

Neoclassical Inequality*

Daniel R. Carroll[†]

Research Department

Federal Reserve Bank of Cleveland

Eric R. Young[‡]

Department of Economics

University of Virginia

December 13, 2016

Abstract

In a model with a worker-capitalist dichotomy, we show that the relationship between inequality (measured as a ratio of incomes for the two types) and growth is complicated; zero growth can raise or lower inequality, depending on parameters. In particular, the elasticity of substitution between capital and labor in production needs to be considerably greater than 1 in order for income inequality be higher with zero growth. If this condition is not met, factor prices adjust strongly causing the fall in the return to capital (the rise in wages) to reduce income inequality. Our results extend to models with endogenous growth and idiosyncratic risk.

Keywords: Inequality; Growth; Elasticity of Substitution

1 Introduction

The effect of low growth on the distribution of income and capital has gained attention recently. Certainly the most prominent example is Thomas Piketty's *Capital in the Twenty-First Century*

*This paper is an expanded version of part of the paper previously circulated as "The Piketty Transition." We thank participants at the UVA-Richmond Fed Research Jamboree, the 7th International Conference of Macroeconomics at GRIPS, Hong Kong University, City University of Hong Kong, and the AEA Meetings for comments. Young also thanks the Batten Institute at the Darden School and the Bankard Fund for Political Economy for financial support and the hospitality of LAEF at UCSB.

[†]Email: Daniel.Carroll@clev.frb.org.

[‡]Corresponding author. Email: ey2d@virginia.edu.

which documents the dynamics of wealth inequality over hundreds of years and across several countries. In addition, Piketty puts forth an economic model to account for these data. Roughly speaking, Piketty's model has two groups of households, laborers and capitalists, who derive all of their income from a single source (labor and capital, respectively); in this environment a natural measure of inequality is capital's share of national income. Under the assumption that capitalists set net saving equal to a constant fraction of net output, a decline in growth to zero leads to an explosion in inequality. This result has been found in models where wealth is accumulated exogenously through multiplicative random shocks (Piketty and Saez 2014) and in models where wealth enters directly into the utility function of households (Piketty and Zucman 2015). Krusell and Smith (2015) show, however, that under more standard assumptions about capitalist saving, the rise in inequality from low growth is less dire than Piketty predicts: with constant gross saving out of gross income, the increase in inequality is substantially smaller (but, it should be noted, still quite large).

Our goal in this paper is to characterize the relationship between growth and inequality in a macroeconomic model minimally extended to generate inequality. Our model has the following ingredients: some households, called capitalists, own claims to the productive technology while other ones, called laborers, do not; both types have an endowment of time that can be rented to firms in return for labor income and their labor productivity is time-invariant. We can analytically characterize many features of the relationship between growth and inequality (measured as the ratio of capitalist income to laborer income), and we use numerical tools to uncover the behavior of income inequality as the long run growth rate of the economy goes to zero; we consider both the short-run and long-run effects (transitions and steady states).

We find that steady states with zero growth generally have lower inequality than steady states with positive growth rates. When growth is low, capitalists discount the future at a lower rate, and thus accumulate more capital; however, this accumulation leads to an abundance of capital relative to labor and results in higher equilibrium wages, both absolutely and relative to capital's return. In terms of inequality, the movement in relative factor prices away from capital and toward labor mitigates the effects of increasing wealth. Generally, the factor price effect dominates the rise the wealth. Zero growth steady states are associated with higher inequality only when the elasticity of substitution in production between capital and labor is considerably greater than 1

(i.e., strong substitutes in production). A high elasticity of substitution mutes the factor price effect by preventing wages from becoming "too large" relative to the return to capital. The elasticity of substitution cannot be arbitrarily high, however. Conditional on a particular capital share in production, balanced growth places a limit on the elasticity of substitution: the higher is capital's share the lower is the upper limit on the elasticity of substitution. We find that the explosive increase in long run inequality only occurs when the elasticity of substitution is very close to its upper bound. Moreover, the values required for this explosion in inequality to obtain are starkly at odds with empirical measurements of the elasticity of substitution given estimates of capital's share of income; to be specific, if capital's share of income is 0.36 (as documented by Gollin 2002 for a large sample of countries) the required elasticity is 1.33, which exceeds nearly every estimate surveyed by Chirinko (2008) except one extreme outlier estimate. Note that these estimates are specifically aimed at measuring the long-run elasticity of substitution, which is the relevant one for our study – short-run elasticities are likely to be even smaller.

We then characterize how the inequality-growth relationship changes in the presence of redistribution via capital income taxation. We consider two cases – either tax revenue is rebated lump-sum to all households (uniform transfers) or only to laborer households (targeted transfers). Not surprisingly, we find that redistribution does reduce inequality, but it operates by shrinking the income of the laborer household by less than that of the capitalist household, rather than by increasing the laborer's income and decreasing the capitalist's. And the minimum elasticity of substitution needed to get higher inequality at zero growth is not sensitive to the value of the capital tax, provided we remain on the upward-sloping portion of the Laffer curve.

Our experiments crucially assume that the growth rate is an exogenous parameter; we do not allow for feedback that goes from interest rates to growth rates. While this assumption has the virtue of not requiring us to take a stand on why growth falls (it is simply a decline in the growth rate of exogenous labor-augmenting technology), it is obviously limiting – if interest rates and growth rates are jointly determined, they may not behave in the same manner. With this point in mind, we extend the model to include a production externality, as in Romer (1986), that renders the social production function constant returns to scale in capital. Once again, we find that zero growth only produces extreme inequality when the elasticity of substitution between capital and labor is well above 1. The key is that the tight link between returns and growth – namely that

falling growth must come with falling returns – still obtains in endogenous growth models.¹

One common feature of the models described above is a tight link between growth rates, interest rates, and time rates of preference. Explosive inequality only arises if the interest rate falls very little as growth falls, but the perfectly-elastic long-run supply curve for capital in those models requires the high elasticity to prevent a large decline. We therefore turn to a class of models in which interest rates are partially disconnected from time rates of preference, namely incomplete market models (Aiyagari 1994). We assume that individual labor productivity is subject to a persistent idiosyncratic shock for both types of households and allow laborers to save using a low-return technology.² We find our results continue to obtain, because a reasonable amount of idiosyncratic risk still leaves the economy operating on a very (but not infinitely) elastic portion of the capital supply curve; put another way, interest rates still must fall because there is not enough "room" to accommodate a large decline in growth.

Finally, we extend the model further by introducing idiosyncratic shocks to capitalist returns. Piketty (2014) notes the importance of preventing declines in returns and suggests that financial innovation divorces the return on capital from the characteristics of the owner/operators; we interpret this innovation as introducing claims that permit returns to be diversified, effectively eliminating the idiosyncratic return risk. Idiosyncratic risk in returns is a key driver of wealth concentration in many models, and has support in the data (see Benhabib, Bisin, and Luo 2016, Hubmer, Krusell, and Smith 2016, De Nardi, Fella, and Yang 2016); what we investigate here is whether it changes our results about the connection between growth and inequality. We find that in the absence of financial innovation none of our results change – inequality does not significantly respond to changes in growth rates. In the presence of financial innovation, however, it does – but it goes in the opposite direction! Since idiosyncratic risk has been eliminated for the capitalist population, within-group inequality falls; thus we get that two measures of inequality, the capitalist-laborer income ratio and the Gini coefficient, actually go in opposite directions.

¹Other types of growth models reduce to similar social production functions; see Hammond and Rodríguez-Clare (1993).

²It turns out not to matter if this technology is money (negative real return) or an inferior type of capital.

2 Model

The model economy is populated by two groups, called capitalists and workers, who are situated in dynasties that live forever and value the utility of descendants; the size of the two groups are $\mu \in (0, 1)$ and $1 - \mu$. Both groups have identical isoelastic preferences over consumption and leisure (non-work time). Our main assumption is that there is no mobility across groups – at some point in the infinite past, some dynasties were lucky enough to get granted access to a productive asset called capital, and some were not, and that situation has persisted.

Both groups supply labor elastically and have constant labor productivities. We normalize the productivity of laborers to 1 and denote by e the labor productivity of capitalists. Throughout the paper, we maintain that $e \geq 1$, and generally will assume $e = 1$. One effective unit of labor earns w units of wage as compensation, and capital pays a gross return $1 + r$.

2.1 Household Problems

2.1.1 Laborers

Given the current stock of capital in the economy, the laborer's problem is a static choice of how many hours, l , to supply at market wage w

$$V(K) = \max_{l \in [0,1]} \left\{ \frac{[wl(1-l)^\theta]^{1-\sigma}}{1-\sigma} + \beta V(K') \right\}. \quad (1)$$

The solution is

$$l^* = \frac{1}{1+\theta} \quad (2)$$

$$x^* = wl^* \quad (3)$$

where x^* is the household's consumption.

2.1.2 Capitalists

A capitalist chooses consumption, hours, and savings to solve the dynamic program

$$v(k, K) = \max_{k', h, c} \left\{ \frac{[c(1-h)^\theta]^{1-\sigma}}{1-\sigma} + \beta(1+g)^{1-\sigma} v(k', K') \right\} \quad (4)$$

subject to

$$c + (1 + g)k' \leq whe + (1 + r)k \quad (5)$$

$$k' \geq 0, c \geq 0, h \in [0, 1).$$

Since the first two boundary conditions will never bind, we ignore them from now on. Taking the first-order conditions and applying the envelope condition produces three conditions:

$$\left[c(1-h)^\theta \right]^{-\sigma} = \beta(1+g)^{-\sigma} \left[c'(1-h')^\theta \right]^{-\sigma} (1+r') \quad (6)$$

$$\left[c(1-h)^\theta \right]^{-\sigma} \left[we(1-h)^\theta - \theta c(1-h)^{\theta-1} \right] \leq 0 \quad (7)$$

$$c + (1 + g)k' = whe + (1 + r)k; \quad (8)$$

the second condition holds with equality if $h > 0$.

