \beta}{\beta}, \tag{21}$$

which through the first-order conditions of the firm determines the steady state wage rate

$$w = (1-\alpha)^{\frac{1}{\nu}}\frac{r+\delta}{\alpha}\left[\left(\frac{r+\delta}{\alpha}\right)^{\frac{\nu}{1-\nu}} - \alpha\right]^{\frac{\nu-1}{\nu}}. \tag{22}$$

Notice that for $\sigma > 0$, the steady state interest rate is increasing in g. If we restrict attention to non-negative growth rates, the interest rate attains its minimum and the wage rate its maximum when $g = 0$, where the interest rate is

$$r_{\min} = \frac{1-\beta}{\beta}$$

and the wage is

$$w_{\max} = (1-\alpha)^{\frac{1}{\nu}}\frac{r_{\min}+\delta}{\alpha}\left[\left(\frac{r_{\min}+\delta}{\alpha}\right)^{\frac{\nu}{1-\nu}} - \alpha\right]^{\frac{\nu-1}{\nu}}. \tag{23}$$

Notice that while $r_{\min}$ is determined only by the discount factor, $w_{\max}$ also depends upon capital's share in production, α, and the elasticity of substitution parameter, ν. Figure 1 plots the steady state wage when $g = 0$. For higher values of ν, the steady state wage increases exponentially, and the slope is increasing in α. Not all combinations of α and ν are permissible since

$$\alpha^{\frac{1}{1-\nu}} < (r_{\min} + \delta)^{\frac{\nu}{1-\nu}} \tag{24}$$

must hold for wages to be real numbers. Given (α, β, δ), the upper bound on ν is $\nu_{\max} = \frac{\log(\alpha)}{\log(r_{\min}+\delta)}$. In order to allow for capital and labor to be either complements or substitutes in production, we impose that $\alpha > \frac{1-\beta}{\beta} + \delta$, which implies $\nu_{\max} > 0$. Under the baseline calibration (see below), $r_{\min} \approx 0.0101$, implying that $\nu_{\max} \approx 0.305$. Thus, balanced growth puts a restriction on the degree to which capital and labor are substitutable; letting $\xi = \frac{1}{1-\nu}$ denote the elasticity, we find $\xi_{\max} = \frac{1}{1-0.305} = 1.438$.

Under appropriate conditions for α and ν, we can find K by imposing the capital market clearing condition at r:

$$\begin{aligned} K &= \left\{ \frac{\left[\frac{r+\delta}{\alpha}\right]^{\frac{\nu}{1-\nu}} - \alpha}{1-\alpha} \right\}^{-\frac{1}{\nu}} N \\ &= \varphi N \end{aligned} \tag{25}$$

where

$$N = \mu h e + \frac{1-\mu}{1+\theta}. \tag{26}$$

Note here that φ is the capital-to-labor ratio. Since under the restrictions on α and ν

$$\frac{d\varphi}{dr} = -\frac{1}{\alpha\ (1-\alpha)\ (1-\nu)} \left(\frac{\left(\frac{r+\delta}{\alpha}\right)^{\frac{\nu}{1-\nu}} - \alpha}{1-\alpha} \right)^{-\frac{1+\nu}{\nu}} \left(\frac{\alpha}{r+\delta} \right)^{\frac{1-2\nu}{1-\nu}} < 0 \tag{27}$$

and $\frac{dr}{dg} > 0$, the capital-labor ratio will be higher in a low-growth economy. Finally, the steady state relative return on capital (as compared to human capital) $r - g$ falls if and only if

$$\sigma (1+g)^{\sigma-1} > \beta,$$

which, near $g = 0$, requires $\sigma \geq 1$. Estimates for σ in the literature run from essentially infinite (Hall 1988, Campbell 1999) to close to one for the subgroup of stock market participants (Guvenen

2006) to significantly below one for that same group (Vissing-Jørgensen 2002). To get $r-g$ to rise with low growth, the latter estimates must not apply; given the survey discussion in Havránek (2015), it seems reasonable to assume that $\sigma \geq 1$ is most plausible even for the capitalists, and he concludes that the median estimate corrected for publication bias is somewhere around 3 with a minimum of roughly 1.2. More recently Crump et al. (2016) use the new Survey of Consumer Expectations to obtain a tight estimate of σ just above one. We will use $\sigma = 2$ as representative of the macroeconomic literature.

Aggregate effective labor is a function of K because the capitalist hours decision depends upon wealth. If wealth is sufficiently high, then the non-negativity constraint on hours binds. We consider both the binding and non-binding cases below.

2.3.1 Case 1: Capitalists Work

Under the assumption that capitalists supply positive hours, we can obtain aggregate capital by substituting the definition of h into the market clearing condition for capital,

$$\begin{aligned} K &= \varphi N \\ &= \frac{\varphi\left(\frac{\mu e+(1-\mu)}{1+\theta}\right)}{1+\varphi\frac{\theta}{1+\theta}\frac{(r-g)}{w}}. \end{aligned} \tag{28}$$

It can be shown that

$$\frac{w}{\varphi} = \left(\frac{r+\delta}{\alpha}\right)^{\xi} - (r+\delta) > 0, \tag{29}$$

where $\xi = \frac{1}{1-\nu}$ is the elasticity of substitution between capital and labor. The strict inequality results from imposing the restriction $\nu < \nu_{\max}$. Multiplying the numerator and denominator of (28) by $\frac{w}{\varphi}$, we arrive at

$$K = \frac{w\left[\mu e + (1-\mu)\right]}{(1+\theta)\left[\left(\frac{r+\delta}{\alpha}\right)^{\xi} - (r+\delta)\right] + \theta\,(r-g)}.$$

Individual capitalist wealth is

$$k = \frac{K}{\mu} = \frac{\left[\mu e + (1-\mu)\right]}{\mu}\frac{w}{r-g}\frac{1}{(1+\theta)\frac{r}{r-g}z\,(g) + \theta}.$$

Here

$$z\,(g) = \frac{\left(\frac{r+\delta}{\alpha}\right)^{\xi} - (r+\delta)}{r} = \frac{\frac{w}{r}}{\frac{K}{N}} > 0, \tag{30}$$

where z depends upon the growth rate through r.

Since z can be expressed as $\frac{wN}{rK}$, we call it the *aggregate factor income (AFI) ratio.* As will be made clear below where we derive a closed-form expression for steady state inequality, the behavior of inequality at low values of g depends crucially upon z. If factor prices are slow to adjust relative to $\frac{K}{N}$, then z will be low and inequality will be high. On the other hand, if factor prices are very responsive to imbalances in $\frac{K}{N}$ then inequality will be low. The elasticity of substitution in production directly governs this responsiveness.

Continuing to solve for inequality, by substituting k into h, we obtain

$$h = \frac{1}{1+\theta}\left[1 - \frac{\mu e + (1-\mu)}{\mu e}\frac{\frac{\theta}{1+\theta}}{\frac{r}{r-g}z\left(g\right) + \frac{\theta}{1+\theta}}\right]. \tag{31}$$

Let steady state income inequality be measured by the ratio of capitalist's income, $y = whe + rk$, to laborer's income, $q = \frac{w}{1+\theta}$, and denote it by ζ.

$$\begin{aligned}\zeta &= \frac{y}{q} \\ &= e + \frac{\mu e + (1-\mu)}{\mu} Q\left(z\left(g\right);\theta\right)\end{aligned}$$

where

$$Q\left(z\left(g\right);\theta\right) = \frac{\frac{r}{r-g} - \frac{\theta}{1+\theta}}{\frac{r}{r-g}z\left(g\right) + \frac{\theta}{1+\theta}}. \tag{32}$$

First, note that inequality increases as the measure of capitalists, μ, decreases. Holding all other parameters constant, the steady state Euler equation implies a unique capital-to-effective labor input ratio, and consequently w and r are invariant to μ. Because each laborer works a constant number of hours, increasing the fraction of laborers in the economy necessarily leads to higher aggregate effective labor supply. In order for the $\frac{K}{N}$ to remain the same, K must rise in proportion to N, and so capitalists' wealth, $k = \frac{K}{\mu}$, also increases. Because factor prices do not change, laborer's income is constant, but capitalists' income increases. Notice that Q is continuous in θ. We now state a series of propositions related to Q which characterize steady state income inequality conditional on g. In the interest of space, all proofs are relegated to the appendix.

Proposition 1. *Inequality is bounded from below by relative labor productivity e.*

It follows immediately from the non-negativity of Q that inequality increases with relative productivity e. We assume that capitalists are at least as productive as laborers, and since e is the lower bound, we can without loss of generality assume that $e = 1$. We make this assumption in the remainder of the paper. This simplifies our expression for inequality to

$$\zeta = 1 + \frac{1}{\mu} Q$$

Given population shares, relative labor productivity, and preferences, the behavior of steady state inequality fundamentally depends on Q. Q is a function of three factors: $\frac{r}{r-g}$, $\frac{\theta}{1+\theta}$, and the aggregate factor income ratio, z. The first term captures the incentive of capitalists to save. This can be seen by substituting $\beta = \frac{1}{1+\rho}$ where ρ is the rate of time preference into (21) and loglinearizing which yields

$$r \approx \sigma g + \rho.$$

Using this identity and remembering that σ is the reciprocal of the intertemporal elasticity of substitution in consumption (IES), $\frac{r}{r-g}$ can be written as approximately equal to

$$\frac{1}{1 - \left(\frac{1}{IES} + \frac{\rho}{g}\right)^{-1}}$$

All else equal, a decline in ρ (i.e., a rise in the discount factor, β) or an increase in the willingness of households to substitute consumption across time (i.e., a decline in σ), both of which induce capitalists to save more, increase $\frac{r}{r-g}$. How the willingness of capitalists to save relates to inequality motivates our next proposition.

Proposition 2. *Holding the aggregate factor income ratio (z) constant, inequality is increasing in capitalists incentive to save, $\frac{r}{r-g}$.*

As $g \to 0$, $\frac{r}{r-g}$ declines monotonically to 1 so it acts to reduce inequality in a zero growth steady state. Since $\frac{r}{r-g}$ can be written as

$$\frac{1}{1 - \left(\frac{r}{g}\right)^{-1}},$$

the above proposition is the same as saying that inequality falls in $\frac{r}{g}$ (again ignoring z) [6]. Of course, $\frac{r}{r-g}$ cannot move without $z\,(g)$ changing as well, since z is a function of both r and g. Proposition (2) points out that for a given θ if steady state inequality *increases* as the growth

[6]This is consistent with the claim that as the gap between r and g increases inequality declines.

rate goes to zero, *it must be due entirely to a decline in the aggregate factor income ratio.* The sign of $\frac{dz}{dg}$ depends on parameters, and so then does the sign of $\frac{dQ}{dg}$. Later in the paper we turn to numerical methods to get a clearer picture of exactly how g affects ζ.

The second term, $\frac{\theta}{1+\theta}$ governs the wealth effect on capitalists hours. Rearranging the steady state equation for capitalist hours

$$\begin{aligned} h^* &= \frac{1}{1+\theta} - \frac{\theta}{1+\theta}\frac{(r-g)}{w}k \\ &= l^* - \frac{\theta}{1+\theta}\frac{(r-g)}{w}k. \end{aligned}$$

Thus, as θ increases, the wealth effect on hours becomes stronger for capitalists. This leads to our third proposition.

Proposition 3. *Steady state inequality is a decreasing function of θ.*

This statement is proven by signing the derivative of Q with respect to θ, and it highlights the importance of the capitalists' hours decision for determining long run income inequality. Intuitively, when capitalists supply labor hours, increases in the wage that produce higher income for laborers produce higher income for capitalists as well. Thus, inequality is higher than it would be if those wage income gains went only to laborers. Consider an increase in the steady state wage from w to Δw. Then inequality becomes

$$\frac{\Delta wh + rk}{\Delta wl}.$$

If the non-negativity constraint on hours binds for capitalist households so $h = 0$, then inequality unambiguously declines from the increase in wages. If on the other hand, $\theta = 0$ so that capitalists and laborers both supply 1 unit of labor, any increase in w is fully realized by both household types. Inequality still declines, but not by as much as when capitalists work fewer hours than 1. Proposition 3 permits us to define the supremum over the set of possible steady state inequality values across g given parameters.

$$\begin{aligned} \sup\left(\zeta\left(g\right)\right) &= 1 + \frac{1}{\mu}Q\left(z\left(g\right);0\right) \\ &= 1 + \frac{1}{\mu}\frac{1}{z\left(g\right)}. \end{aligned}$$

That is for a steady state growth rate g, inequality can be no larger than the above affine function of the inverse of the aggregate factor income ratio.

While we do not explicitly model labor income taxation, lower θ values have the same effect as higher average (flat) labor income taxes. This can be seen by substituting $\tilde{w} = (1-\tau)\, w$ in place of w in the definitions of labor income above. Proposition 3 then can be interpreted as consistent with the literature on the difference between the US and Europe in average hours worked whether the source of the difference arises from fiscal policy as in [23] or social preferences for leisure as in [1].

2.3.2 Case 2: Capitalists Do Not Work

The expressions are simpler when capitalists do not work. When $h = 0$, N is fixed at $\frac{1-\mu}{1+\theta}$, and $K = \varphi\left(\frac{1-\mu}{1+\theta}\right)$:

$$k = \frac{K}{\mu} = \frac{1-\mu}{\mu}\varphi\frac{1}{1+\theta},$$

so

$$\zeta = \frac{1-\mu}{\mu}\frac{\varphi}{w}r.$$

Substituting in (29), inequality can be written as a function of the rental rate and the model parameters,

$$\begin{aligned}\zeta &= \left[\frac{1}{\mu}-1\right]\frac{r}{\left(\frac{r+\delta}{\alpha}\right)^{\xi}-(r+\delta)} \\ &= \left[\frac{1}{\mu}-1\right]\frac{1}{z\,(g)} \\ &= \left[\frac{1}{\mu}-1\right]Q\left(z\,(g)\,;0\right)\end{aligned}$$

If capitalists do not work, then inequality is just a multiple of the aggregate factor income ratio.

We now turn to the role played by the production function in determining steady state inequality. Since according to equation (21), the steady state interest rate is invariant to either α or ξ, we can uncover the effect of these parameters on inequality by signing the derivative of Q with respect to each.

Proposition 4. *Steady state inequality is an increasing function of α and ξ.*

A larger value of α implies a higher return to capital relative to labor. Remember that although r is not a function of α or ξ, w is. Thus, for a given elasticity of substitution, higher α implies higher capitalist income and therefore higher inequality. For ξ, as the elasticity of substitution between capital and labor increases, factor prices become less responsive to imbalances in the relative supply of both factors. As ξ approaches values near its maximum imposed by balanced growth, the increase $\frac{K}{N}$ dominates the rise in $\frac{w}{r}$, and inequality becomes explosively large at low values of g. We will explore just how close ξ has to be to its maximum for explosive growth to obtain in a later numerical experiment.