Capitalists choose consumption c , work effort h , and capital holdings k' to maximize lifetime utility; we have already incorporated growth in labor productivity g in the usual method to ensure the (normalized) wealth of the capitalist remains bounded over time (see King, Plosser, and Rebelo 1988 for details on how this normalization is done). We require that the inverse of the intertemporal elasticity of substitution satisfies $\sigma > 0$.³

We can obtain the aggregate capital stock and labor input by summing over all individuals.

$$K = \mu k \quad (9)$$

$$N = \mu h e + (1 - \mu) l. \quad (10)$$

Note the asymmetry – capitalists supply all the capital, but labor is (at least in principle) supplied by both types; note also that aggregate labor input is in terms of "effective" units of labor (hours weighted by productivity) and that both types' hours are perfect substitutes in the production of effective hours.

³Moll (2014) shows that the model is completely analytically tractable if $\sigma = 1$ for capitalist households. We survey briefly the literature on estimating σ later in the paper, as this parameter plays an important role in some of our results.

2.2 The Firm

The supply side of our economy consists of a single firm employing a constant returns to scale production technology (nothing would change if we had a large number of identical firms, except notation would be more tedious):

$$Y = (\alpha K^\nu + (1 - \alpha) N^\nu)^{\frac{1}{\nu}}, \quad (11)$$

where $\alpha \in (0, 1)$ is the "share" of capital in production and $\nu \leq 1$ governs the elasticity of substitution. If $\nu = 1$, capital and labor are perfectly substitutable, so that the firm will employ only the cheaper factor. If $\nu = -\infty$, capital and labor are perfect complements, and therefore will be employed in fixed ratios (given by $\frac{\alpha}{1-\alpha}$). If $\nu = 0$, we get the Cobb-Douglas case where the shares of capital and labor income in total income will be fixed at α and $1 - \alpha$.⁴ Profit maximization yields

$$r = \alpha \left(\alpha + (1 - \alpha) \left(\frac{K}{N} \right)^{-\nu} \right)^{\frac{1-\nu}{\nu}} - \delta \quad (12)$$

$$w = (1 - \alpha) \left(\alpha \left(\frac{K}{N} \right)^\nu + 1 - \alpha \right)^{\frac{1-\nu}{\nu}}. \quad (13)$$

Note that both the rental rate and the wage rate are related to the capital-labor ratio, but not to the levels of capital and labor.⁵

Finally, there are aggregate conditions that relate supply and demand in each of three markets – the markets for capital, labor, and "goods". First, the firm must hire all the capital and labor supplied by households (these conditions are ensured by variations in r and w). Second, the supply of goods must be sufficient to cover the consumption of capitalists, the consumption of workers, and the investment by capitalists into new capital:

$$\mu c + (1 - \mu) x + \mu (k' - (1 - \delta) k) = Y. \quad (14)$$

Walras's law ensures that the goods market condition will be satisfied provided both the labor and capital markets clear.

⁴We experimented with a form of capital-skill complementarity where capitalist hours are a complement to capital and laborer hours are a substitute to the composite of capital and capitalist hours. We found that our results are not changed; further details are available upon request.

⁵The assumption that firms operate in perfectly competitive goods markets is not restrictive; assuming monopolistic competition does not change our results. It only changes the market clearing condition for capital by subtracting the value of profits from the savings of capitalists when determining the value of aggregate capital.

2.2.1 General Equilibrium

A recursive competitive equilibrium is a set of household functions

$$\{V(K), v(k, K), h(k, K), c(k, K), k'(k, K), l(k, K), x(k, K)\},$$

price functions $r(K)$ and $w(K)$, and aggregate labor $N(K)$ such that

1. Given pricing functions, the household functions solve the capitalist and laborer problems;
2. Given pricing functions the firm maximizes profit by demanding K and $N(K)$;
3. Markets clear:

$$K = \mu k'(K, K)$$

$$N(K) = \mu h(K, K) e + (1 - \mu) l(K, K)$$

$$Y(K) = \mu c(K, K) + (1 - \mu) x(K, K) + \mu(1 + g) k'(K, K) - (1 - \delta) \mu K.$$

2.3 Steady State

The balanced growth path is characterized by the system of equations

$$1 = \beta(1 + g)^{-\sigma}(1 + r) \tag{15}$$

$$h = \max \left\{ \frac{we - \theta(r - g)k}{(1 + \theta)we}, 0 \right\} \tag{16}$$

$$c = whe + (r - g)k \tag{17}$$

$$l = \frac{1}{1 + \theta} \tag{18}$$

$$x = w(r)l \tag{19}$$

$$w(r) = (1 - \alpha)^{\frac{1}{\nu}} \frac{r + \delta}{\alpha} \left[\left(\frac{r + \delta}{\alpha} \right)^{\frac{\nu}{1-\nu}} - \alpha \right]^{\frac{\nu-1}{\nu}}. \tag{20}$$

The steady state Euler equation pins down r ,

$$r = \frac{(1 + g)^\sigma - \beta}{\beta}, \tag{21}$$

which through the first-order conditions of the firm determines the steady state wage rate

$$w = (1 - \alpha)^{\frac{1}{\nu}} \frac{r + \delta}{\alpha} \left[\left(\frac{r + \delta}{\alpha} \right)^{\frac{\nu}{1-\nu}} - \alpha \right]^{\frac{\nu-1}{\nu}}. \tag{22}$$

Notice that for $\sigma > 0$, the steady state interest rate is increasing in g . If we restrict attention to non-negative growth rates, the interest rate attains its minimum and the wage rate its maximum when $g = 0$, where the interest rate is

$$r_{\min} = \frac{1 - \beta}{\beta}$$

and the wage is

$$w_{\max} = (1 - \alpha)^{\frac{1}{\nu}} \frac{r_{\min} + \delta}{\alpha} \left[\left(\frac{r_{\min} + \delta}{\alpha} \right)^{\frac{\nu}{1-\nu}} - \alpha \right]^{\frac{\nu-1}{\nu}}. \quad (23)$$

Notice that while r_{\min} is determined only by the discount factor, w_{\max} also depends upon capital share's in production, α , and the elasticity of substitution parameter, ν . Figure 1 plots the steady state wage when $g = 0$. For higher values of ν , the steady state wage increases exponentially, and the slope is increasing in α . Not all combinations of α and ν are permissible since

$$\alpha^{\frac{1}{1-\nu}} < (r_{\min} + \delta)^{\frac{\nu}{1-\nu}} \quad (24)$$

must hold for wages to be real numbers. Given (α, β, δ) , the upper bound on ν is $\nu_{\max} = \frac{\log(\alpha)}{\log(r_{\min} + \delta)}$. In order to allow for capital and labor to be either complements or substitutes in production, we impose that $\alpha > \frac{1-\beta}{\beta} + \delta$, which implies $\nu_{\max} > 0$. Under the baseline calibration (see below), $r_{\min} \approx 0.0101$, implying that $\nu_{\max} \approx 0.305$. Thus, balanced growth puts a restriction on the degree to which capital and labor are substitutable; letting $\xi = \frac{1}{1-\nu}$ denote the elasticity, we find $\xi_{\max} = \frac{1}{1-0.305} = 1.438$.

Under appropriate conditions for α and ν , we can find K by imposing the capital market clearing condition at r :

$$\begin{aligned} K &= \left\{ \frac{\left[\frac{r+\delta}{\alpha} \right]^{\frac{\nu}{1-\nu}} - \alpha}{1 - \alpha} \right\}^{-\frac{1}{\nu}} N \\ &= \varphi N \end{aligned} \quad (25)$$

where

$$N = \mu h e + \frac{1 - \mu}{1 + \theta}. \quad (26)$$

Note here that φ is the capital-to-labor ratio. Since under the restrictions on α and ν

$$\frac{d\varphi}{dr} = -\frac{1}{\alpha(1-\alpha)(1-\nu)} \left(\frac{\left(\frac{r+\delta}{\alpha}\right)^{\frac{\nu}{1-\nu}} - \alpha}{1-\alpha} \right)^{-\frac{1+\nu}{\nu}} \left(\frac{\alpha}{r+\delta} \right)^{\frac{1-2\nu}{1-\nu}} < 0 \quad (27)$$

and $\frac{dr}{dg} > 0$, the capital-labor ratio will be higher in a low-growth economy. Finally, the steady state relative return on capital (as compared to human capital) $r - g$ falls if and only if

$$\sigma(1+g)^{\sigma-1} > \beta,$$

which, near $g = 0$, requires $\sigma \geq 1$. Estimates for σ in the literature run from essentially infinite (Hall 1988, Campbell 1999) to close to one for the subgroup of stock market participants (Güvenen 2006) to significantly below one for that same group (Vissing-Jørgensen 2002). To get $r - g$ to rise with low growth, the latter estimates must not apply; given the survey discussion in Havránek (2015), it seems reasonable to assume that $\sigma \geq 1$ is most plausible even for the capitalists, and he concludes that the median estimate corrected for publication bias is somewhere around 3 with a minimum of roughly 1.2. We will use $\sigma = 2$ as representative of the macroeconomic literature.

Aggregate effective labor is a function of K because the capitalist hours decision depends upon wealth. If wealth is sufficiently high, then the non-negativity constraint on hours binds. We consider both the binding and non-binding cases below.