Finally, we give a necessary condition for inequality to be higher in a zero growth steady state than it is in a low growth steady state.

Proposition 5. *If $\zeta(g) < \zeta(0)$, then the elasticity of substitution between capital and labor is greater than* 1.

Intuitively, this proposition says that in order for zero growth to lead to a steady state with greater inequality, factor prices $\frac{w}{r}$ must rise by less than $\frac{K}{N}$, which only happens if the elasticity of substitution between labor and capital is above 1. Moreover, positive depreciation increases the required degree of substitutability. The trade-off can be seen most easily by rearranging the AFI ratio as

$$z(g) = \frac{\frac{w}{r}}{\frac{K}{N}} = \frac{\frac{w}{r+\delta}}{\chi \frac{K}{N}},$$

where $\chi = \frac{r}{r+\delta}$, and setting $\xi = 1$ (i.e., the production function is Cobb-Douglas). In that case,

$$z(g) = \frac{(1-\alpha)}{\alpha} \frac{1}{\chi(g)}.$$

Since

$$z(0) > z(g),$$

$\zeta(g) > \zeta(0)$. Without depreciation, χ would be 1, and $z(g)$ would be constant which again would imply that $\zeta(g) > \zeta(0)$. In the literature, Rognlie (2015) focuses on the distinction between gross and net elasticities, showing that net elasticities are always smaller; as noted in the Introduction, Piketty focuses on net saving out of net income, and thus needs a very large gross elasticity to reverse this result and obtain $\zeta(g) < \zeta(0)$.

2.3.3 How Income Inequality Changes with Growth

In the steady state income inequality can be decomposed into the sum of two ratios,

$$\zeta \propto \frac{rk}{wl} + \frac{h}{l}.$$

The first term is the ratio of capital income to laborer's income. If capitalists do not work, then income inequality is proportional the ratio of capital income to labor income in the economy. When capitalists supply positive hours, some algebra shows that

$$\begin{aligned}\zeta &\propto \frac{w - \theta\left(r - g\right)k + \left(1 + \theta\right)rk}{w} \\ &= 1 + \frac{r + \theta g}{w}k.\end{aligned}$$

Regardless of capitalists' hours, whether inequality rises or falls depends solely upon the product of wealth, k, with $\frac{r+\theta g}{w}$. This product behaves in the same way as the factor price ratio $\frac{r}{w}$.

Because the closed-form expressions for steady state inequality do not yield unambiguous results for the effect of g on inequality, we use a computer to evaluate the expressions and plot the results for long run growth rates between 0 and 10 percent. To analyze the model numerically, we need to assign values to the structural parameters of the model. Here, we pick a reasonable set of values for some parameters, where reasonable means "gives rise to aggregates roughly consistent with US post-war averages." These numbers are $\beta = 0.99$, $\sigma = 2$, $\delta = 0.025$, $\theta = 1.25$, $g = 0.02$, and $e = 1$. Because the production parameters α and ν (or analogously, ξ) are central to the results, we study our model over a wide range of values. In cases where capital and labor are more substitutable than the usual Cobb-Douglas case ($\xi = 1$); the value of ν satisfies the restrictions needed to have a steady state growth path.[7] Figures 2-3 show the steady state ratio of capitalist income to laborer income for growth rates between 0 and 10 percent for the baseline capital share of income in production and for a higher value.

Within a small neighborhood of zero growth, the zero growth steady states is associated with lower, rather than higher, inequality, even if capital and labor are substitutes. Moreover, a high level of steady state inequality only obtains whenever the elasticity of substitution is close to the maximum conditional on α.

For all of the parameter values, wealth decreases with growth. Figures 4-5 plot $k\left(g\right)$ for several values of ξ and of α. As the elasticity of substitution between capital and labor increases, the level of capital in the zero-growth steady state becomes very large, especially when capital's

[7] Thus, our productivity growth should be interpreted as purely labor-augmenting or as exogenously-accumulating human capital (see King, Plosser, and Rebelo 1988); with Cobb-Douglas it does not matter whether the productivity growth affects capital, labor, or both.

share, α, is high. If we ignored the general equilibrium effect on prices, savings behavior alone would suggest extreme inequality; however, the factor price ratio also responds to g. Because $w(r)$ is decreasing in r and r is increasing in g,

$$\frac{dw}{dg} < 0.$$

Therefore as the long-run growth rate declines so does the factor price ratio. Numerical results show that unless ξ is close to $\xi_{\max}(\alpha) = \frac{1}{1-\nu_{\max}}$, the declining factor price ratio more than offsets wealth near $g = 0$, so capital income to laborer income declines in the neighborhood of zero growth.

To make the key point clear, Figure (10) displays ζ in the steady state as a function of g; the kink occurs when the capitalist's labor supply hits zero. At higher levels of g, where capitalists supply positive hours, inequality is decreasing in g, but the effects are not very large. In this region, the rise in $\frac{r}{r-g}$ discussed in Proposition (2) is offset by a rise in $z(g)$. Remember that $z(g) = \frac{w}{r}/\frac{K}{N}$. By (27), we know that the denominator of $z(g)$ is falling. As capital becomes scarce relative to labor, the numerator falls as well. Because ξ is not sufficiently larger than 1 in the baseline, the factor price ratio declines faster than the capital-labor ratio. The dashed line below inequality, labeled $\zeta(h=0)$ in the figure, shows the behavior of $z(g)$ more clearly. It plots $\left[\frac{1}{\mu} - 1\right] Q(z(g); 0)$, the appropriate measure when capitalists do not work, which only depends on $z(g)$. Notice that this line declines much faster in g than does actual inequality.

Continuing toward $g = 0$, we see that steady state income inequality rises slightly after the kink before descending sharply. To the left of the kink, capitalists do not work, so $\frac{r}{r-g}$ has no direct effect on ζ. Initially, $z(g)$ is still falling and without a corresponding decline in $\frac{r}{r-g}$ income inequality rises. Eventually, as g moves closer to 0 and $\frac{K}{N}$ climbs steeply, the aggressive adjustment for the factor price takes over and $z(g)$ rises. As before, the dashed line below inequality shows a counterfactual measure, this time using the measure for when capitalists work. Notice that it falls more sharply than ζ, highlighting the combined effects of $z(g)$ and $\frac{r}{r-g}$ falling.

We also plot in Figure (11) the dynamics of ζ starting from the steady state with $g = 0.02$; inequality initially jumps up due to changes in labor supply, then declines monotonically as capital accumulates. In the first period, income inequality jumps for two reasons. First, because only capitalists supply labor elastically, the increase in N_t is due entirely to capitalists. The wage falls in response, but not in the same proportion as N_t rises, so capitalist's labor income increases.

Second, although K_t is inelastic, r_t increases, pushing up capital income. Therefore, both sources of capitalist's income rise while laborer's income falls slightly because of lower wages. After the initial surge in inequality, capitalists accumulate wealth over time, so wages rise and the return on capital falls. In the new steady state, income inequality is well *below* its original level.

2.3.4 Conditions for High Inequality with Zero Growth

We have shown that with exogenous growth, zero growth implies extreme levels of income inequality, capital relative to output, and capital share of output only when the capital share and the elasticity of substitution in the production function are high. Next we ask just how large must these values be and what is implied by empirically plausible estimates in order to produce extreme inequality and high capital share of income? We solve the model numerically for a wide range of (α, ξ) combinations and find that for a capital share, α, extreme inequality only emerges as ξ approaches the upper bound placed upon it by balanced growth. Figures 6-7 plot contour maps $\frac{K}{Y}$ and ζ, respectively, for (α, ξ) combinations when $g = 0$.[8] Locations farther to the north and east (i.e., higher capital share in production and higher elasticity of substitution) are associated with both greater $\frac{K}{Y}$ (and, consequently, $\frac{(r+\delta)K}{Y}$). This pattern persists until the (α, ξ) combination violates the condition imposed by balanced growth, that is $\nu > \frac{\log(\alpha)}{\log(r_{\min}+\delta)}$, where $\xi > \xi_{\max}(\alpha)$. For instance, at $\alpha = 0.36$, the maximum elasticity of substitution is roughly 1.44.[9] At this maximum value, $\frac{K}{Y}$ is 2.8 times larger than it is when the elasticity of substitution is 1. Likewise, capital share of income goes from 0.36 to 0.99. Notice, however, that for most of the permissible region, $\frac{K}{Y}$ is much lower and $\frac{(r+\delta)K}{Y}$ is not especially high. Again when $\alpha = 0.36$, an elasticity of substitution of 1.2 produces a $\frac{K}{Y}$ only 60 percent larger than the ratio with unit elasticity, and $\frac{(r+\delta)K}{Y}$ is 0.57.[10], [11].

Because the parameter θ plays an important role for the amount of steady state inequality,

[8]Because when $g = 0$, $r = r_{\min}$, a plot of $\frac{(r+\delta)K}{Y}$ looks the same as a plot of $\frac{K}{Y}$.

[9]Gollin (2002) finds that capital's share of income is roughly one-third, once one takes careful account of self-employment income; see also Gomme and Rupert (2007).

[10]For reference, in the 2013 wave of the Survey of Consumer Finances, the ratio of average real income of the top quintile to the median is about 5.84. Income in this calculation is measured by the SCF variable "INCOME."

[11]Piketty and Zucman (2015) give an zero-growth example with high inequality and capital share. That example assumes $\alpha = 0.21$, substantially smaller than conventional estimates. To reach extreme values of $\frac{K}{Y}$ and $\frac{(r+\delta)K}{Y}$ with this lower capital share of production, the elasticity of substitution must be 1.87.

we reproduce figure 7 for $\theta = 0$ and $\theta = 10$, in figures 8 and 9, respectively.[12] When $\theta = 0$, inequality is at a maximum. This greatly expands the region of (α, ξ) values for which inequality rises above 1.5 (relative to the baseline), but it has almost no discernible effect on the region of high inequality. When θ is set to 10, the contour map looks almost identical to that for the baseline. Thus, the value of θ is most important for determining the region over which low to moderate inequality arises under zero growth, but is not a significant factor behind very high inequality.

Under the baseline value for capital's share of production, α, inequality becomes extreme near an elasticity of 1.44. How reasonable is an elasticity of substitution of 1.44? Chirinko (2008) surveys estimates from 31 studies. Only two studies support an elasticity above 1.5, and only three additional studies find evidence for an elasticity above one, while the median is significantly below one.[13] The most extreme estimate in Chirinko (2008) is 3.4, based on Mexican data and long-run tax reforms; it is unclear whether such an estimate should be applied to the questions at hand. More recently, Karabarbounis and Neiman (2014), Rognlie (2015), and Semieniuk (2014) argue that the aggregate elasticity of substitution is likely less than one. We want to draw particular attention to Rognlie (2015), who demonstrates that the net elasticity – the elasticity substitution between capital and labor for net output – is always smaller than the one for gross output, which is the one estimated in practice. We conclude that a growth model where the growth rate is exogenous will not predict explosive inequality if the aggregate production function is calibrated to match the data.

2.4 Capital Income Taxation

Next we consider the role for fiscal policy to mitigate the effects of inequality. Our goal here is not to characterize optimal taxation, but rather to examine the connection between fiscal policy and the growth-inequality relationship[14]. We consider two redistributionary fiscal policies. Each

[12]Remember that inequality is decreasing in θ so values between 0 and 10 are contained in this exercise.

[13]Palivos (2008) provides two additional references that find elasticities above one, based on abandoning the CES structure in favor of a production function with a variable elasticity. These elasticities are just barely above one, however.

[14]See Farhi and Werning (2014), Krusell (2002), and Straub and Werning (2015) for discussions of optimal policy in related models.

policy taxes capital income at a flat rate, τ, and transfers the revenues lump-sum. In the first case, the transfer is uniform (i.e., equally shared among all households) while in the second case the transfer is targeted to only laborer households.

2.4.1 Uniform transfer

Letting T denote the lump-sum transfer, in a steady state a laborer household works

$$l^* = \frac{1}{1+\theta} - \frac{\theta}{1+\theta}\frac{T}{w}$$

and consumes

$$x^* = \frac{w+T}{1+\theta}.$$

Government budget balance requires

$$T = \tau r K$$

and thus as capitalists save more, they produce an ever larger wealth effect on laborer hours.

The solution to the capitalist's problem is more complicated, but once again we can divide our results into two cases: capitalists work and capitalists do not work, where the steady state hours decision for a capitalist is

$$h^* = \max\left\{\frac{we - \theta\left[(1-\tau)\,r - g\right]k - \theta T}{(1+\theta)\,we}, 0\right\}.$$

Now equation (21) changes slightly

$$r = \frac{(1+g)^{\sigma} - \beta}{(1-\tau)\,\beta}$$

so

$$r_{\min} = \frac{1}{1-\tau}\frac{1-\beta}{\beta}. \tag{33}$$

This expression makes clear the important distinction between both the zero capital tax case and the one here: given preferences β and σ, the long-run growth rate determines the unique steady state *after-tax* return on capital,

$$\tilde{r} \equiv (1-\tau)\,r = \frac{(1+g)^{\sigma} - \beta}{\beta}.$$

The general equilibrium expressions for household decisions are too complicated to be signed analytically so we solve for them numerically. Figures (12) and Figure (13) show the steady-state

Laffer curves for our economy. As capital's share of production, α, increases the peak increases and moves to the left (i.e., the maximum is greater and occurs at a lower tax rate); increasing the elasticity of substitution has the same effect. For the remainder of the discussion, we will assume that τ is always to the left of the peak so $\frac{dT}{d\tau} > 0$.

Figure (14) shows how inequality in a zero growth steady state is affected by the capital income tax. For all values of ξ, inequality declines in the tax level. To understand this result, it is helpful to point out a few features of the model. First, k falls in τ and thus so does after-tax capital income. Although this result is not surprising, it is worth highlighting because it helps to understand how hours behave. It is clear from (33) that $\frac{dr_{\min}}{d\tau} > 0$, from which it follows that $\frac{dw_{\max}}{d\tau} < 0$ and $\frac{d}{d\tau}\left(\frac{K}{N}\right) < 0$. Second, laborer hours, $l^* = \frac{1}{1+\theta} - \mu\frac{\theta}{1+\theta}\frac{T}{w_{\max}}$, necessarily fall as τ increases so laborer's pre-transfer income is reduced by the tax. Meanwhile, the response of capitalist hours to an increase in the tax depends upon the value of ξ. When the lower bound on hours in not binding,

$$h^* = \frac{1}{1+\theta}\left[1 - (1+\mu-\tau)\,\theta\frac{r_{\min}}{w_{\max}}k\right].$$

Differentiating with respect to τ,

$$\frac{dh^*}{d\tau} = \theta\frac{r_{\min}}{w_{\max}}k - \theta\,(1+\mu-\tau)\left[\frac{r_{\min}}{w_{\max}}\frac{dk}{d\tau} - \frac{d\frac{r_{\min}}{w_{\max}}}{d\tau}k\right].$$

Because capitalists pay the entire tax but only receive a fraction of the revenues, higher tax rates produce a negative wealth effect, pushing hours up. An additional negative wealth effect comes from the downward adjustment in wealth ($\frac{dk}{d\tau} < 0$). On the other hand, greater τ increases the factor price ratio $\frac{r_{\min}}{w_{\max}}$, which encourages capitalists to consume more leisure. In order for the net effect on hours to be negative, factor prices must be very responsive to the tax, meaning that capital and labor are strong *complements* in production. Figure (16) plots the tax/hours curve across various different values of ξ; if hours decline they do not decline much unless the tax rate is past the peak of the Laffer curve, while for high values of ξ the rise in capitalist hours is quite pronounced.

The capital income tax reduces inequality. After-tax income falls for both types of households, but capitalist income is reduced by a greater percentage. For capitalists, capital income (net of taxes and transfers) and labor income decline, despite the general tendency for capitalists to increase hours in response to the tax. The decline in the equilibrium wage more than offsets the positive hours response..

Finally, adding capital income taxes does not alter our finding that ξ must be significantly greater than 1 in order for steady inequality to be higher under zero growth. Figure (15), plots the cutoff value of ξ after which $\zeta(0) > \zeta(g)$ $(g > 0)$ under the baseline calibration. For tax rates from 0 to 70 percent, the required elasticity of substitution is between 1.3 and 1.4, which is a relatively small range and still well above most estimates. Also, except for a small range of values below 20 percent, the required ξ is increasing with τ.

2.4.2 Targeted Transfer

Now consider the consequences of giving the transfer only to laborer households. Government budget balance implies

$$T = \frac{\tau r K}{1-\mu},$$

so the wealth effect on hours will be stronger for both household types: negative for laborers and positive for capitalists.[15] If the parameter values are such that capitalists do not work, then

$$K = \frac{(1-\mu)\frac{w(\tilde{r})}{\tilde{r}}}{(1+\theta)\,z\,(g;\tau)+\theta\tau},$$

$$N = \frac{(1-\mu)\,z\,(g;\tau)}{(1+\theta)\,z\,(g;\tau)+\theta\tau}$$

$$T = \tau\frac{w\,(\tilde{r})}{(1+\theta)\,z\,(g;\tau)+\theta\tau}$$

where

$$z\,(g;\tau) = \frac{\left[\left(\frac{\frac{\tilde{r}}{1-\tau}+\delta}{\alpha}\right)^{\xi} - \left(\frac{\tilde{r}}{1-\tau}+\delta\right)\right]}{\frac{\tilde{r}}{1-\tau}}$$

is the equivalent of (30) with taxation.

[15]The Laffer curves for the targeted transfer case are very similar to those for the uniform transfer case, so we do not explicitly plot them.

Some algebra shows that inequality is

$$\begin{aligned}\zeta &= \frac{1-\mu}{\mu}\frac{(1-\tau)}{\frac{\left(\frac{r+\delta}{\alpha}\right)^{\xi}-(r+\delta)}{r}+\tau} \\ &= \frac{1-\mu}{\mu}\frac{1-\tau}{z\left(g;\tau\right)+\tau}.\end{aligned}$$

Furthermore,

$$\frac{d\zeta}{d\tau} = \frac{\widetilde{r}^{2}}{\alpha\mu}\frac{\xi}{(1-\tau)^{2}}\left(-\frac{1-\mu}{\left(\left(\frac{\frac{\widetilde{r}}{1-\tau}+\delta}{\alpha}\right)^{\xi}-(\widetilde{r}+\delta)\right)^{2}}\right)\left(\frac{\frac{\widetilde{r}}{1-\tau}+\delta}{\alpha}\right)^{\xi-1} < 0;$$

that is, inequality falls as the tax rate rises (again, supposing we are on the upward-sloping side of the Laffer curve).

In contrast, if capitalists are supplying labor,

$$K = \frac{(1-\mu+\mu e)\,\frac{w(\widetilde{r})}{\widetilde{r}}}{(1+\theta)\,z\left(g;\tau\right)+\theta\frac{(\widetilde{r}-g)}{\widetilde{r}}},$$

$$N = \frac{(1-\mu+\mu e)\,z\left(g;\tau\right)}{(1+\theta)\,z\left(g;\tau\right)+\theta\frac{(\widetilde{r}-g)}{\widetilde{r}}},$$

and

$$T = \frac{\tau}{1-\mu}\left(\frac{(1-\mu+\mu e)\,w\left(\widetilde{r}\right)}{(1+\theta)\,z\left(g;\tau\right)+\theta\frac{(\widetilde{r}-g)}{\widetilde{r}}}\right).$$

Inequality is

$$\zeta = \frac{1-\mu}{\mu}\left\{\frac{\mu\left[(1+\theta)\,z\left(g;\tau\right)+1+\theta-\tau\right]e+(1-\mu)\left(1-\tau+\theta\frac{g}{r}\right)}{\mu e\tau+(1-\mu)\left[(1+\theta)\,z\left(g;\tau\right)+\theta\frac{(r-g)}{r}+\tau\right]}\right\}. \tag{34}$$

The expression in Equation (34) is formidable and, as a result, the derivative cannot easily be signed; however, we can show that inequality decreases in the tax rate locally at 0.

Proposition 6. *A targeted transfer decreases inequality. In other words,* $\zeta\left(\tau\right) < \zeta\left(0\right)$.

The proof can be found in the appendix. As was the case with uniform transfers, a targeted transfer does not undo our result that a high elasticity of substitution between capital and labor is necessary for higher inequality in a zero-growth steady state.

3 Endogenous Growth

Here we consider two extensions that allow for long-run growth to be endogenous. As noted above, the positive comovement between growth rates and returns is crucial – as growth falls, returns also fall, and thus unless capital moves a lot relative to labor inequality cannot move significantly. We ask the natural question now – if growth is endogenous, does this positive comovement still obtain?

3.1 Endogenous Growth without Productivity Growth

One model of endogenous growth fits directly into our above framework – if $\nu > 0$, as we have noted before is important for our results, the marginal product of capital can be bounded away from zero.[16] Suppose there is no exogenous productivity growth. Then we have

$$\lim_{K\to\infty}\left\{\frac{\partial Y}{\partial K}\right\} = A\alpha^{\frac{1}{\nu}}$$

so that

$$\begin{aligned} r &= A\alpha^{\frac{1}{\nu}} - \delta \\ w &= (1-\alpha)\, A\alpha^{\frac{1-\nu}{\nu}}, \end{aligned}$$

which are both constant; note that our result above regarding the existence of a balanced-growth path is related to the requirement that r can approach zero. Using the Euler equation of the capitalist and supposing that parameters are such that $h = 0$, we get

$$1 + g = \frac{c'}{c} = \left(\beta\left(1 + A\alpha^{\frac{1}{\nu}} - \delta\right)\right)^{\frac{1}{\sigma}};$$

the resource constraint then requires that worker consumption, capital, and output both asymptotically grow at this same constant rate. However, this condition is impossible, since wages are constant and hours are bounded, meaning that the economy cannot display balanced growth. In this case, inequality explodes to infinity over time if and only if $g > 0$. Since

$$\frac{\partial g}{\partial \nu} = -\frac{1}{\sigma}\left(\beta\left(1 + A\alpha^{\frac{1}{\nu}} - \delta\right)\right)^{\frac{1-\sigma}{\sigma}}\frac{\alpha^{\frac{1}{\nu}}}{\nu^2}\beta A\ln(\alpha) > 0,$$

[16] Jones and Manuelli (1990) discuss this model and related ones in which balanced growth obtains only asymptotically.

the more substitutable capital is for labor the faster inequality explodes, just as in the exogenous growth case. Moreover, $\frac{\partial r}{\partial g} > 0$, so even if parameters are such that $g = 0$, the return on capital will be low and inequality will be low as well.

3.2 Romer Externality Model

Next we consider Romer (1986) where a production externality renders the production function, in the aggregate, constant returns to scale in capital. For exposition, we restrict our analysis again to the case where capitalists do not supply labor.[17] The household problem is not meaningfully different from the exogenous growth case above so we focus on the firm's problem. The individual firm solves the problem

$$\max_{K,N} \left\{ A \left(\alpha K^{\nu} + (1-\alpha)(SN)^{\nu}\right)^{\frac{1}{\nu}} - rK - wN \right\}$$

taking as given the externality term S. We suppose that

$$S = K$$

in equilibrium, which leads to the factor price conditions

$$r = \alpha A \left(\alpha + (1-\alpha) N^{\nu}\right)^{\frac{1-\nu}{\nu}} - \delta$$
$$w = (1-\alpha) A \left(\alpha N^{-\nu} + 1 - \alpha\right)^{\frac{1-\nu}{\nu}} K.$$

We can define a 'deflated wage' (relative to K) and label it $\tilde{w}$:

$$\tilde{w} = (1-\alpha) A \left(\alpha N^{-\nu} + 1 - \alpha\right)^{\frac{1-\nu}{\nu}}.$$

The Euler equation of the capitalist determines the endogenous growth rate:

$$1 + g = \beta \left(1 + \alpha A \left(\alpha + (1-\alpha) \left(\frac{1-\mu}{1+\theta} \right)^{\nu} \right)^{\frac{1-\nu}{\nu}} - \delta \right).$$

Because this economy does not feature transitional dynamics, it is always on the balanced-growth path. Along this balanced-growth path, it is easy to see that $\frac{\partial g}{\partial A} > 0$, and therefore $\frac{\partial g}{\partial r} > 0$. We can therefore expect that our previous results will hold – if a decline in TFP leads to lower

[17] As long as the wealth effect reduces hours, in any environment where capital income becomes explosively large capitalists would not work anyway.

growth it will also lead to lower returns, and therefore the quantitative effect on inequality will depend critically on the production elasticity.

To see this result formally, long-run income inequality when capitalists do not work is

$$\zeta = \frac{\frac{K}{N}}{\frac{w}{r+\delta}} = \frac{\frac{1}{N}}{\frac{\widetilde{w}}{r+\delta}}.$$

As before,

$$\zeta = \frac{\frac{1}{N}}{\frac{\widetilde{w}}{r+\delta}} = \frac{1-\mu}{\mu}\frac{\alpha}{1-\alpha}N^{\frac{1}{\xi}-1}$$

where

$$N = \frac{1-\mu}{1+\theta}$$

is constant and less than 1. Once again, long-run income inequality will only be high if the elasticity of substitution is well above 1. Moreover, the upper bound on inequality, $\frac{\frac{K}{N}}{\frac{w}{r}}$, does not depend upon A so even if growth changes due to a productivity shock, inequality cannot rise too far as a result.

4 Conclusion

This two-household model has shown that the parameters that primarily govern the behavior of inequality in a zero growth steady state are related to production. The capital share and the elasticity of substitution between capital and labor control how quickly both the steady state wage rate and wealth rise as g nears zero. In addition, they change the response of hours. Generally, steady state hours are higher when $g = 0$, but if both α and ν are sufficiently high, hours are lower (perhaps zero) in low growth steady states and rise as g increases. Unless the capital and labor are sufficiently strong substitutes in production, steady state inequality is lower under zero growth than it is in a steady state with positive growth. Steady states with extreme values of capital to income, capital's share of income, and income inequality only arise when the elasticity of substitution between capital and labor and capital share in production are jointly very near to values which violate balanced growth. These results continue to hold in the presence of endogenous growth.

In ongoing work we are exploring environments where interest rates, growth rates, and rates of time preference are not so tightly linked, in particular models with idiosyncratic risk. Our preliminary findings regarding these models is that explosive inequality can obtain, but it requires that idiosyncratic risk be very large (to open a big gap between the three rates) and that precautionary savings should decline in response to changes in growth (perhaps through financial development).

References

[1] Alesina, A., Glaeser, E., Sacerdote, B., 2015. "Work and Leisure in the United States and Europe: Why so Different?" NBER macroeconomics annual 20, 1-64.

[2] Campbell, J.Y., 1999. "Asset Prices, Consumption and the Business Cycle." In Taylor, J.B., Woodford, M., (eds.) Handbook of Macroeconomics 1, North-Holland.

[3] Chirinko, R.S., 2008. "σ: The Long and Short of It." Journal of Macroeconomics 30(2), 671-686. https://doi.org/10.1016/j.jmacro.2007.10.010

[4] Crump, R.K., Eusepi, S., Tambalotti, A., Topa, G., 2015. "Subjective Intertemporal Substitution." Federal Reserve Bank of New York Staff Report 734.

[5] Farhi, E., Werning, I., 2014. "Bequest Taxation and $r - g$". Manuscript.

[6] Gollin, D., 2002. "Getting Income Shares Right." Journal of Political Economy 110(2), 458-474. https://doi.org/10.1086/338747

[7] Gomme, P., Rupert, P., 2007. "Theory, Measurement, and Calibration of Macroeconomic Models." Journal of Monetary Economics 54(2), 460-497. https://doi.org/10.1016/j.jmoneco.2005.09.005

[8] Guvenen, F., 2006. "Reconciling Conflicting Evidence on the Elasticity of Intertemporal Substitution: A Macroeconomic Perspective." Journal of Monetary Economics 53(7), 1451-1472. https://doi.org/10.1016/j.jmoneco.2005.06.001

[9] Hall, R.E., 1988. "Intertemporal Substitution in Consumption." Journal of Political Economy 96(2), 339-357. https://doi.org/10.1086/261539

[10] Hammond, P.J., Rodríguez-Clare, A., 1993. "On Endogenizing Long-Run Growth." Scandinavian Journal of Economics 95(4), 391-425. https://doi.org/10.2307/3440904

[11] Havránek, T., 2015. "Measuring Intertemporal Substitution: The Importance of Method Choices and Selective Reporting." Journal of the European Economic Association 13(6), 1180-1204. https://doi.org/10.1111/jeea.12133

[12] Jones, L.E., Manuelli, R., 1990. "A Convex Model of Equilibrium Growth: Theory and Policy Implications." Journal of Political Economy 98(5), 1008-1038. https://doi.org/10.1086/261717

[13] Karabarbounis, L., Neiman, B., 2014. "Capital Depreciation and Labor Shares Around the World: Measurement and Implications." National Bureau of Economic Research Working Paper 20606.