2.3.1 Case 1: Capitalists Work

Under the assumption that capitalists supply positive hours, we can obtain aggregate capital by substituting the definition of h into the market clearing condition for capital,

$$\begin{aligned} K &= \varphi N \\ &= \frac{\varphi \left(\frac{\mu e + (1-\mu)}{1+\theta} \right)}{1 + \varphi \frac{\theta}{1+\theta} \frac{(r-g)}{w}}. \end{aligned} \quad (28)$$

It can be shown that

$$\frac{w}{\varphi} = \left(\frac{r+\delta}{\alpha} \right)^{\xi} - (r+\delta) > 0, \quad (29)$$

where $\xi = \frac{1}{1-\nu}$ is the elasticity of substitution between capital and labor. The strict inequality results from imposing the restriction $\nu < \nu_{\max}$. Multiplying the numerator and denominator of

(28) by $\frac{w}{\varphi}$, we arrive at

$$K = \frac{w [\mu e + (1 - \mu)]}{(1 + \theta) \left[\left(\frac{r+\delta}{\alpha} \right)^\xi - (r + \delta) \right] + \theta (r - g)}.$$

Individual capitalist wealth is

$$k = \frac{K}{\mu} = \frac{[\mu e + (1 - \mu)]}{\mu} \frac{w}{r - g} \frac{1}{(1 + \theta) \frac{r}{r-g} z(g) + \theta},$$

where

$$z(g) = \frac{\left(\frac{r+\delta}{\alpha} \right)^\xi - (r + \delta)}{r} = \frac{\frac{w}{r}}{\frac{K}{N}} > 0. \quad (30)$$

Two of three factors in determining the behavior of steady state inequality, the factor price ratio and the capital-to-labor ratio, are expressed in the function $z(g)$. The sensitivity of the factor price ratio relative to the capital-to-labor ratio affects how inequality responds near zero growth; and this sensitivity depends directly upon the elasticity of substitution between capital and labor.

Substituting k into h ,

$$h = \frac{1}{1 + \theta} \left[1 - \frac{\mu e + (1 - \mu)}{\mu e} \frac{\frac{\theta}{1+\theta}}{\frac{r}{r-g} z(g) + \frac{\theta}{1+\theta}} \right]. \quad (31)$$

We can now derive steady state income inequality, ζ , measured by the ratio of capitalist's income, $y = whe + rk$, to laborer's income, $q = \frac{w}{1+\theta}$:

$$\begin{aligned} \zeta &= \frac{y}{q} \\ &= e + \frac{\mu e + (1 - \mu)}{\mu} Q(z(g); \theta) \end{aligned}$$

where

$$Q(z(g); \theta) = \frac{\frac{r}{r-g} - \frac{\theta}{1+\theta}}{\frac{r}{r-g} z(g) + \frac{\theta}{1+\theta}}.$$

Inequality increases as the measure of capitalists, μ , decreases. Holding all other parameters constant, the steady state Euler equation implies a unique capital-to-effective labor input ratio, and consequently w and r are invariant to μ . Because laborer's hours are constant, a lower μ necessarily implies higher effective labor supply. Therefore, K must rise in proportion to N , and so capitalists' wealth, $k = \frac{K}{\mu}$, also increases. Because factor prices do not change, laborer's income is constant, but capitalists' income increases.

Given population share, relative labor productivity, and preferences, the behavior of inequality fundamentally depends on the term $Q(z(g); \theta)$. Notice that $Q(z(g); \theta)$ is continuous in θ . We now state a series of propositions which characterize steady state income inequality conditional on g . In the interest of space, all proofs are relegated to the appendix.

Proposition 1. *Inequality is bounded from below by relative labor productivity e .*

It follows immediately from the non-negativity of Q that inequality increases with relative productivity e . We assume that capitalists are at least as productive as laborers, and since e is the lower bound, we can without loss of generality assume that $e = 1$. We make this assumption in the remainder of the paper.

Proposition 2. *Holding $z(g)$ constant, inequality is increasing in $\frac{r}{r-g}$.*

As $g \rightarrow 0$, $\frac{r}{r-g}$ declines monotonically to 1 so it acts to reduce inequality in a zero growth steady state. Note that since $\frac{r}{r-g}$ can be written as

$$\frac{1}{1 - \left(\frac{r}{g}\right)^{-1}},$$

the above proposition is the same as saying that inequality falls in $\frac{r}{g}$ (again ignoring $z(g)$)⁶. Of course, $\frac{r}{r-g}$ cannot move without changing $z(g)$ as well, since z is a function of both r and g . It is useful though to understand that if inequality *increases* as g goes to zero, it must result entirely from a decline in $z(g)$. The sign of $\frac{dz}{dg}$ depends on parameters, and so then is the sign of $\frac{dQ}{dg}$. Later in the paper we therefore turn to numerical methods to get a clearer picture of exactly how g affects ζ .

Proposition 3. *Steady state inequality is greater when the preference for leisure is weak (i.e., θ is small).*

This statement is proven by signing the derivative of Q with respect to θ and highlights the importance of the capitalists' hours decision for determining long run income inequality.

⁶This is consistent with the claim that as the gap between r and g increases inequality declines.

2.3.2 Case 2: Capitalists Do Not Work

The expressions are simpler when capitalists do not work. When $h = 0$, N is fixed at $\frac{1-\mu}{1+\theta}$, and

$$K = \varphi \left(\frac{1-\mu}{1+\theta} \right):$$

$$k = \frac{K}{\mu} = \frac{1-\mu}{\mu} \varphi \frac{1}{1+\theta},$$

so

$$\zeta = \frac{1-\mu}{\mu} \frac{\varphi}{w} r.$$

Substituting in (29), inequality can be written as a function of the rental rate and the model parameters,

$$\begin{aligned} \zeta &= \left[\frac{1}{\mu} - 1 \right] \frac{r}{\left(\frac{r+\delta}{\alpha} \right)^\xi - (r+\delta)} \\ &= \left[\frac{1}{\mu} - 1 \right] \frac{1}{z(g)} \\ &= \left[\frac{1}{\mu} - 1 \right] Q(z(g); 0) \end{aligned}$$

Having solved for inequality in terms of Q , we can, for a fixed growth rate, bound steady state inequality and order it over θ .

Proposition 4. *For a given growth rate, g , $\zeta \in \left[\left(\frac{1}{\mu} - 1 \right) Q(z(g); 0), 1 + \frac{1}{\mu} Q(z(g); 0) \right]$.*

Because $\varphi = \frac{K}{N}$, we can re-write $Q(z(g), 0)$ as the capital-to-labor ratio divided by the factor price ratio.

$$Q(z(g), 0) = \frac{1}{z(g)} = \frac{\frac{K}{N}}{\frac{w}{r}} = \frac{rK}{wN}.$$

The market clearing conditions for capital and labor imply

$$\frac{r+\delta}{w} = \frac{MPK}{MPN} = \frac{1-\alpha}{\alpha} \left(\frac{K}{N} \right)^{\frac{1}{\xi}}.$$

We can now express the bounds on inequality in terms of the capital-to-labor ratio and the steady state interest rate:

$$\zeta \in \left[\frac{1-\mu}{\mu} \frac{\alpha}{1-\alpha} \chi \left(\frac{K}{N} \right)^{\left(1-\frac{1}{\xi}\right)}, 1 + \frac{1}{\mu} \frac{\alpha}{1-\alpha} \chi \left(\frac{K}{N} \right)^{\left(1-\frac{1}{\xi}\right)} \right],$$

where $\chi = \frac{r}{r+\delta}$.

Proposition 5. *Steady state inequality is an increasing function of α and ξ .*

Finally, we state a necessary condition for inequality to be higher in a zero growth steady state than it is in a near zero growth steady state.

Proposition 6. *For $e = 1$, if $\zeta(g) < \zeta(0)$, then the elasticity of substitution is greater than 1.*

Intuitively, this proposition says that in order for zero growth to lead to a steady state with greater inequality, factor prices $\frac{w}{r}$ must rise by less than $\frac{K}{N}$, which only happens if the elasticity of substitution between labor and capital is above 1. Moreover, positive depreciation increases the required degree of substitutability. The tradeoff can be seen most easily when $z(g)$ is written as the ratio

$$z(g) = \frac{\frac{w}{r}}{\frac{K}{N}} = \frac{\frac{w}{r+\delta}}{\chi \frac{K}{N}}$$

and $\xi = 1$ (i.e., the production function is Cobb-Douglas). In that case,

$$z(g) = \frac{(1-\alpha)}{\alpha} \frac{1}{\chi(g)}$$

and

$$z(0) > z(g)$$

so $\zeta(g) > \zeta(0)$. Without depreciation, χ would be 1, and $z(g)$ would be constant which again would imply that $\zeta(g) > \zeta(0)$. In the literature, Rognlie (2015) focuses on the distinction between gross and net elasticities, showing that net elasticities are always smaller; as noted in the Introduction, Piketty focuses on net saving out of net income, and thus needs a very large gross elasticity to reverse this result and obtain $\zeta(g) < \zeta(0)$.

2.3.3 How Income Inequality Changes with Growth

In the steady state income inequality can be decomposed into the sum of two ratios,

$$\zeta \propto \frac{rk}{wl} + \frac{h}{l}e.$$

The first term is the ratio of capital income to laborer's income. If capitalists do not work, then income inequality is proportional the ratio of capital income to labor income in the economy.

When capitalists supply positive hours, some algebra shows that

$$\begin{aligned} \zeta &\propto \frac{we - \theta(r-g)k + (1+\theta)rk}{w} \\ &= e + \frac{r+\theta g}{w}k. \end{aligned}$$

Regardless of capitalists' hours, whether inequality rises or falls depends solely upon the product of wealth, k , and $\frac{r+\theta g}{w}$, which behaves in the same way as the factor price ratio $\frac{r}{w}$.

Because the closed-form expressions for steady state inequality do not yield unambiguous results for the effect of g on inequality, we use a computer to evaluate the expressions and plot the results for long run growth rates between 0 and 10 percent. To analyze the model numerically, we need to assign values to the structural parameters of the model. Here, we pick a reasonable set of values for some parameters, where reasonable means "gives rise to aggregates roughly consistent with US post-war averages." These numbers are $\beta = 0.99$, $\sigma = 2$, $\alpha = 0.36$, $\delta = 0.025$, $\theta = 1.25$, $g = 0.02$, and $e = 1$. Finally, to give Piketty's argument a stronger case, we set $\nu = 0.1$ ($\xi \approx 1.11$) so that capital and labor are more substitutable than the usual Cobb-Douglas case ($\xi = 1$); this value of ν satisfies the restrictions needed to have a steady state growth path.⁷ Figures 2-3 show the steady state ratio of capitalist income to laborer income for growth rates between 0 and 10 percent for the baseline capital share of income in production and for a higher value.

In most cases, within a neighborhood of zero growth, inequality is falling rather than rising, even if capital and labor are substitutes. High inequality only obtains whenever the elasticity of substitution is close to the maximum conditional on α .

For all of the parameter values, wealth decreases with growth. Figures 4-5 plot $k(g)$ for several values of ξ and of α . As the elasticity of substitution between capital and labor increases, the level of capital in the zero-growth steady state becomes very large, especially when capital's share, α , is high. If we ignored the general equilibrium effect on prices, savings behavior alone would suggest extreme inequality; however, the factor price ratio also responds to g . Because $w(r)$ is decreasing in r and r is increasing in g ,

$$\frac{dw}{dg} < 0.$$

Therefore as the long-run growth rate declines so does the factor price ratio. Numerical results show that unless ξ is close to $\xi_{\max}(\alpha) = \frac{1}{1-\nu_{\max}}$, the declining factor price ratio more than offsets wealth near $g = 0$, so capital income to laborer income declines in the neighborhood of zero growth.

⁷Thus, our productivity growth should be interpreted as purely labor-augmenting or as exogenously-accumulating human capital (see King, Plosser, and Rebelo 1988); with Cobb-Douglas it does not matter whether the productivity growth affects capital, labor, or both.

To make the key point clear, Figure (8) displays ζ in the steady state as a function of g ; the kink occurs when the capitalist's labor supply hits zero. At higher levels of g , where, under the baseline calibration, capitalists supply positive hours, inequality is decreasing in g , but the effects are not very large. In this region, the rise in $\frac{r}{r-g}$ discussed in Proposition (2) is offset by a rise in $z(g)$. Remember that $z(g) = \frac{w}{r} / \frac{K}{N}$. By (27), we know that the denominator of $z(g)$ is falling. As capital becomes scarce relative to labor, the numerator falls as well. Because ξ is not sufficiently larger than 1 in the baseline, the factor price ratio declines faster than the capital-labor ratio. The dashed line below inequality, labeled $\zeta(h=0)$ in the figure, shows the behavior of $z(g)$ more clearly. It plots $\left[\frac{1}{\mu} - 1\right] Q(z(g); 0)$, the appropriate measure when capitalists do not work, which only depends on $z(g)$. Notice that this line declines much faster in g than does actual inequality.

Continuing toward $g = 0$, we see that steady state income inequality rises slightly after the kink before descending sharply. To the left of the kink, capitalists do not work, so $\frac{r}{r-g}$ has no direct effect on ζ . Initially, $z(g)$ is still falling and without a corresponding decline in $\frac{r}{r-g}$ income inequality rises. Eventually, as g moves closer to 0 and $\frac{K}{N}$ climbs steeply, the aggressive adjustment for the factor price takes over and $z(g)$ rises. As before, the dashed line below inequality shows a counterfactual measure, this time using the measure for when capitalists work. Notice that it falls more sharply than ζ , highlighting the combined effects of $z(g)$ and $\frac{r}{r-g}$ falling.

We also plot in Figure (9) the dynamics of ζ starting from the steady state with $g = 0.02$; inequality initially jumps up due to changes in labor supply, then declines monotonically as capital accumulates. In the first period, income inequality jumps for two reasons. First, because only capitalists supply labor elastically, the increase in N_t is due entirely to capitalists. The wage falls in response, but not in the same proportion as N_t rises, so capitalist's labor income increases. Second, although K_t is inelastic, r_t increases, pushing up capital income. Therefore, both sources of capitalist's income rise while laborer's income falls slightly because of lower wages. After the initial surge in inequality, capitalists accumulate wealth over time, so wages rise and the return on capital falls. In the new steady state, income inequality is well *below* its original level.

2.3.4 Conditions for High Inequality with Zero Growth

We have shown that with exogenous growth, zero growth implies extreme levels of income inequality, capital relative to output, and capital share of output only when the capital share and the elasticity of substitution in the production function are high. Next we ask just how large must these values be and what is implied by empirically plausible estimates in order to produce extreme inequality and high capital share of income? We solve the model numerically for a wide range of (α, ξ) combinations and find that for a capital share, α , extreme inequality only emerges as ξ approaches the upper bound placed upon it by balanced growth. Figures 6-7 plot contour maps $\frac{K}{Y}$ and ζ , respectively, for (α, ξ) combinations when $g = 0$.⁸ Locations farther to the north and east (i.e., higher capital share in production and higher elasticity of substitution) are associated with both greater $\frac{K}{Y}$ (and, consequently, $\frac{(r+\delta)K}{Y}$). This pattern persists until the (α, ξ) combination violates the condition imposed by balanced growth, that is $\nu > \frac{\log(\alpha)}{\log(r_{\min} + \delta)}$, where $\xi > \xi_{\max}(\alpha)$. For instance, at $\alpha = 0.36$, the maximum elasticity of substitution is roughly 1.44.⁹ At this maximum value, $\frac{K}{Y}$ is 2.8 times larger than it is when the elasticity of substitution is 1. Likewise, capital share of income goes from 0.36 to 0.99. Notice, however, that for most of the permissible region, $\frac{K}{Y}$ is much lower and $\frac{(r+\delta)K}{Y}$ is not especially high. Again when $\alpha = 0.36$, an elasticity of substitution of 1.2 produces a $\frac{K}{Y}$ only 60 percent larger than the ratio with unit elasticity, and $\frac{(r+\delta)K}{Y}$ is 0.57.^{10, 11}

How reasonable is an elasticity of substitution of 1.44? Chirinko (2008) surveys estimates from 31 studies. Only two studies support an elasticity above 1.5, and only three additional studies find evidence for an elasticity above one, while the median is significantly below one.¹²

⁸Because when $g = 0$, $r = r_{\min}$, a plot of $\frac{(r+\delta)K}{Y}$ looks the same as a plot of $\frac{K}{Y}$.

⁹Gollin (2002) finds that capital's share of income is roughly one-third, once one takes careful account of self-employment income; see also Gomme and Rupert (2007).

¹⁰For reference, in the 2013 wave of the Survey of Consumer Finances, the ratio of average real income of the top quintile to the median is about 5.84. Income in this calculation is measured by the SCF variable "INCOME."

¹¹Piketty and Zucman (2015) give an zero-growth example with high inequality and capital share. That example assumes $\alpha = 0.21$, substantially smaller than conventional estimates. To reach extreme values of $\frac{K}{Y}$ and $\frac{(r+\delta)K}{Y}$ with this lower capital share of production, the elasticity of substitution must be 1.87.

¹²Palivos (2008) provides two additional references that find elasticities above one, based on abandoning the CES structure in favor of a production function with a variable elasticity. These elasticities are just barely above one, however.

The most extreme estimate in Chirinko (2008) is 3.4, based on Mexican data and long-run tax reforms; it is unclear whether such an estimate should be applied to the questions at hand. More recently, Karabarbounis and Neiman (2014), Rognlie (2015), and Semieniuk (2014) argue that the aggregate elasticity of substitution is likely less than one. We want to draw particular attention to Rognlie (2015), who demonstrates that the net elasticity – the elasticity substitution between capital and labor for net output – is always smaller than the one for gross output, which is the one estimated in practice. We conclude that a growth model where the growth rate is exogenous will not predict explosive inequality if the aggregate production function is calibrated to match the data.

2.4 Capital Income Taxation

Next we consider the role for fiscal policy to mitigate the effects of inequality. As noted in the Introduction, the optimal fiscal policy under commitment here involves a positive capital income tax, even in the long-run (Straub and Werning 2015).¹³ Our goal here is not to characterize optimal taxation, but rather to examine the connection between fiscal policy and the growth-inequality relationship. We consider two redistributionary fiscal policies. Each policy taxes capital income at a flat rate, τ , and transfers the revenues lump-sum. In the first case, the transfer is uniform (i.e., equally shared among all households) while in the second case the transfer is targeted to only laborer households.

2.4.1 Uniform transfer

Letting T denote the lump-sum transfer, in a steady state a laborer household works

$$l^* = \frac{1}{1 + \theta} - \frac{\theta}{1 + \theta} \frac{T}{w}$$

and consumes

$$x^* = \frac{w + T}{1 + \theta}.$$

Government budget balance requires

$$T = \tau r K$$

¹³The optimal policy without commitment will likely also involve a positive capital income tax; see Krusell (2002) for a simple model of redistribution without commitment. Farhi and Werning (2014) explore the connection between optimal estate taxation and $r - g$ based inequality.

and thus as capitalists save more, they produce an ever larger wealth effect on laborer hours.

The solution to the capitalist's problem is more complicated, but once again we can divide our results into two cases: capitalists work and capitalists do not work, where the steady state hours decision for a capitalist is

$$h^* = \max \left\{ \frac{we - \theta [(1 - \tau)r - g]k - \theta T}{(1 + \theta)we}, 0 \right\}.$$

Now equation (21) changes slightly

$$r = \frac{(1 + g)^\sigma - \beta}{(1 - \tau)\beta}$$

so

$$r_{\min} = \frac{1}{1 - \tau} \frac{1 - \beta}{\beta}. \quad (32)$$

This expression makes clear the important distinction between both the zero capital tax case and the one here: given preferences β and σ , the long-run growth rate determines the unique steady state *after-tax* return on capital,

$$\tilde{r} \equiv (1 - \tau)r = \frac{(1 + g)^\sigma - \beta}{\beta}.$$

The general equilibrium expressions for household decisions are too complicated to be signed analytically so we solve for them numerically. Figures (10) and Figure (11) show the steady-state Laffer curves for our economy. As capital's share of production, α , increases the peak increases and moves to the left (i.e., the maximum is greater and occurs at a lower tax rate); increasing the elasticity of substitution has the same effect. For the remainder of the discussion, we will assume that τ is always to the left of the peak so $\frac{dT}{d\tau} > 0$.