[14] King, R.G., Plosser, C.I., Rebelo, S.T., 1988. "Production, Growth, and Business Cycles I. The Basic Neoclassical Model." Journal of Monetary Economics 21(2-3), 195-232. https://doi.org/10.1016/0304-3932(88)90030-X

[15] Krusell, P., 2002. "Time-Consistent Redistribution." European Economic Review 46(4-5), 755-769. https://doi.org/10.1016/S0014-2921(01)00211-2

[16] Krusell, P., Smith, A.A., Jr., 2015. "Is Piketty's 'Second Law of Capitalism' Fundamental?" Journal of Political Economy 123(4), 725-748. https://doi.org/10.1086/682574

[17] McGrattan, E.R., Prescott, E.C., 2013. "On Financing Retirement with an Aging Population." Federal Reserve Bank of Minneapolis Staff Report 472.

[18] Moll, B., 2014. "Productivity Losses from Financial Frictions: Can Self-Financing Undo Capital Misallocation?" American Economic Review 104(10), 3186-3221. https://doi.org/10.1257/aer.104.10.3186

[19] Palivos, T., 2008. "Comment on 'σ: The Long and Short of It'." Journal of Macroeconomics 30(2), 687-690. https://doi.org/10.1016/j.jmacro.2007.09.004

[20] Piketty, T., 2014. Capital in the Twenty-First Century. Belknap Press. https://doi.org/10.4159/9780674369542

[21] Piketty, T., Saez, E., 2014, "Inequality in the Long Run." Science 344(6186), 838-843. https://doi.org/10.1126/science.1251936

[22] Piketty, T., Zucman, G., 2015, "Wealth and Inheritance in the Long Run." Elsevier Handbook of Income Distribution, 2, Elsevier.

[23] Prescott, E.C., 2004. "Why do Americans Work so much more than Europeans?" No. w10316. National Bureau of Economic Research, 2004.

[24] Rognlie, M., 2015. "Deciphering the Fall and Rise in the Net Capital Share." Forthcoming, Brookings Papers on Economic Activity.

[25] Romer, P.M., 1986. "Increasing Returns and Long-Run Growth." Journal of Political Economy 94(5), 1002-1037. https://doi.org/10.1086/261420

[26] Semieniuk, G., 2014. "Piketty's Elasticity of Substitution: A Critique." Schwartz Center for Economic Policy Analysis Working Paper 2014-8.

[27] Straub, L., Werning, I., 2015. "Positive Long-Run Capital Taxation: Chamley-Judd Revisited." Manuscript.

[28] Vissing-Jørgensen, A., 2002. "Limited Asset Market Participation and the Elasticity of Intertemporal Substitution." Journal of Political Economy 110(4), 825-853. https://doi.org/10.1086/340782

5 Appendix: Proofs

Proof of Proposition 1

Proof. Because $e > 0$ and $\mu \in (0,1)$, we only need to show that

$$Q\left(z\left(g\right);\theta\right) > 0.$$

Notice first, that $\frac{\theta}{1+\theta} \in [0,1)$. If $\sigma \geq 1$, then the $\min \frac{r}{r-g} = 1$ for non-negative growth rates, since the $\lim_{g\to\infty} \frac{r}{r-g} = 1$. Finally, $z\left(g\right) > 0$, so for finite θ, $Q\left(z\left(0\right);\theta\right) > 0$, and $\min \zeta = e$. □

Proof of Proposition 2

Proof. It is immediate that

$$\begin{aligned}
\frac{dQ}{d\left[\frac{r}{r-g}\right]} &= \frac{\frac{r}{r-g}z\left(g\right) + \frac{\theta}{1+\theta} - z\left(g\right)\left(\frac{r}{r-g} - \frac{\theta}{1+\theta}\right)}{\left[\frac{r}{r-g}z\left(g\right) + \frac{\theta}{1+\theta}\right]^2} \\
&= \frac{\frac{\theta}{1+\theta} + z\left(g\right)\frac{\theta}{1+\theta}}{\left[\frac{r}{r-g}z\left(g\right) + \frac{\theta}{1+\theta}\right]^2} \\
&> 0.
\end{aligned}$$

□

Proof of Proposition 3

Proof. A larger θ implies less willingness to work on the part of households. The steady state interest rate and $z\left(g\right)$ are independent of θ. Then

$$\begin{aligned}
\frac{dQ}{d\theta} &= \frac{dQ}{d\frac{\theta}{1+\theta}}\frac{d\frac{\theta}{1+\theta}}{d\theta} \\
&= \left[-\frac{\frac{r}{r-g}z\left(g\right) + \frac{r}{r-g}}{\left(\frac{r}{r-g}z\left(g\right) + \frac{\theta}{1+\theta}\right)^2}\right]\frac{1}{\left(1+\theta\right)^2} \\
&< 0
\end{aligned}$$

Inequality rises as θ falls. □

Proof of Proposition 4

Proof. Fix θ. Then

$$Q\left(z\left(g\right);\theta\right)=\frac{\frac{r}{r-g}-\frac{\theta}{1+\theta}}{\frac{r}{r-g}z\left(g\right)+\frac{\theta}{1+\theta}}$$

Because steady state r is invariant to changes in α and ξ, all that matters for inequality is $z\left(g\right)$.

$$\begin{aligned}
z\left(g\right) &= \frac{\left[\left(\frac{r(g)+\delta}{\alpha}\right)^{\xi}-\left(r\left(g\right)+\delta\right)\right]}{r\left(g\right)} \\
&= \frac{1}{\chi}\frac{\left[\alpha^{-\xi}\left(r\left(g\right)+\delta\right)^{\xi}-\left(r\left(g\right)+\delta\right)\right]}{r\left(g\right)+\delta} \\
&= \frac{1}{\chi}\left[\alpha^{-\xi}\left(r\left(g\right)+\delta\right)^{\xi-1}-1\right].
\end{aligned}$$

Because $\xi\geq 0$, a greater capital's share decreases z and increases steady state inequality. Likewise steady state inequality will be greater for larger elasticities of substitution since

$$\frac{dz\left(g\right)}{d\xi}=\frac{1}{\chi}\left\{-\alpha^{-\xi}\left(r\left(g\right)+\delta\right)^{\xi-1}\log\left(r\left(g\right)+\delta\right)\left[1-\frac{\log\left(\alpha\right)}{\log\left(r\left(g\right)+\delta\right)}\right]\right\}<0$$

where the strict inequality holds since $r+\delta<\alpha<1$. □

Proof of Proposition 5

Proof. First, in order for $\zeta\left(g\right)<\zeta\left(0\right)$, $z\left(0\right)<z\left(g\right)$. There are four cases to consider: (1) $h\left(g\right)=0,\ h\left(0\right)=0$; (2) $h\left(g\right)>0,\ h\left(0\right)>0$; (3) $h\left(g\right)>0,\ h\left(0\right)=0$; and (4) $h\left(g\right)=0,\ h\left(0\right)>0$. It will be shown that the fourth case cannot obtain. The first case is obvious since $\zeta\left(g\right)<\zeta\left(0\right)$ implies

$$\frac{1}{z\left(g\right)}<\frac{1}{z\left(0\right)}$$

since z is non-negative. For the second case, $\zeta\left(g\right)<\zeta\left(0\right)$ implies

$$\frac{\frac{r(g)}{r(g)-g}-\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z\left(g\right)+\frac{\theta}{1+\theta}}<\frac{1-\frac{\theta}{1+\theta}}{z\left(0\right)+\frac{\theta}{1+\theta}}.$$

By Proposition 2,

$$\frac{1-\frac{\theta}{1+\theta}}{z\left(g\right)+\frac{\theta}{1+\theta}}<\frac{\frac{r(g)}{r(g)-g}-\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z\left(g\right)+\frac{\theta}{1+\theta}}$$

so

$$\frac{1-\frac{\theta}{1+\theta}}{z\left(g\right)+\frac{\theta}{1+\theta}}<\frac{1-\frac{\theta}{1+\theta}}{z\left(0\right)+\frac{\theta}{1+\theta}}.$$

For the third case,

$$1+\frac{1}{\mu}\frac{\frac{r(g)}{r(g)-g}-\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z\left(g\right)+\frac{\theta}{1+\theta}}<\left(\frac{1-\mu}{\mu}\right)\frac{1}{z\left(0\right)}.$$

If this is true, then by Proposition 2 and Proposition 3

$$1+\frac{1}{\mu}\frac{\frac{r(g)}{r(g)-g}-\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z\left(g\right)+\frac{\theta}{1+\theta}}<1+\frac{1}{\mu}\frac{1-\frac{\theta}{1+\theta}}{z\left(0\right)+\frac{\theta}{1+\theta}}<1+\frac{1}{\mu}\frac{\frac{r(g)}{r(g)-g}-\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z\left(0\right)+\frac{\theta}{1+\theta}}.$$

Now for the final case, which we will show implies a contradiction. From (31),

$$h=\max\left\{0,\frac{1}{1+\theta}\left[1-\frac{1}{\mu}\frac{\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z\left(g\right)+\frac{\theta}{1+\theta}}\right]\right\}.$$

Because $h\left(g\right)=0$ and $h\left(0\right)>0$

$$\frac{1}{\mu}\frac{\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z\left(g\right)+\frac{\theta}{1+\theta}}>1>\frac{1}{\mu}\frac{\frac{\theta}{1+\theta}}{z\left(0\right)+\frac{\theta}{1+\theta}}$$

so

$$z\left(0\right)>\frac{r\left(g\right)}{r\left(g\right)-g}z\left(g\right)>z\left(g\right).$$

We can see from the (31), for each growth rate, there is a $\bar{\theta}\left(g\right)$ such that

$$\begin{aligned}h\left(g\right)&=0\;\forall\;\theta>\bar{\theta}\left(g\right)\\&>0\;\forall\;\theta<\bar{\theta}\left(g\right).\end{aligned}$$

In order for $h\left(0\right)>h\left(g\right)=0$ at θ, it would have to be the case that $\bar{\theta}\left(g\right)<\bar{\theta}\left(0\right)$, and $\theta\in\left[\bar{\theta}\left(g\right),\bar{\theta}\left(0\right)\right]$. Additionally, $\zeta\left(g\right)$ must be at its minimum

$$\zeta\left(g\right)=\left(\frac{\mu-1}{\mu}\right)\frac{1}{z\left(g\right)}$$

while

$$\zeta\left(0\right)>\left(\frac{\mu-1}{\mu}\right)\frac{1}{z\left(0\right)}.$$

We know from case (1), that $\min\left[\zeta\left(g\right)\right]<\min\left[\zeta\left(0\right)\right]$, which implies that $z\left(g\right)<z\left(0\right)$. Because z is invariant to θ, it cannot be the case that $z\left(0\right)>z\left(g\right)$, so case (4) cannot obtain. Simply put, $\bar{\theta}\left(0\right)<\bar{\theta}\left(g\right)$. Therefore whenever $\zeta\left(g\right)<\zeta\left(0\right)$, $z\left(g\right)>z\left(0\right)$. Now to close the proof

$$z\left(g\right)=\frac{1}{\chi\left(g\right)}\left[\alpha^{-\xi}\left(r\left(g\right)+\delta\right)^{\xi-1}-1\right]$$

where again $\chi = \frac{r(g)}{r(g)+\delta}$. If $\zeta(0) > \zeta(g)$, then

$$\frac{1}{\chi(0)}\left[\alpha^{-\xi}(r(0)+\delta)^{\xi-1} - 1\right] < \frac{1}{\chi(g)}\left[\alpha^{-\xi}(r(g)+\delta)^{\xi-1} - 1\right]$$

Because χ is increasing in g,

$$(r(0)+\delta)^{\xi-1} < (r(g)+\delta)^{\xi-1}$$

Now since $r(g) > r(0) = r_{\min}$, this condition can only hold if $\xi > 1$. $\square$

Proof of Proposition 6

Proof. There are two cases: $h = 0$ or $h > 0$. In the first case,

$$\begin{aligned}
\zeta &= \frac{1-\mu}{\mu}\frac{1-\tau}{z(g;\tau)+\tau} \\
&= \frac{1-\mu}{\mu}\frac{(1-\tau)}{\frac{\left(\frac{\frac{\tilde{r}}{1-\tau}+\delta}{\alpha}\right)^{\xi} - \left(\frac{\tilde{r}}{1-\tau}+\delta\right)}{\frac{\tilde{r}}{1-\tau}} + \tau}
\end{aligned}$$

Differentiating with respect to τ,

$$\frac{d\zeta}{d\tau} = -\frac{\tilde{r}^2}{\alpha\mu}\frac{\xi}{(1-\tau)^2}\left(\frac{1-\mu}{\left(\left(\frac{\frac{\tilde{r}}{1-\tau}+\delta}{\alpha}\right)^{\xi} - (\tilde{r}+\delta)\right)^2}\right)\left(\frac{\frac{\tilde{r}}{1-\tau}+\delta}{\alpha}\right)^{\xi-1} \leq 0\ \forall\tau,\ w.e.\ iff\ \xi \geq 0,$$

When $h > 0$,

$$\begin{aligned}
\zeta &= \frac{we - \theta((1-\tau)r - g)k}{w+T} + (1+\theta)\frac{(1-\tau)rk}{w+T} \\
&= \frac{\frac{w}{k}e + (1-\tau)r + \theta g}{\frac{w}{k} + \frac{\mu}{1-\mu}\tau r}
\end{aligned}$$

Converting this to after-tax returns using

$$\frac{\tilde{r}}{1-\tau} = r$$

and simplifying yields

$$\zeta = \frac{A(\tau)e + \tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu}\frac{\tau}{1-\tau}\tilde{r}}$$

where

$$A(\tau) = \frac{\mu}{((1-\mu)+\mu e)}\left((1+\theta)\left(\left(\frac{\frac{\tilde{r}}{1-\tau}+\delta}{\alpha}\right)^{\xi} - \left(\frac{\tilde{r}}{1-\tau}+\delta\right)\right) + \theta\left(\frac{\tilde{r}}{1-\tau}-g\right)\right)$$
$$= \frac{\mu}{((1-\mu)+\mu e)}\left((1+\theta)\left(\frac{\frac{\tilde{r}}{1-\tau}+\delta}{\alpha}\right)^{\xi} - \frac{\tilde{r}}{1-\tau} - (1+\theta)\delta - \theta g\right)$$

We want to show $\zeta(\tau) < \zeta(0)\,,\ \tau > 0$. Notice that since $\frac{\tau}{1-\tau}\tilde{r}$ is monotonically increasing in τ,

$$\frac{A(\tau)\,e + \tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu}\frac{\tau}{1-\tau}\tilde{r}} \leq \frac{A(\tau)\,e}{A(\tau)} + \frac{\tilde{r}+\theta g}{A(\tau) + \frac{\mu}{1-\mu}\frac{\tau}{1-\tau}\tilde{r}}, \quad w.e.\ iff\ \tau = 0.$$

Next we show that

$$e + \frac{\tilde{r}+\theta g}{A(\tau) + \frac{\mu}{1-\mu}\frac{\tau}{1-\tau}\tilde{r}} < e + \frac{\tilde{r}+\theta g}{A(0)}$$

when $\tau > 0$. In other words,

$$A(\tau) + \frac{\mu}{1-\mu}\frac{\tau}{1-\tau}\tilde{r} > A(0)\,.$$

Substituting in for $A(\tau)$ and $A(0)$ and simplifying, we have

$$(1+\theta)\left(\frac{\frac{\tilde{r}}{1-\tau}+\delta}{\alpha}\right)^{\xi} - \frac{\tilde{r}}{1-\tau} + \frac{((1-\mu)+\mu e)}{1-\mu}\frac{\tau}{1-\tau}\tilde{r} > (1+\theta)\left(\frac{\tilde{r}+\delta}{\alpha}\right)^{\xi} - \tilde{r}$$
$$(1+\theta)\left(\frac{\frac{\tilde{r}}{1-\tau}+\delta}{\alpha}\right)^{\xi} - \frac{\left(1-\frac{((1-\mu)+\mu e)}{1-\mu}\tau\right)}{1-\tau}\tilde{r} > (1+\theta)\left(\frac{\tilde{r}+\delta}{\alpha}\right)^{\xi} - \tilde{r}$$
$$(1+\theta)\left(\frac{\frac{\tilde{r}}{1-\tau}+\delta}{\alpha}\right)^{\xi} - \tilde{r} > (1+\theta)\left(\frac{\tilde{r}+\delta}{\alpha}\right)^{\xi} - \tilde{r}$$
$$\frac{\tilde{r}}{1-\tau} > \tilde{r}$$

which is obviously true. The penultimate step holds because

$$\frac{\left(1-\frac{((1-\mu)+\mu e)}{1-\mu}\tau\right)}{1-\tau} < 1$$

Therefore if $\tau > 0$

$$\begin{aligned}\zeta(\tau) &= \frac{A(\tau)e + \tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu}\frac{\tau}{1-\tau}\tilde{r}} \\ &< \frac{A(\tau)e}{A(\tau)} + \frac{\tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu}\frac{\tau}{1-\tau}\tilde{r}} \\ &= e + \frac{\tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu}\frac{\tau}{1-\tau}\tilde{r}} \\ &< e + \frac{\tilde{r} + \theta g}{A(0)} \\ &= \zeta(0)\end{aligned}$$