Figure (12) shows how inequality in a zero growth steady state is affected by the capital income tax. For all values of ξ , inequality declines in the tax level. To understand this result, it is helpful to point out a few features of the model. First, k falls in τ and thus so does after-tax capital income. Although this is not surprising, it is worth highlighting because it helps to understand how hours behave. It is clear from (32) that $\frac{dr_{\min}}{d\tau} > 0$, from which it follows that $\frac{dw_{\max}}{d\tau} < 0$ and $\frac{d}{d\tau} \left(\frac{K}{N} \right) < 0$. Second, laborer hours, $l^* = \frac{1}{1 + \theta} - \mu \frac{\theta}{1 + \theta} \frac{T}{w_{\max}}$, necessarily fall as τ increases so laborer's pre-transfer income is reduced by the tax. Meanwhile, the response of capitalist hours

to an increase in the tax depends upon the value of ξ . When the lower bound on hours is not binding,

$$h^* = \frac{1}{1 + \theta} \left[1 - (1 + \mu - \tau) \theta \frac{r_{\min}}{w_{\max}} k \right].$$

Differentiating with respect to τ ,

$$\frac{dh^*}{d\tau} = \theta \frac{r_{\min}}{w_{\max}} k - \theta (1 + \mu - \tau) \left[\frac{r_{\min}}{w_{\max}} \frac{dk}{d\tau} - \frac{d}{d\tau} \frac{r_{\min}}{w_{\max}} k \right].$$

Because capitalists pay the entire tax but only receive a fraction of the revenues, higher tax rates produce a negative wealth effect, pushing hours up. An additional negative wealth effect comes from the downward adjustment in wealth ($\frac{dk}{d\tau} < 0$). On the other hand, greater τ increases the factor price ratio $\frac{r_{\min}}{w_{\max}}$, which encourages capitalists to consume more leisure. In order for the net effect on hours to be negative, factor prices must be very responsive to the tax, meaning that capital and labor are strong *complements* in production. Figure (14) plots the tax/hours curve across various different values of ξ ; if hours decline they do not decline much unless the tax rate is past the peak of the Laffer curve, while for high values of ξ the rise in capitalist hours is quite pronounced.

The capital income tax reduces inequality. After-tax income falls for both types of households, but capitalist income is reduced by a greater percentage. For capitalists, capital income (net of taxes and transfers) and labor income decline. This holds in spite of the general tendency for capitalists to increase hours in response to the tax. The decline in the equilibrium wage more than offsets the positive hours response..

Finally, adding capital income taxes does not alter our finding that ξ must be significantly greater than 1 in order for steady inequality to be higher under zero growth. Figure (13), plots the cutoff value of ξ after which $\zeta(0) > \zeta(g)$ ($g > 0$) under the baseline calibration. For tax rates from 0 to 70 percent, the required elasticity of substitution is between 1.3 and 1.4, which is a relatively small range and still well above most estimates. Also, except for a small range of values below 20 percent, the required ξ is increasing with τ .

2.4.2 Targeted Transfer

Now consider the consequences of giving the transfer only to laborer households. Government budget balance implies

$$T = \frac{\tau r K}{1 - \mu},$$

so the wealth effect on hours will be stronger for both household types: negative for laborers and positive for capitalists.¹⁴ If the parameter values are such that capitalists do not work, then

$$\begin{aligned} K &= \frac{(1 - \mu) \frac{w(\tilde{r})}{\tilde{r}}}{(1 + \theta) z(g; \tau) + \theta \tau}, \\ N &= \frac{(1 - \mu) z(g; \tau)}{(1 + \theta) z(g; \tau) + \theta \tau} \\ T &= \tau \frac{w(\tilde{r})}{(1 + \theta) z(g; \tau) + \theta \tau} \end{aligned}$$

where

$$z(g; \tau) = \frac{\left[\left(\frac{\frac{\tilde{r}}{1-\tau} + \delta}{\alpha} \right)^\xi - \left(\frac{\tilde{r}}{1-\tau} + \delta \right) \right]}{\frac{\tilde{r}}{1-\tau}}$$

is the equivalent of (30) with taxation.

Some algebra shows that inequality is

$$\begin{aligned} \zeta &= \frac{1 - \mu}{\mu} \frac{(1 - \tau)}{\frac{\left(\frac{r+\delta}{\alpha} \right)^\xi - (r+\delta)}{r} + \tau} \\ &= \frac{1 - \mu}{\mu} \frac{1 - \tau}{z(g; \tau) + \tau}. \end{aligned}$$

Furthermore,

$$\frac{d\zeta}{d\tau} = \frac{\tilde{r}^2}{\alpha \mu (1 - \tau)^2} \left(- \frac{1 - \mu}{\left(\left(\frac{\frac{\tilde{r}}{1-\tau} + \delta}{\alpha} \right)^\xi - (\tilde{r} + \delta) \right)^2} \right) \left(\frac{\tilde{r}}{1-\tau} + \delta \right)^{\xi-1} < 0;$$

that is, inequality falls as the tax rate rises (again, supposing we are on the upward-sloping side of the Laffer curve).

¹⁴The Laffer curves for the targeted transfer case are very similar to those for the uniform transfer case, so we do not explicitly plot them.

In contrast, if capitalists are supplying labor,

$$K = \frac{(1 - \mu + \mu e) \frac{w(\tilde{r})}{\tilde{r}}}{(1 + \theta) z(g; \tau) + \theta \frac{(\tilde{r}-g)}{\tilde{r}}},$$

$$N = \frac{(1 - \mu + \mu e) z(g; \tau)}{(1 + \theta) z(g; \tau) + \theta \frac{(\tilde{r}-g)}{\tilde{r}}},$$

and

$$T = \frac{\tau}{1 - \mu} \left(\frac{(1 - \mu + \mu e) w(\tilde{r})}{(1 + \theta) z(g; \tau) + \theta \frac{(\tilde{r}-g)}{\tilde{r}}} \right).$$

Inequality is

$$\zeta = \frac{1 - \mu}{\mu} \left\{ \frac{\mu [(1 + \theta) z(g; \tau) + 1 + \theta - \tau] e + (1 - \mu) (1 - \tau + \theta \frac{g}{r})}{\mu e \tau + (1 - \mu) [(1 + \theta) z(g; \tau) + \theta \frac{(r-g)}{r} + \tau]} \right\}. \quad (33)$$

The expression in Equation (33) is formidable and, as a result, the derivative cannot easily be signed; however, we can show that inequality decreases in the tax rate locally at 0.

Proposition 7. *A targeted transfer decreases inequality. In other words, $\zeta(\tau) < \zeta(0)$.*

The proof can be found in the appendix. As was the case with uniform transfers, a targeted transfer does not undo our result that a high elasticity of substitution between capital and labor is necessary for higher inequality in a zero-growth steady state.

3 Endogenous Growth

Here we consider two extensions that allow for long-run growth to be endogenous. As noted above, the positive comovement between growth rates and returns is crucial – as growth falls, returns also fall, and thus unless capital moves a lot relative to labor inequality cannot move significantly. We ask the natural question now – if growth is endogenous, does this positive comovement still obtain?

3.1 Endogenous Growth without productivity growth

One model of endogenous growth fits directly into our above framework – if $\nu > 0$, as we have noted before is important for our results, the marginal product of capital can be bounded away

from zero.¹⁵ Suppose there is no exogenous productivity growth. Then we have

$$\lim_{K \rightarrow \infty} \left\{ \frac{\partial Y}{\partial K} \right\} = A\alpha^{\frac{1}{\nu}}$$

so that

$$r = A\alpha^{\frac{1}{\nu}} - \delta$$

$$w = (1 - \alpha) A\alpha^{\frac{1-\nu}{\nu}},$$

which are both constant; note that our result above regarding the existence of a balanced-growth path is related to the requirement that r can approach zero. Using the Euler equation of the capitalist, we get

$$1 + g = \frac{C'}{C} = \left(\beta \left(1 + A\alpha^{\frac{1}{\nu}} - \delta \right) \right)^{\frac{1}{\sigma}};$$

the resource constraint then requires that worker consumption, capital, and output both asymptotically grow at this same constant rate. However, this condition is impossible, since wages are constant and hours are bounded, meaning that the economy cannot display balanced growth. In this case, inequality explodes to infinity over time if and only if $g > 0$. Since

$$\frac{\partial g}{\partial \nu} = -\frac{1}{\sigma} \left(\beta \left(1 + A\alpha^{\frac{1}{\nu}} - \delta \right) \right)^{\frac{1-\sigma}{\sigma}} \frac{\alpha^{\frac{1}{\nu}}}{\nu^2} \beta A \ln(\alpha) > 0,$$

the more substitutable capital is for labor the faster inequality explodes, just as in the exogenous growth case. Moreover, $\frac{\partial r}{\partial g} > 0$, so even if parameters are such that $g = 0$, the return on capital will be low and inequality will be low as well.

3.2 Romer Externality Model

Next we consider Romer (1986) where a production externality renders the production function, in the aggregate, constant returns to scale in capital. For exposition, we restrict our analysis to the case where capitalists do not supply labor.¹⁶ The household problem is not meaningfully

¹⁵Jones and Manuelli (1990) discuss this model and related ones in which balanced growth obtains only asymptotically.