$\square$

Figure 1: Equilibrium Wage in $g = 0$ Steady State

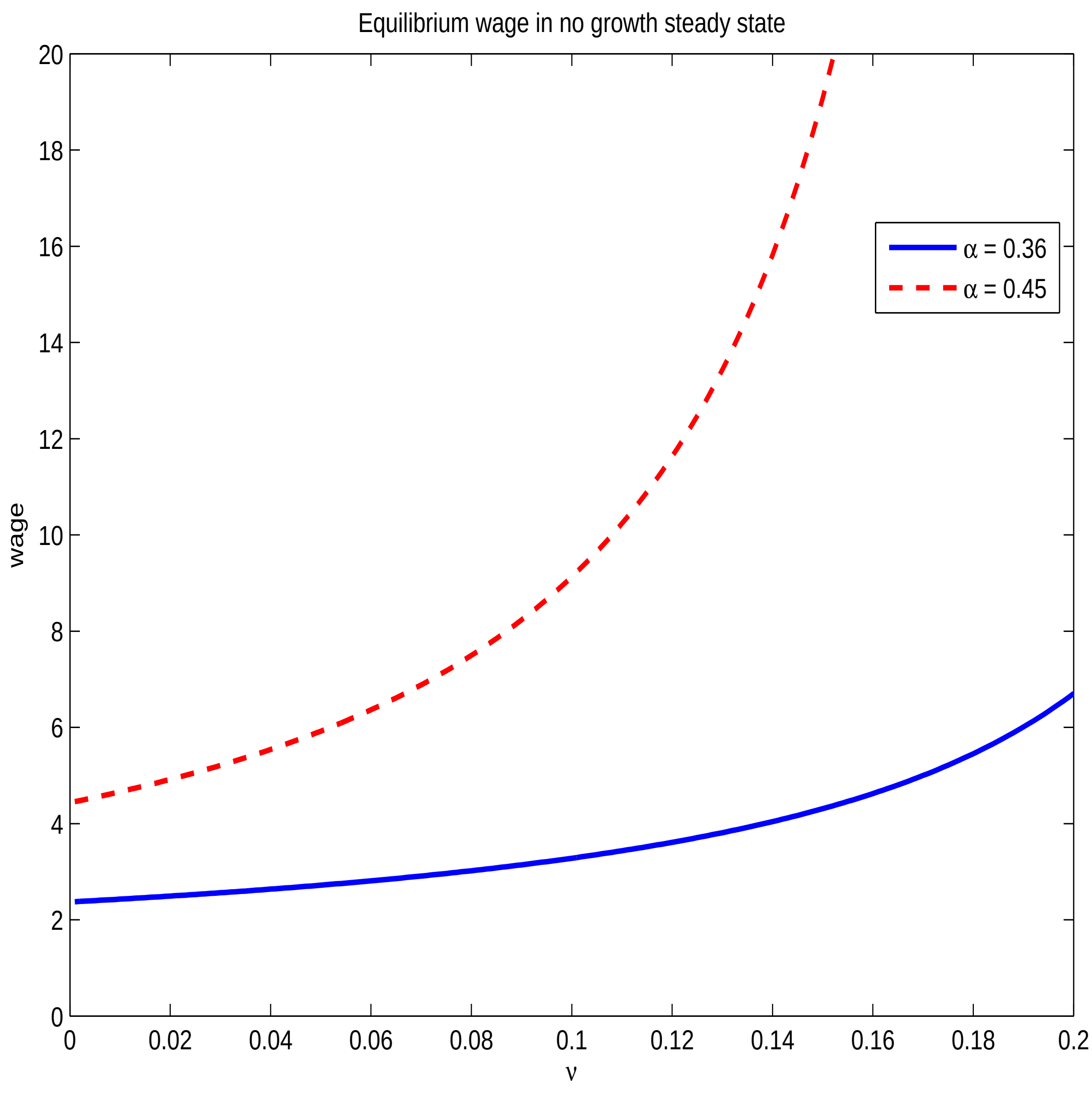


Figure 2: Steady State Inequality

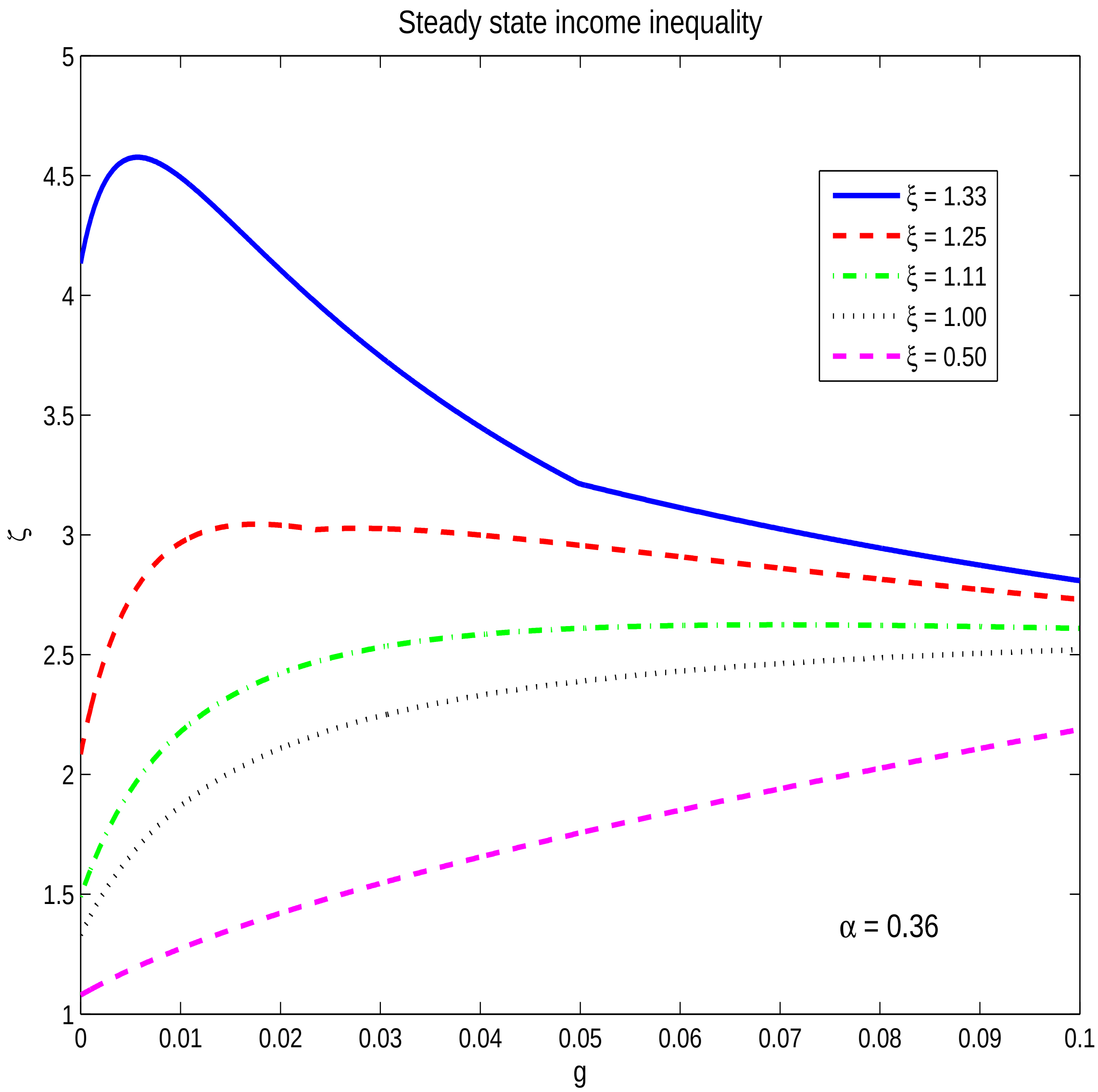


Figure 3: Steady State Inequality

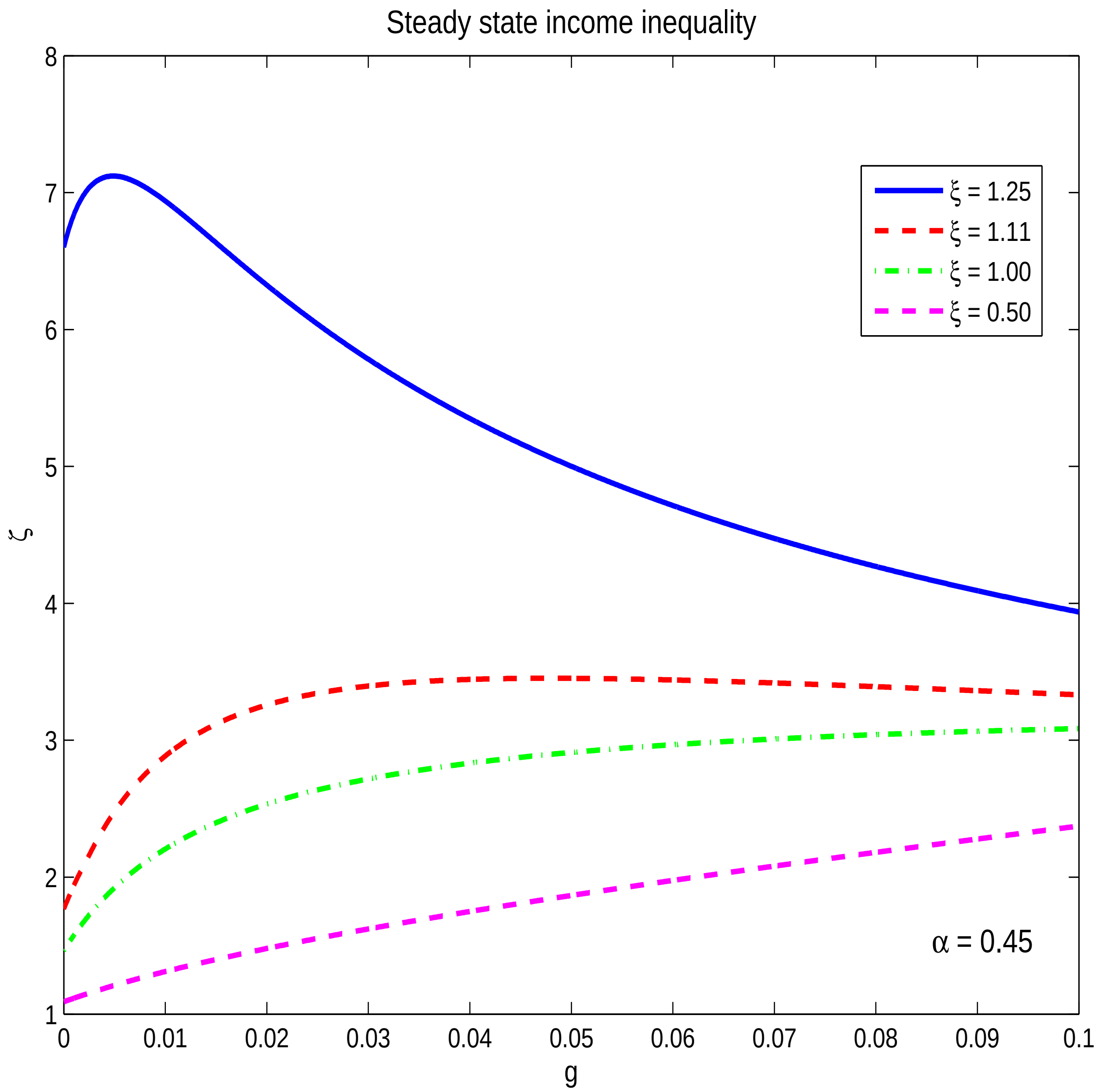


Figure 4: Steady State Wealth

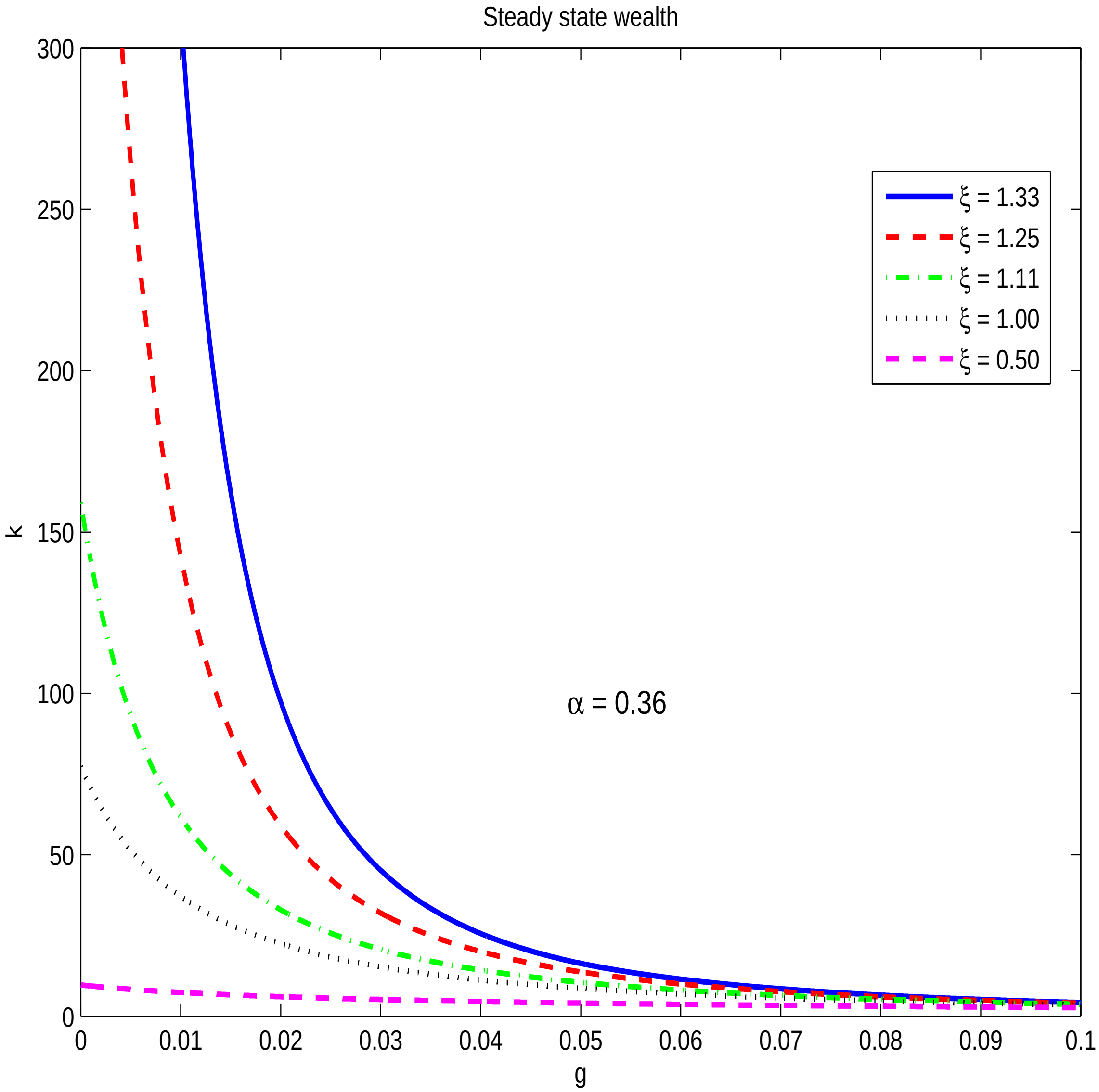


Figure 5: Steady State Wealth

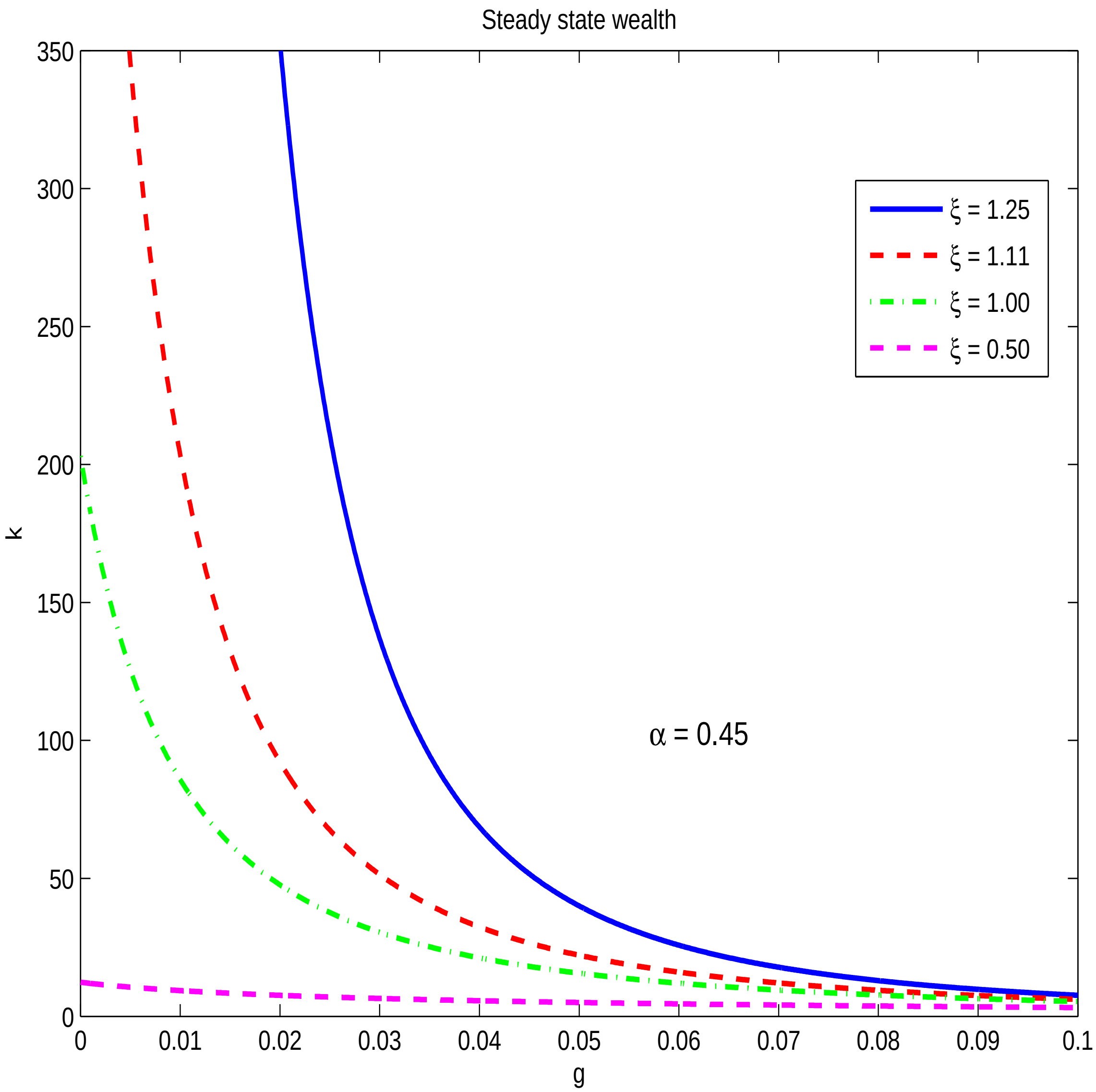


Figure 6: Capital to Income Ratio in a Zero Growth Steady State

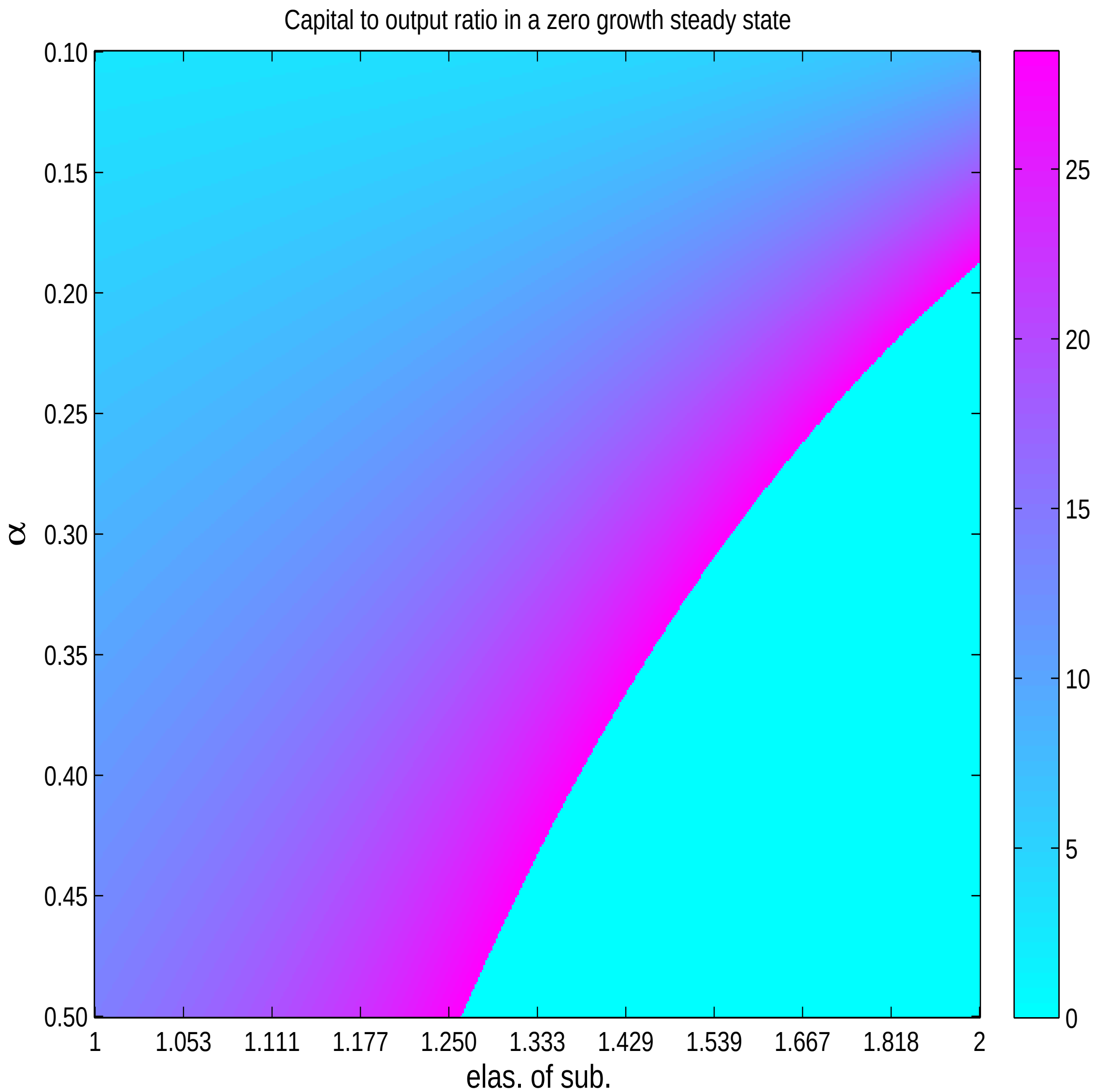
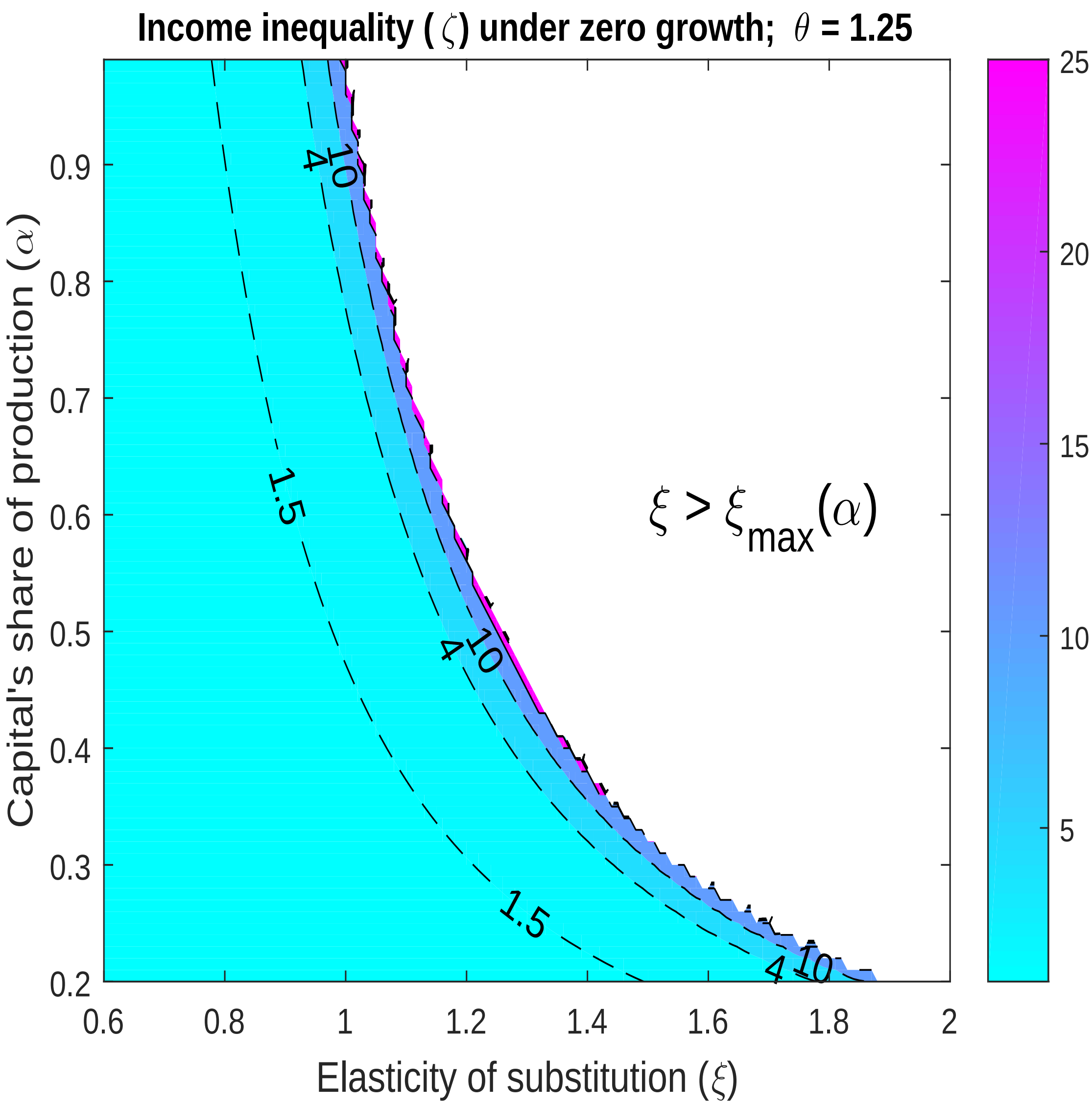
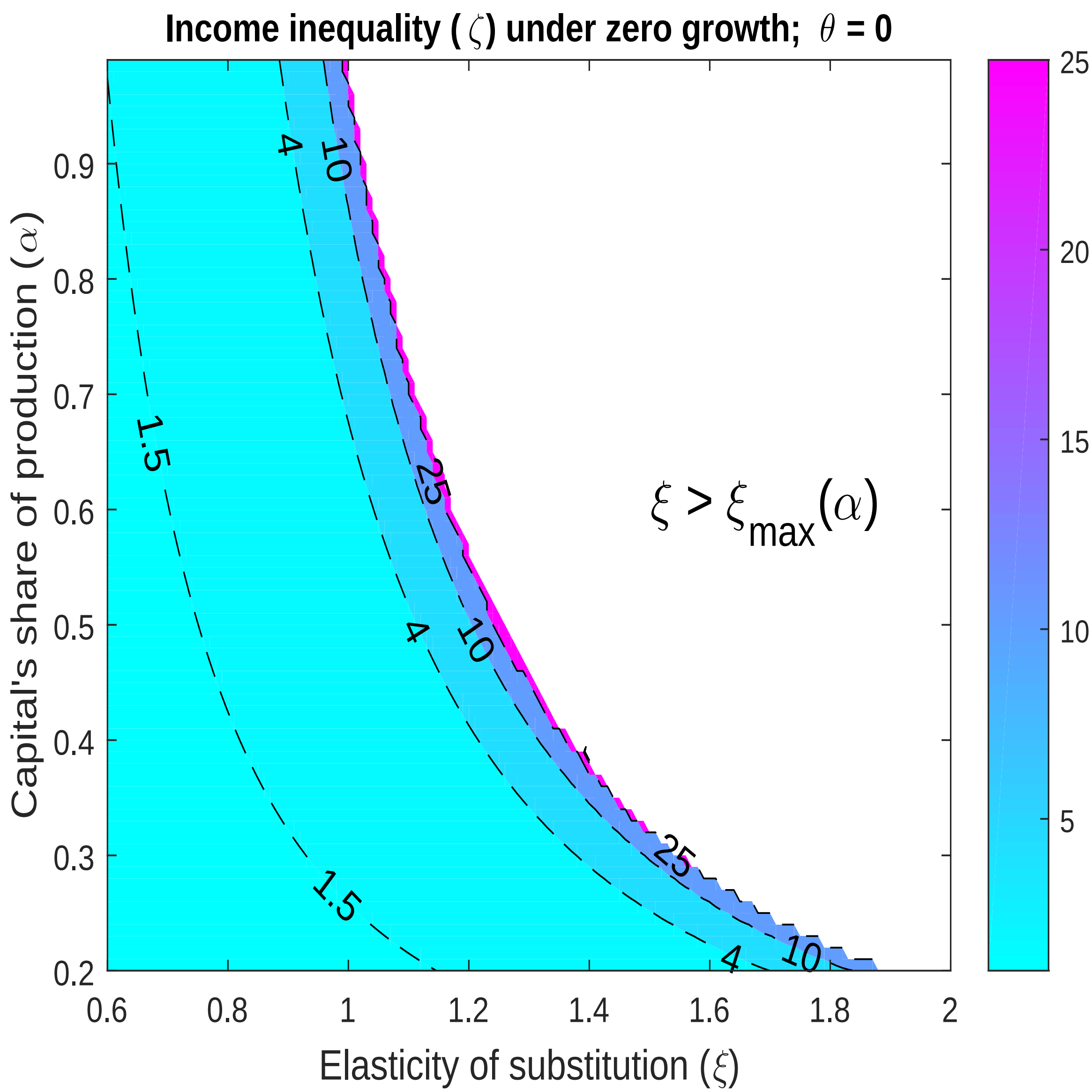