¹⁶As long as the wealth effect reduces hours, in any environment where capital income becomes explosively large capitalists would not work anyway.

different from the exogenous growth case above so we focus on the firm's problem. The individual firm solves the problem

$$\max_{K,N} \left\{ A (\alpha K^\nu + (1 - \alpha) (SN)^\nu)^{\frac{1}{\nu}} - rK - wN \right\}$$

taking as given the externality term S . We suppose that

$$S = K$$

in equilibrium, which leads to the factor price conditions

$$\begin{aligned} r &= \alpha A (\alpha + (1 - \alpha) N^\nu)^{\frac{1-\nu}{\nu}} - \delta \\ w &= (1 - \alpha) A (\alpha N^{-\nu} + 1 - \alpha)^{\frac{1-\nu}{\nu}} K. \end{aligned}$$

We can define a 'deflated wage' (relative to K) and label it \tilde{w} :

$$\tilde{w} = (1 - \alpha) A (\alpha N^{-\nu} + 1 - \alpha)^{\frac{1-\nu}{\nu}}.$$

The Euler equation of the capitalist determines the endogenous growth rate:

$$1 + g = \beta \left(1 + \alpha A \left(\alpha + (1 - \alpha) \left(\frac{1 - \mu}{1 + \theta} \right)^\nu \right)^{\frac{1-\nu}{\nu}} - \delta \right).$$

Because this economy does not feature transitional dynamics, it is always on the balanced-growth path. Along this balanced-growth path, it is easy to see that $\frac{\partial g}{\partial A} > 0$, and therefore $\frac{\partial g}{\partial r} > 0$. We can therefore expect that our previous results will hold – if a decline in TFP leads to lower growth it will also lead to lower returns, and therefore the quantitative effect on inequality will depend critically on the production elasticity.

To see this result formally, long-run income inequality when capitalists do not work is

$$\zeta = \frac{\frac{K}{N}}{\frac{w}{r+\delta}} = \frac{\frac{1}{N}}{\frac{\tilde{w}}{r+\delta}}.$$

As before,

$$\zeta < \frac{\frac{1}{N}}{\frac{\tilde{w}}{r+\delta}} = \frac{1 - \mu}{\mu} \frac{\alpha}{1 - \alpha} N^{\frac{1}{\xi} - 1}$$

where

$$N = \frac{1 - \mu}{1 + \theta}$$

is constant and less than 1. Once again, long-run income inequality will only be high if the elasticity of substitution is well above 1. Moreover, the upper bound on inequality, $\frac{K}{N}$, does not depend upon A so even if growth changes due to a productivity shock, inequality cannot rise too far as a result.

4 The Model with Idiosyncratic Risk

We have shown that in environments without idiosyncratic risk, where the interest rate may be very sensitive to the growth rate, income inequality only explodes under high values of the elasticity of substitution between capital and labor, values which are at odds with most measurements from the data. It seems that if there is any chance for inequality to become extreme, the tight link between the interest rate and the growth rate must be severed. Idiosyncratic risk can break this link by introducing a strong precautionary saving motive, which reduces the elasticity of saving with respect to interest rates in the long run. To this end, we introduce idiosyncratic risk for wages.

We now suppose that e and s follow identical, highly persistent AR(1) processes in logs:

$$\log(e') = 0.95 \log(e) + 0.1\eta'$$

where η' is a standard normal random variable. The definition of equilibrium for this model is a straightforward extension of the model without idiosyncratic risk and is omitted.¹⁷ We allow workers to save using a storage technology with a negative net return of -0.02 , consistent with the US postwar inflation rate, but we can permit other forms of saving; provided that the laborer's savings technology is inferior to the capitalist one, our qualitative results are unchanged.

Due to the special relationship between r and w we can solve this model by finding a single number, namely the return r , such that at that given rate the household's supply of capital and labor, if hired entirely by the firm, lead to a marginal product of capital (net of depreciation) equal to r itself (that is simply Equation 12). We can draw a picture of the steady state as the intersection of the "demand curve" corresponding to the right-hand-side of Equation 12 and

¹⁷There is significant debate over the appropriate form for the labor productivity process, as well as the value of the coefficients. Our results are not sensitive to changes in the parameters of the process for e (and s). They are also not sensitive to allowing capitalists and workers to have different mean productivities.

a "supply curve" that links the aggregate capital/labor ratio (as chosen by households) to the return; see Figure 15.¹⁸

We focus on measuring inequality using Lorenz curves and the Gini coefficient. We present in Figure 16 the Lorenz curves both for our model and for the recent US, using the Survey of Consumer Finances 2007 sample. Our model does a reasonable job of fitting the US Lorenz curve; the middle part would be matched better if we allowed occasional transits between capitalists and workers. Our model is also consistent with mean wealth relative to income – the ratio of K/Y is 6 in the benchmark, consistent with the measurement in McGrattan and Prescott (2013) as well as the number used by Piketty (2014).

Making comparisons across Lorenz curves is difficult, since they could cross multiple times. When making comparisons we will use the Gini coefficient, which is obtained by integrating the area between the perfect-equality line and the actual Lorenz curve. Larger Gini coefficients translate into more unequal distributions. In terms of Gini coefficients, our model does a reasonable job reproducing the extreme inequality observed in the US – our Gini coefficient is even slightly larger, at 0.84, than the US at 0.8. Comparing the two model curves we see that inequality actually drops as g goes to zero, at least in the long run, but not much – the dashed-line curve lies everywhere above the solid one but they are quite close together (the new Gini coefficient is 0.83). r drops because the capital-output ratio roughly doubles while aggregate labor input remains roughly constant, leading to a large increase in the capital/labor ratio and a concomitant decline in returns.

Because the long run comparisons could be misleading if inequality temporarily rises or r takes a long time to fall, we explicitly compute the transition path as the economy moves from the initial growth path with $g = 0.02$ to the one with $g = 0$. We focus on four variables, namely the capital/output ratio, the return to capital, capital's share of income, and the Gini coefficient on wealth. As seen in Figure 17, the transition takes over 100 years to complete for the mean capital stock (which is all that matters for r and $\frac{K}{Y}$), and takes even longer for the Gini coefficient. The Gini coefficient first drops a bit, then recovers, but the quantitative size of the movements are small, meaning that the steady state is not hiding substantial inequality dynamics.

¹⁸We use standard numerical methods to solve for the steady state and the transitional dynamics; a technical appendix outlines the details and is available upon request.

In a working paper version Krusell and Smith (2015), the authors also find small effects of g on inequality, although their model features only one type of household and shocks to household discount factors drive much of the inequality.¹⁹ Similarly, in a complicated life-cycle with bequests and a highly-skewed idiosyncratic income process, De Nardi, Fella, and Yang (2016) find that changes in population growth do not materially affect inequality.²⁰ Both papers use Cobb-Douglas production functions, and our results suggest that they are nonstarters as a result.

4.1 Financial Innovation

The model with idiosyncratic risk still displays the same behavior in terms of returns and therefore inequality – while $\frac{K}{Y}$ rises significantly, r falls and therefore $\frac{rK}{Y}$ increases but not substantially. In contrast, Piketty maintains that r will not fall, meaning that the increase in $\frac{K}{Y}$ will translate directly into an increase in $\frac{rK}{Y}$. It is clear from inspecting Figure 15 that the model cannot reconcile a decline in g with a constant r , since there is no "room" between r and the effective discount factor of the capitalists ($\beta(1+g)^{-\sigma}$) – if g drops by a nontrivial amount (say, from the postwar average of 2 percent to zero), r must fall in the new steady state. To this end, we might ask how one might prevent r from falling with g ; while idiosyncratic risk in returns will certainly put some additional space between returns and time rates of preferences, a decline in g by itself would still reduce r . Here, we note that Piketty (2014) suggests a number of possible mechanisms that would prevent r from falling (or even cause it to increase). We pick one of these proposed mechanisms here – financial innovation:

In addition, capital markets may become more and more sophisticated and more and more "perfect" in the sense used by economists (meaning that the return on capital will become increasingly disconnected from the individual characteristics of the owner). (Piketty 2014, page 376).

¹⁹That part of the paper apparently did not survive the publication process.

²⁰This paper notes that $r - g$ is not really important. If $r - g$ rises, it matters whether r is rising or g is falling; it just doesn't matter very much.

We explore this idea by assuming that, in the initial steady state, the capitalist is exposed to an iid idiosyncratic shock to end-of-period wealth u , changing the dynamic program to

$$v(k, e) = \max_{k', h, c} \left\{ \frac{(c(1-h)^\theta)^{1-\sigma}}{1-\sigma} + \beta(1+g)^{1-\sigma} E[v(u'k', e')] \right\}$$

$$c + (1+g)k' \leq (r+1-\delta)k + weh$$

$$k' \geq 0$$

$$h \geq 0$$

$$c \geq 0.$$

We set the standard deviation of u to be 0.1, equal to the innovation variance of the wage shock; while one can quibble with our quantitative choice, recent work has demonstrated the presence of significant idiosyncratic return variation and our qualitative results are not affected by this choice.²¹ Figure 18 shows that the initial steady state now has a substantially lower r and a larger gap between the discount factor and the return (a symptom of increased market incompleteness). A decline in idiosyncratic risk will make the asset supply curve shift to the left as precautionary motives are blunted, a force which will increase r and work against the decline in g .

The transition with financial innovation looks very similar qualitatively to the benchmark model, with one clear exception – there is now a substantial *decrease* in the Gini coefficient on wealth. Thus, inequality as measured by $(\frac{rK}{Y})$ and by standard measures (Gini) move in opposite directions. Figures 19 and 20 show the Lorenz curves and transitional dynamics; the decrease in inequality occurs because there is less inequality in wealth within the capitalist population.²²

5 Conclusion

This two-household model has shown that the parameters that primarily govern the behavior of inequality in a zero growth steady state are related to production. The capital share and the

²¹See Hubmer, Krusell, and Smith (2016) or Benhabib, Bisin, and Luo (2016) for evidence of idiosyncratic risk in returns.

²²The "blip" in the dynamic path for the Gini coefficient is not a numerical artifact – it is an upward jump followed by a quick but continuous decline back to the previous transition path. We have not been able to figure out where this blip comes from, but it clearly does not affect the results we emphasize.

elasticity of substitution between capital and labor control how quickly both the steady state wage rate and wealth rise as g nears zero. In addition, they change the response of hours. Generally, steady state hours are higher when $g = 0$, but if both α and ν are sufficiently high, hours are lower (perhaps zero) in low growth steady states and rise as g increases. Unless the capital and labor are sufficiently strong substitutes in production, steady state inequality is lower under zero growth than it is in a steady state with positive growth. Steady states with extreme values of capital to income, capital's share of income, and income inequality only arise when the elasticity of substitution between capital and labor and capital share in production are jointly very near to values which violate balanced growth. These results continue to hold in the presence of endogenous growth and idiosyncratic risk.