Figure 7: Income Inequality in a Zero Growth Steady State





Income inequality (ζ) under zero growth; θ = 0
Capital's share of production (α)
Elasticity of substitution (ξ)
$\xi > \xi_{max}(\alpha)$
1.5
4
10
25
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0.6
0.8
1
1.2
1.4
1.6
1.8
2
5
10
15
20
25

Figure 9: Income Inequality in a Zero Growth Steady State

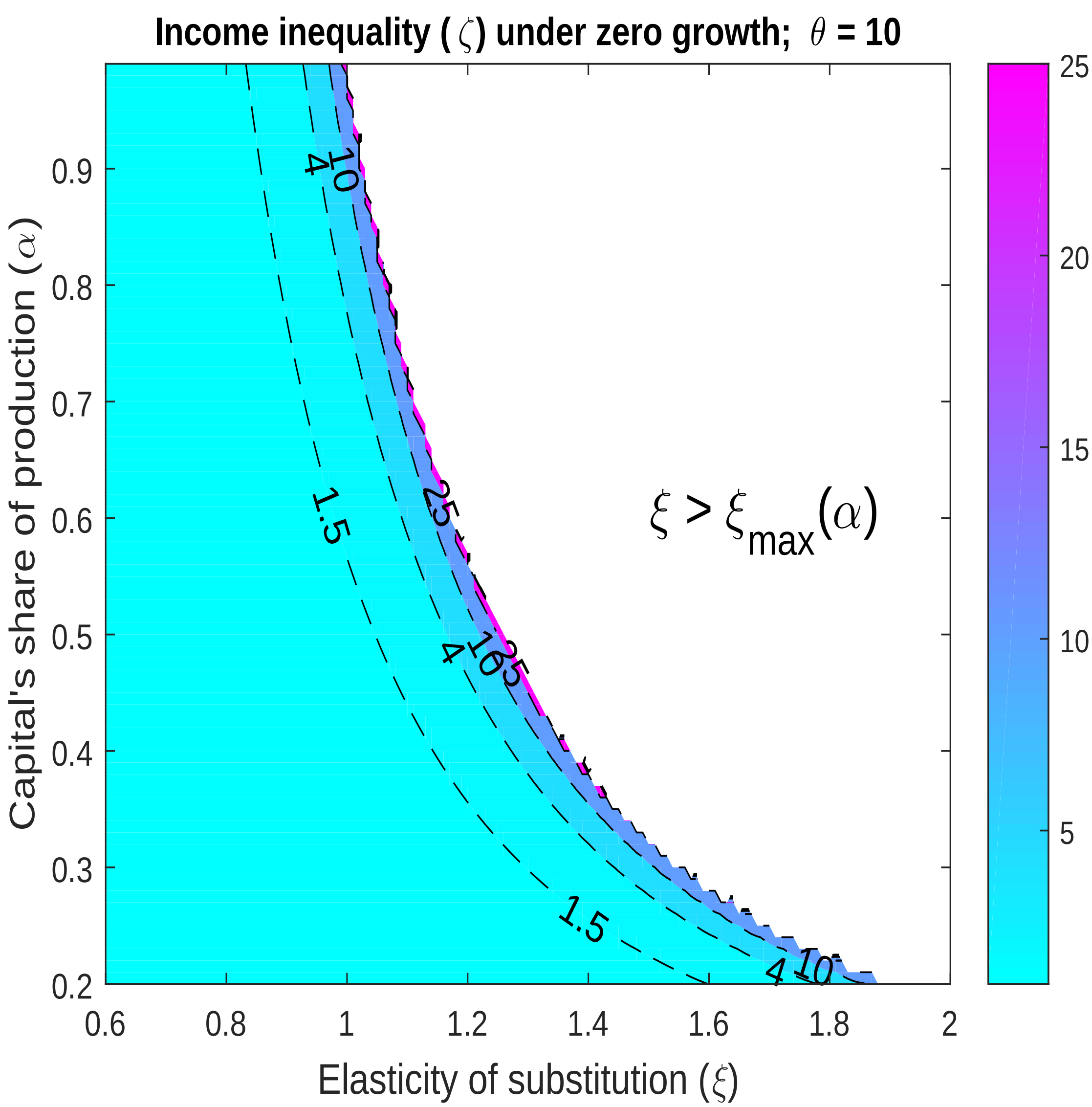


Figure 10: Steady State Inequality

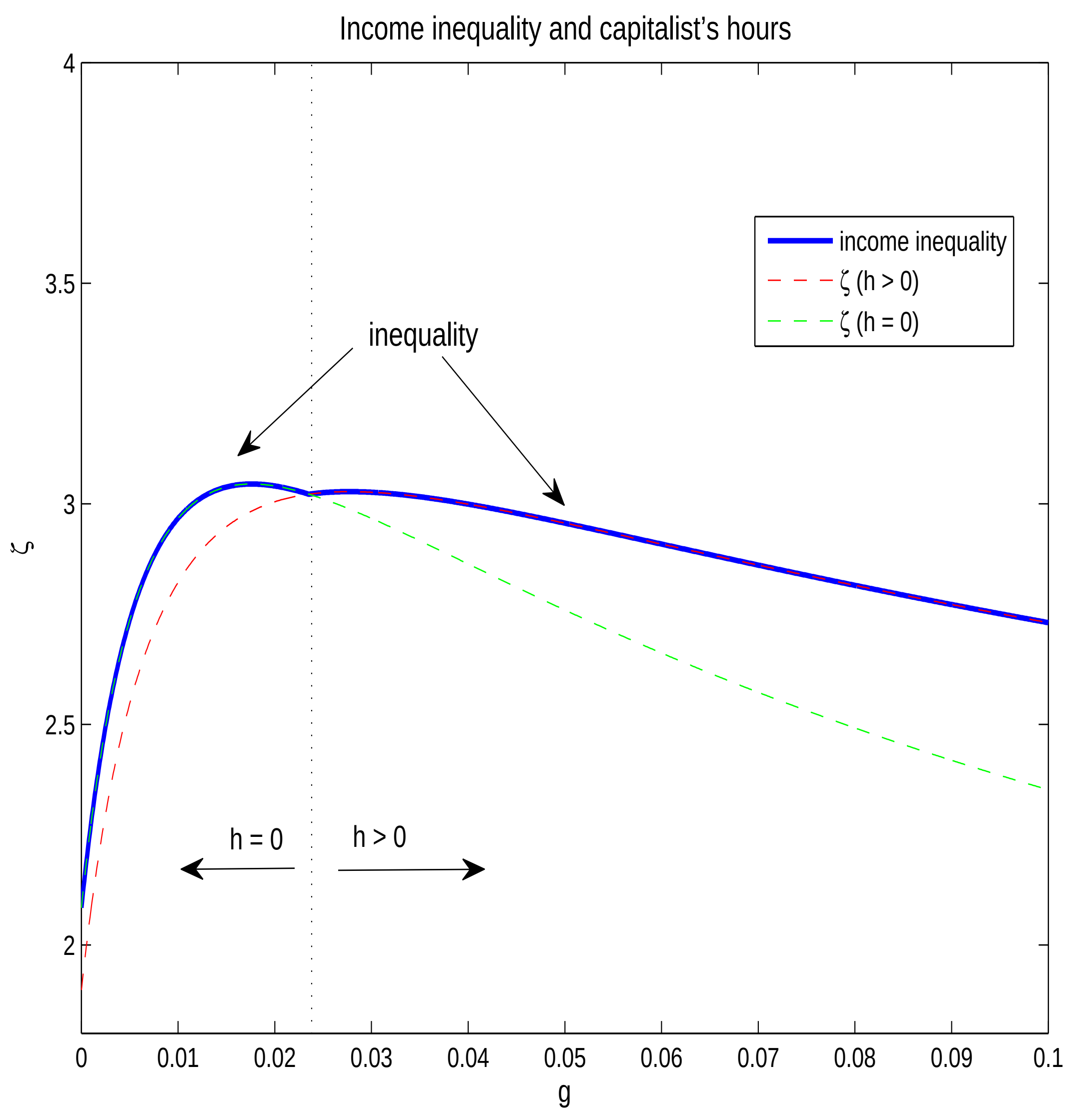


Figure 11: Dynamics of Inequality

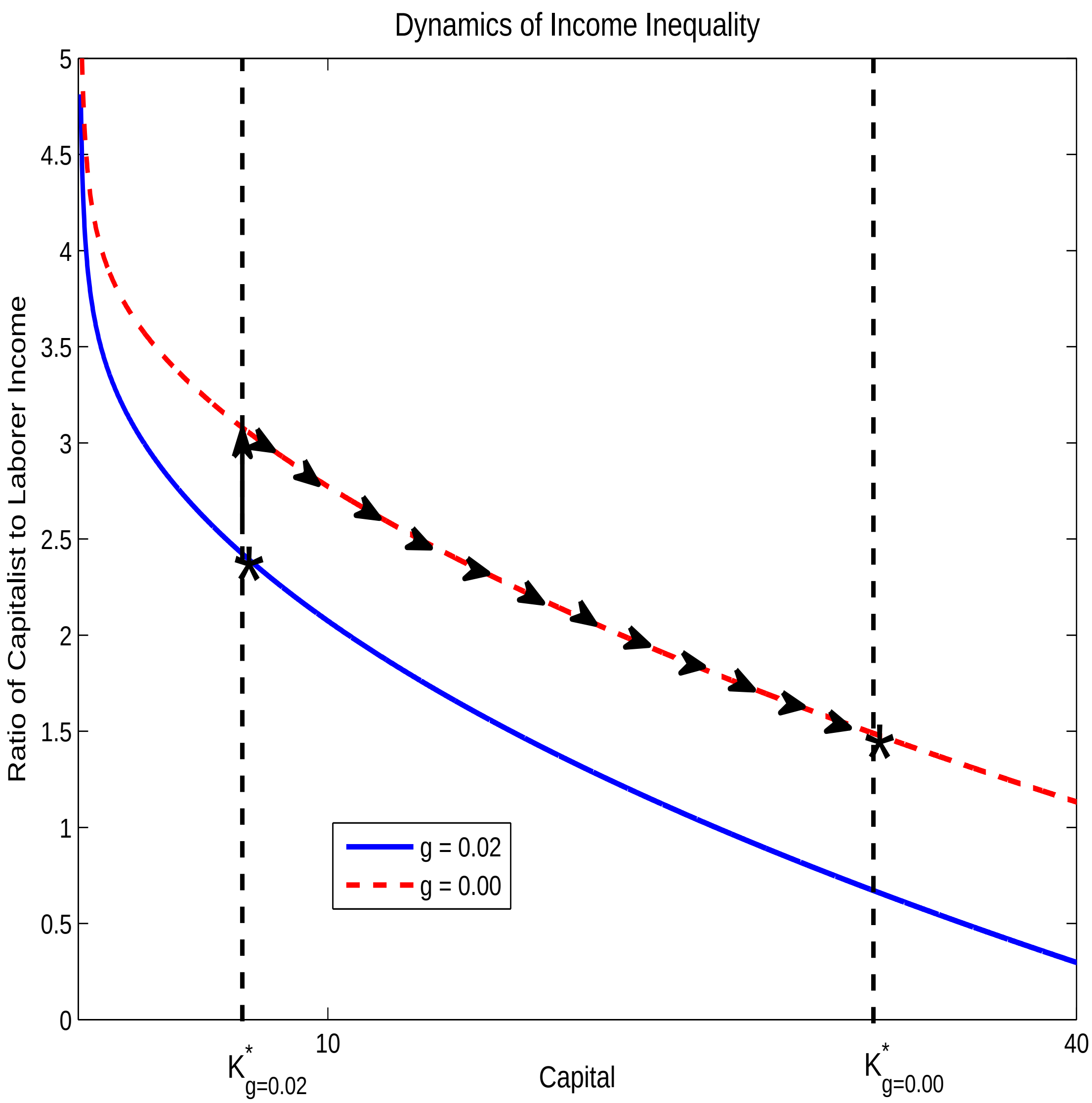


Figure 12: Laffer Curve

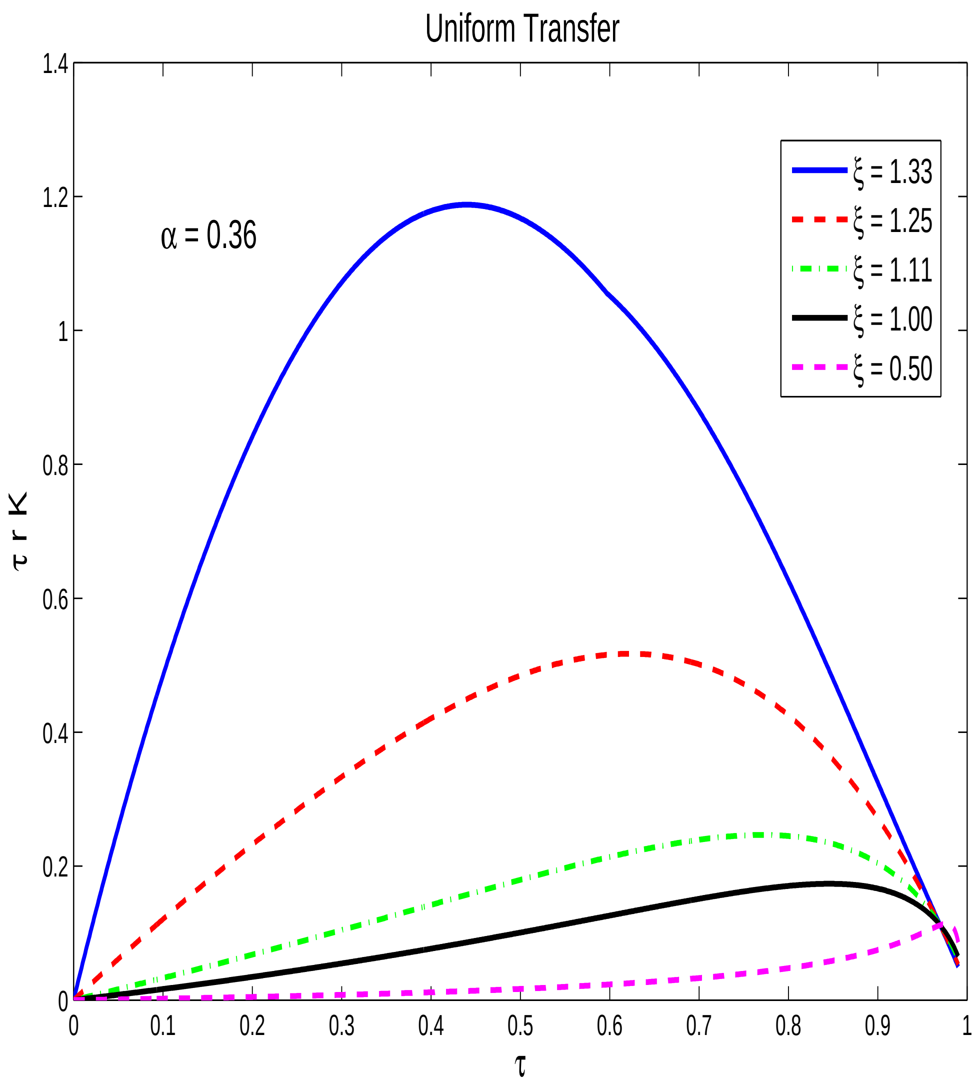


Figure 13: Laffer Curve

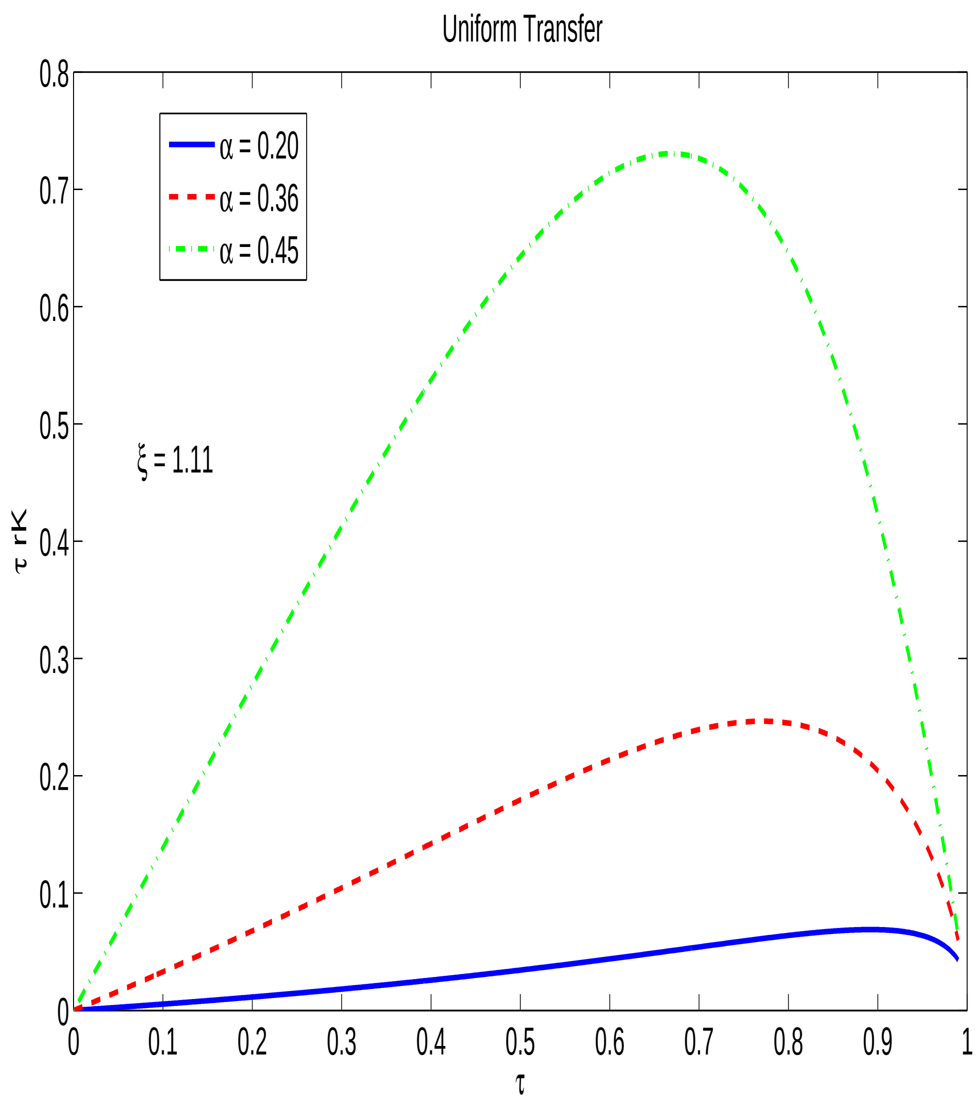


Figure 14: Long run inequality under capital income taxation and zero growth

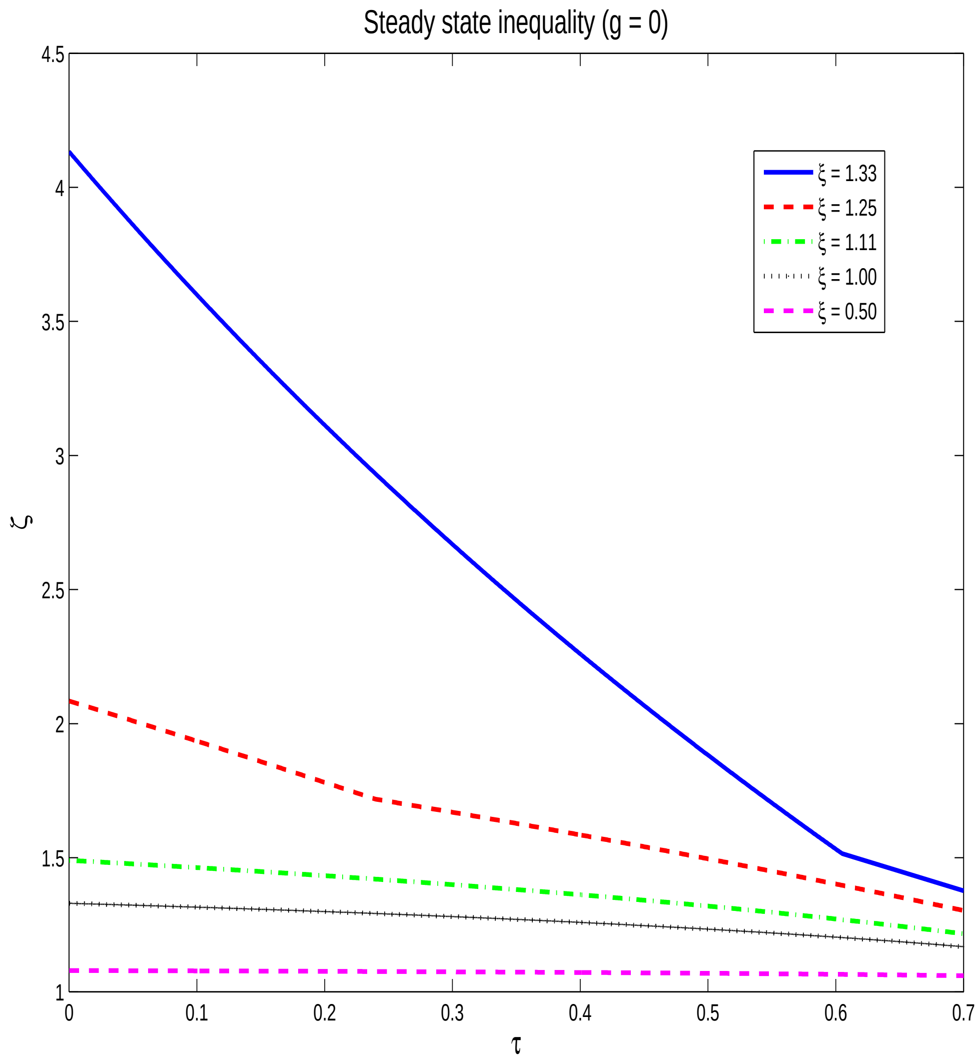


Figure 15: Necessary elasticity of substitution to cause higher inequality with zero growth

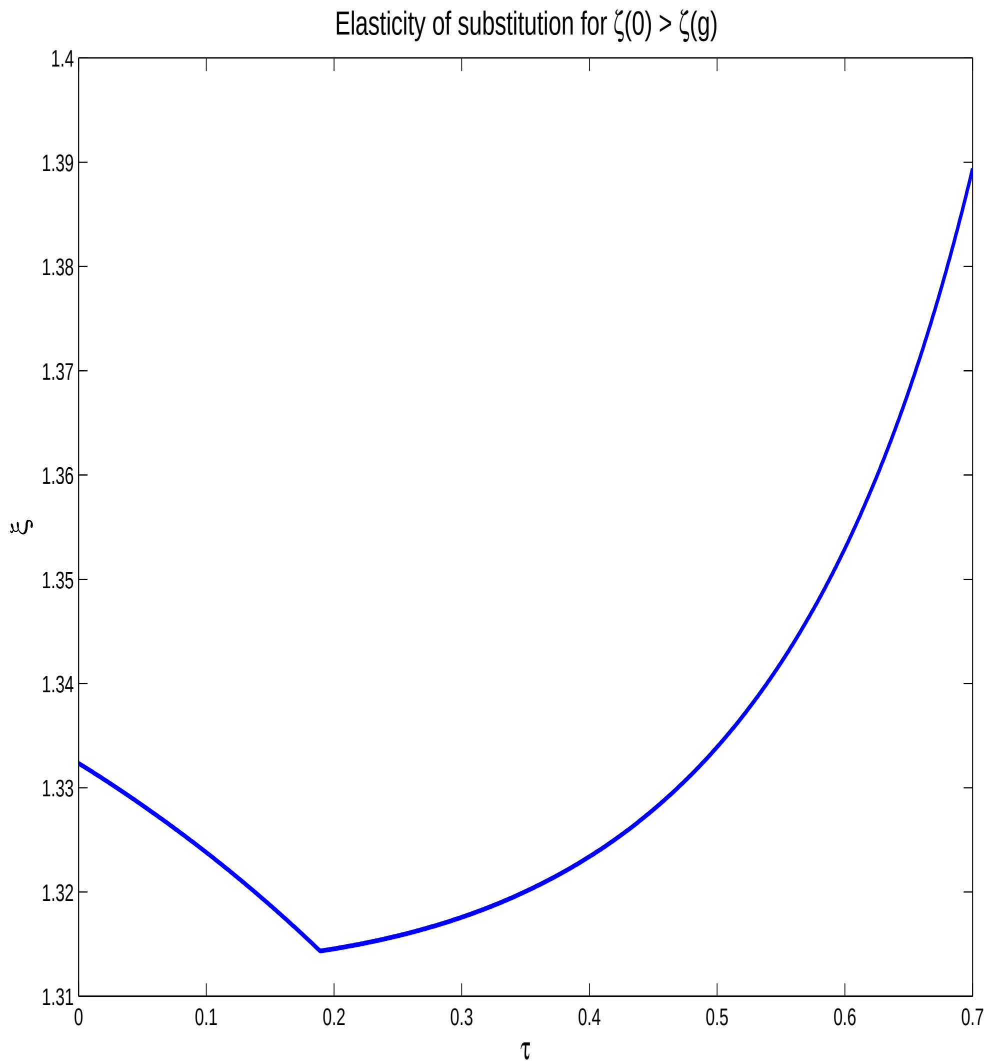


Figure 16: Capitalist Hours