References

- [1] Aiyagari, S.R., 1994. "Uninsured Idiosyncratic Risk and Aggregate Saving." *Quarterly Journal of Economics* 109(3), 659-684.
- [2] Benhabib, J., Bisin, A., Luo, M., 2016. "Wealth Distribution and Social Mobility in the US: A Quantitative Approach." Manuscript.
- [3] Campbell, J.Y., 1999. "Asset Prices, Consumption and the Business Cycle." In Taylor, J.B., Woodford, M., (eds.) *Handbook of Macroeconomics* 1, North-Holland.
- [4] Chirinko, R.S., 2008. " σ : The Long and Short of It." *Journal of Macroeconomics* 30(2), 671-686.
- [5] De Nardi, C., Fella, G., Yang, F., 2016. "Piketty's Book and Macro Models of Wealth Inequality." National Bureau of Economic Research Working Paper 21730.
- [6] Farhi, E., Werning, I., 2014. "Bequest Taxation and $r - g$ ". Manuscript.
- [7] Gollin, D., 2002. "Getting Income Shares Right." *Journal of Political Economy* 110(2), 458-474.
- [8] Gomme, P., Rupert, P., 2007. "Theory, Measurement, and Calibration of Macroeconomic Models." *Journal of Monetary Economics* 54(2), 460-497.

- [9] Guvenen, F., 2006. "Reconciling Conflicting Evidence on the Elasticity of Intertemporal Substitution: A Macroeconomic Perspective." *Journal of Monetary Economics* 53(7), 1451-1472.
- [10] Hall, R.E., 1988. "Intertemporal Substitution in Consumption." *Journal of Political Economy* 96(2), 339-357.
- [11] Hammond, P.J., Rodríguez-Clare, A., 1993. "On Endogenizing Long-Run Growth." *Scandinavian Journal of Economics* 95(4), 391-425.
- [12] Havránek, T., 2015. "Measuring Intertemporal Substitution: The Importance of Method Choices and Selective Reporting." *Journal of the European Economic Association* 13(6), 1180-1204.
- [13] Hubmer, J., Krusell, P., Smith, A.A., Jr., 2016. "The Historical Evolution of the Wealth Distribution: A Quantitative-Theoretic Investigation." Manuscript.
- [14] Jones, L.E., Manuelli, R., 1990. "A Convex Model of Equilibrium Growth: Theory and Policy Implications." *Journal of Political Economy* 98(5), 1008-1038.
- [15] Karabarbounis, L., Neiman, B., 2014. "Capital Depreciation and Labor Shares Around the World: Measurement and Implications." National Bureau of Economic Research Working Paper 20606.
- [16] King, R.G., Plosser, C.I., Rebelo, S.T., 1988. "Production, Growth, and Business Cycles I. The Basic Neoclassical Model." *Journal of Monetary Economics* 21(2-3), 195-232.
- [17] Krusell, P., 2002. "Time-Consistent Redistribution." *European Economic Review* 46(4-5), 755-769.
- [18] Krusell, P., Smith, A.A., Jr., 2015. "Is Piketty's 'Second Law of Capitalism' Fundamental?" *Journal of Political Economy* 123(4), 725-748.
- [19] McGrattan, E.R., Prescott, E.C., 2013. "On Financing Retirement with an Aging Population." Federal Reserve Bank of Minneapolis Staff Report 472.

- [20] Moll, B., 2014. "Productivity Losses from Financial Frictions: Can Self-Financing Undo Capital Misallocation?" *American Economic Review* 104(10), 3186-3221.
- [21] Palivos, T., 2008. "Comment on ' σ : The Long and Short of It'." *Journal of Macroeconomics* 30(2), 687-690.
- [22] Piketty, T., 2014. *Capital in the Twenty-First Century*. Belknap Press.
- [23] Piketty, T., Saez, E., 2014, "Inequality in the Long Run." *Science* 344(6186), 838-843.
- [24] Piketty, T., Zucman, G., 2015, "Wealth and Inheritance in the Long Run." *Elsevier Handbook of Income Distribution*, 2, Elsevier.
- [25] Rognlie, M., 2015. "Deciphering the Fall and Rise in the Net Capital Share." Forthcoming, *Brookings Papers on Economic Activity*.
- [26] Romer, P.M., 1986. "Increasing Returns and Long-Run Growth." *Journal of Political Economy* 94(5), 1002-1037.
- [27] Semieniuk, G., 2014. "Piketty's Elasticity of Substitution: A Critique." *Schwartz Center for Economic Policy Analysis Working Paper* 2014-8.
- [28] Straub, L., Werning, I., 2015. "Positive Long-Run Capital Taxation: Chamley-Judd Revisited." *Manuscript*.
- [29] Vissing-Jørgensen, A., 2002. "Limited Asset Market Participation and the Elasticity of Intertemporal Substitution." *Journal of Political Economy* 110(4), 825-853.

6 Appendix: Proofs

Proof of Proposition 1

Proof. Because $e, s > 0$ and $\mu \in (0, 1)$, we only need to show that

$$Q(z(g); \theta) > 0.$$

Notice first, that $\frac{\theta}{1+\theta} \in [0, 1)$. If $\sigma \geq 1$, then the $\min \frac{r}{r-g} = 1$ for non-negative growth rates, since the $\lim_{g \rightarrow \infty} \frac{r}{r-g} = 1$. Finally, $z(g) > 0$, so for finite θ , $Q(z(0); \theta) > 0$, and $\min \zeta = \frac{e}{s}$. \square

Proof of Proposition 2

Proof. It is immediate that

$$\begin{aligned} \frac{dQ}{d\left[\frac{r}{r-g}\right]} &= \frac{\frac{r}{r-g}z(g) + \frac{\theta}{1+\theta} - z(g) \left(\frac{r}{r-g} - \frac{\theta}{1+\theta}\right)}{\left[\frac{r}{r-g}z(g) + \frac{\theta}{1+\theta}\right]^2} \\ &= \frac{\frac{\theta}{1+\theta} + z(g) \frac{\theta}{1+\theta}}{\left[\frac{r}{r-g}z(g) + \frac{\theta}{1+\theta}\right]^2} \\ &> 0. \end{aligned}$$

\square

Proof of Proposition 3

Proof. A larger θ implies less willingness to work on the part of households. The steady state interest rate and $z(g)$ are independent of θ . Then

$$\begin{aligned} \frac{dQ}{d\theta} &= \frac{dQ}{d\frac{\theta}{1+\theta}} \frac{d\frac{\theta}{1+\theta}}{d\theta} \\ &= \left[-\frac{\frac{r}{r-g}z(g) + \frac{r}{r-g}}{\left(\frac{r}{r-g}z(g) + \frac{\theta}{1+\theta}\right)^2} \right] \frac{1}{(1+\theta)^2} \\ &< 0 \end{aligned}$$

Inequality rises as θ falls.

\square

Proof of Proposition 4

Proof. Consider $\bar{\theta}$ such that for all $\theta > \bar{\theta}$, $h = 0$, and all $\theta \leq \bar{\theta}$, $h > 0$. Let $0 < \theta_1 < \bar{\theta} < \theta_2$. Then since $\frac{\theta}{1+\theta} \in [0, 1)$ is continuous, strictly monotonic, and decreasing in θ , and since Q is decreasing in θ ,

$$\begin{aligned}\zeta_{\theta_2} &= \left[\frac{1-\mu}{\mu} \right] Q(z(g); 0) \\ &\leq \zeta_{\bar{\theta}} \\ &= 1 + \frac{1}{\mu} Q(z(g); \bar{\theta}) \\ &< 1 + \frac{1}{\mu} Q(z(g); \theta_1) \\ &< 1 + \frac{1}{\mu} Q(z(g); 0) \\ &= 1 + \frac{1}{1-\mu} \zeta_{\theta_2},\end{aligned}$$

where the first equality holds because $h = 0$ at θ_2 , and the second line follows from the continuity of Q in θ . Because capitalists do not work in a steady state with $\theta > \bar{\theta}$, inequality will be unaffected by increasing θ beyond $\bar{\theta}$. Therefore, for any $\theta \geq 0$, ζ_{θ} is bounded below by the $\left[\frac{1-\mu}{\mu} \right] \frac{1}{z(g)}$ and above by $1 + \frac{1}{\mu} \frac{1}{z(g)}$. \square

Proof of Proposition 5

Proof. Fix θ . Then

$$Q(z(g); \theta) = \frac{\frac{r}{r-g} - \frac{\theta}{1+\theta}}{\frac{r}{r-g} z(g) + \frac{\theta}{1+\theta}}$$

Because steady state r is invariant to changes in α and ξ , all that matters for inequality is $z(g)$.

$$\begin{aligned}z(g) &= \frac{\left[\left(\frac{r(g)+\delta}{\alpha} \right)^{\xi} - (r(g) + \delta) \right]}{r(g)} \\ &= \frac{1}{\chi} \frac{\left[\alpha^{-\xi} (r(g) + \delta)^{\xi} - (r(g) + \delta) \right]}{r(g) + \delta} \\ &= \frac{1}{\chi} \left[\alpha^{-\xi} (r(g) + \delta)^{\xi-1} - 1 \right].\end{aligned}$$

Because $\xi \geq 0$, a greater capital's share decreases z and increases steady state inequality. Likewise steady state inequality will be greater for larger elasticities of substitution since

$$\frac{dz(g)}{d\xi} = \frac{1}{\chi} \left\{ -\alpha^{-\xi} (r(g) + \delta)^{\xi-1} \log(r(g) + \delta) \left[1 - \frac{\log(\alpha)}{\log(r(g) + \delta)} \right] \right\} < 0$$

where the strict inequality holds since $r + \delta < \alpha < 1$. □

Proof of Proposition 6

Proof. First, in order for $\zeta(g) < \zeta(0)$, $z(0) < z(g)$. There are four cases to consider: (1) $h(g) = 0$, $h(0) = 0$; (2) $h(g) > 0$, $h(0) > 0$; (3) $h(g) > 0$, $h(0) = 0$; and (4) $h(g) = 0$, $h(0) > 0$. It will be shown that the fourth case cannot obtain. The first case is obvious since $\zeta(g) < \zeta(0)$ implies

$$\frac{1}{z(g)} < \frac{1}{z(0)}$$

since z is non-negative. For the second case, $\zeta(g) < \zeta(0)$ implies

$$\frac{\frac{r(g)}{r(g)-g} - \frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g) + \frac{\theta}{1+\theta}} < \frac{1 - \frac{\theta}{1+\theta}}{z(0) + \frac{\theta}{1+\theta}}.$$

By Proposition 2,

$$\frac{1 - \frac{\theta}{1+\theta}}{z(g) + \frac{\theta}{1+\theta}} < \frac{\frac{r(g)}{r(g)-g} - \frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g) + \frac{\theta}{1+\theta}}$$

so

$$\frac{1 - \frac{\theta}{1+\theta}}{z(g) + \frac{\theta}{1+\theta}} < \frac{1 - \frac{\theta}{1+\theta}}{z(0) + \frac{\theta}{1+\theta}}.$$

For the third case,

$$1 + \frac{1}{\mu} \frac{\frac{r(g)}{r(g)-g} - \frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g) + \frac{\theta}{1+\theta}} < \left(\frac{1 - \mu}{\mu} \right) \frac{1}{z(0)}.$$

If this is true, then by Proposition 2 and Proposition 3

$$1 + \frac{1}{\mu} \frac{\frac{r(g)}{r(g)-g} - \frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g) + \frac{\theta}{1+\theta}} < 1 + \frac{1}{\mu} \frac{1 - \frac{\theta}{1+\theta}}{z(0) + \frac{\theta}{1+\theta}} < 1 + \frac{1}{\mu} \frac{\frac{r(g)}{r(g)-g} - \frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(0) + \frac{\theta}{1+\theta}}.$$

Now for the final case, which we will show implies a contradiction. From (31),

$$h = \max \left\{ 0, \frac{1}{1+\theta} \left[1 - \frac{1}{\mu} \frac{\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g) + \frac{\theta}{1+\theta}} \right] \right\}.$$

Because $h(g) = 0$ and $h(0) > 0$

$$\frac{1}{\mu} \frac{\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g) + \frac{\theta}{1+\theta}} > 1 > \frac{1}{\mu} \frac{\frac{\theta}{1+\theta}}{z(0) + \frac{\theta}{1+\theta}}$$

so

$$z(0) > \frac{r(g)}{r(g) - g} z(g) > z(g).$$

We can see from the (31), for each growth rate, there is a $\bar{\theta}(g)$ such that

$$\begin{aligned} h(g) &= 0 \quad \forall \theta > \bar{\theta}(g) \\ &> 0 \quad \forall \theta < \bar{\theta}(g). \end{aligned}$$

In order for $h(0) > h(g) = 0$ at θ , it would have to be the case that $\bar{\theta}(g) < \bar{\theta}(0)$, and $\theta \in [\bar{\theta}(g), \bar{\theta}(0)]$. Additionally, $\zeta(g)$ must be at its minimum

$$\zeta(g) = \left(\frac{\mu - 1}{\mu} \right) \frac{1}{z(g)}$$

while

$$\zeta(0) > \left(\frac{\mu - 1}{\mu} \right) \frac{1}{z(0)}.$$

We know from case (1), that $\min[\zeta(g)] < \min[\zeta(0)]$, which implies that $z(g) < z(0)$. Because z is invariant to θ , it cannot be the case that $z(0) > z(g)$, so case (4) cannot obtain. Simply put, $\bar{\theta}(0) < \bar{\theta}(g)$. Therefore whenever $\zeta(g) < \zeta(0)$, $z(g) > z(0)$. Now to close the proof

$$z(g) = \frac{1}{\chi(g)} \left[\alpha^{-\xi} (r(g) + \delta)^{\xi-1} - 1 \right]$$

where again $\chi = \frac{r(g)}{r(g) + \delta}$. If $\zeta(0) > \zeta(g)$, then

$$\frac{1}{\chi(0)} \left[\alpha^{-\xi} (r(0) + \delta)^{\xi-1} - 1 \right] < \frac{1}{\chi(g)} \left[\alpha^{-\xi} (r(g) + \delta)^{\xi-1} - 1 \right]$$

Because χ is increasing in g ,

$$(r(0) + \delta)^{\xi-1} < (r(g) + \delta)^{\xi-1}$$

Now since $r(g) > r(0) = r_{\min}$, this condition can only hold if $\xi > 1$. □

Proof of Proposition 7

Proof. There are two cases: $h = 0$ or $h > 0$. In the first case,

$$\begin{aligned} \zeta &= \frac{1 - \mu}{\mu} \frac{1 - \tau}{z(g; \tau) + \tau} \\ &= \frac{1 - \mu}{\mu} \frac{(1 - \tau)}{\frac{\left(\frac{\bar{r}}{\alpha} + \delta \right)^{\xi} - \left(\frac{\bar{r}}{1 - \tau} + \delta \right)}{\frac{\bar{r}}{1 - \tau}} + \tau} \end{aligned}$$

Differentiating with respect to τ ,

$$\frac{d\zeta}{d\tau} = -\frac{\tilde{r}^2}{\alpha\mu(1-\tau)^2} \left(\frac{1-\mu}{\left(\left(\frac{\tilde{r}}{1-\tau} + \delta \right)^\xi - (\tilde{r} + \delta) \right)^2} \right) \left(\frac{\tilde{r}}{1-\tau} + \delta \right)^{\xi-1} \leq 0 \quad \forall \tau, \text{ w.e. iff } \xi \geq 0,$$

When $h > 0$,

$$\begin{aligned} \zeta &= \frac{we - \theta((1-\tau)r - g)k}{w+T} + (1+\theta) \frac{(1-\tau)rk}{w+T} \\ &= \frac{\frac{w}{k}e + (1-\tau)r + \theta g}{\frac{w}{k} + \frac{\mu}{1-\mu}\tau r} \end{aligned}$$

Converting this to after-tax returns using

$$\frac{\tilde{r}}{1-\tau} = r$$

and simplifying yields

$$\zeta = \frac{A(\tau)e + \tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu} \frac{\tau}{1-\tau} \tilde{r}}$$

where

$$\begin{aligned} A(\tau) &= \frac{\mu}{((1-\mu) + \mu e)} \left((1+\theta) \left(\left(\frac{\tilde{r}}{1-\tau} + \delta \right)^\xi - \left(\frac{\tilde{r}}{1-\tau} + \delta \right) \right) + \theta \left(\frac{\tilde{r}}{1-\tau} - g \right) \right) \\ &= \frac{\mu}{((1-\mu) + \mu e)} \left((1+\theta) \left(\frac{\tilde{r}}{1-\tau} + \delta \right)^\xi - \frac{\tilde{r}}{1-\tau} - (1+\theta)\delta - \theta g \right) \end{aligned}$$

We want to show $\zeta(\tau) < \zeta(0)$, $\tau > 0$. Notice that since $\frac{\tau}{1-\tau}\tilde{r}$ is monotonically increasing in τ ,

$$\frac{A(\tau)e + \tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu} \frac{\tau}{1-\tau} \tilde{r}} \leq \frac{A(\tau)e}{A(\tau)} + \frac{\tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu} \frac{\tau}{1-\tau} \tilde{r}}, \text{ w.e. iff } \tau = 0.$$

Next we show that

$$e + \frac{\tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu} \frac{\tau}{1-\tau} \tilde{r}} < e + \frac{\tilde{r} + \theta g}{A(0)}$$

when $\tau > 0$. In other words,

$$A(\tau) + \frac{\mu}{1-\mu} \frac{\tau}{1-\tau} \tilde{r} > A(0).$$

Substituting in for $A(\tau)$ and $A(0)$ and simplifying, we have

$$\begin{aligned}
(1+\theta) \left(\frac{\tilde{r}}{1-\tau} + \delta \right)^\xi - \frac{\tilde{r}}{1-\tau} + \frac{((1-\mu)+\mu e)}{1-\mu} \frac{\tau}{1-\tau} \tilde{r} &> (1+\theta) \left(\frac{\tilde{r}+\delta}{\alpha} \right)^\xi - \tilde{r} \\
(1+\theta) \left(\frac{\tilde{r}}{1-\tau} + \delta \right)^\xi - \frac{\left(1 - \frac{((1-\mu)+\mu e)\tau}{1-\mu} \right)}{1-\tau} \tilde{r} &> (1+\theta) \left(\frac{\tilde{r}+\delta}{\alpha} \right)^\xi - \tilde{r} \\
(1+\theta) \left(\frac{\tilde{r}}{1-\tau} + \delta \right)^\xi - \tilde{r} &> (1+\theta) \left(\frac{\tilde{r}+\delta}{\alpha} \right)^\xi - \tilde{r} \\
\frac{\tilde{r}}{1-\tau} &> \tilde{r}
\end{aligned}$$

which is obviously true. The penultimate step holds because

$$\frac{\left(1 - \frac{((1-\mu)+\mu e)\tau}{1-\mu} \right)}{1-\tau} < 1$$

Therefore if $\tau > 0$

$$\begin{aligned}
\zeta(\tau) &= \frac{A(\tau)e + \tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu} \frac{\tau}{1-\tau} \tilde{r}} \\
&< \frac{A(\tau)e}{A(\tau)} + \frac{\tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu} \frac{\tau}{1-\tau} \tilde{r}} \\
&= e + \frac{\tilde{r} + \theta g}{A(\tau) + \frac{\mu}{1-\mu} \frac{\tau}{1-\tau} \tilde{r}} \\
&< e + \frac{\tilde{r} + \theta g}{A(0)} \\
&= \zeta(0)
\end{aligned}$$

□

Figure 1: Equilibrium Wage in $g = 0$ Steady State

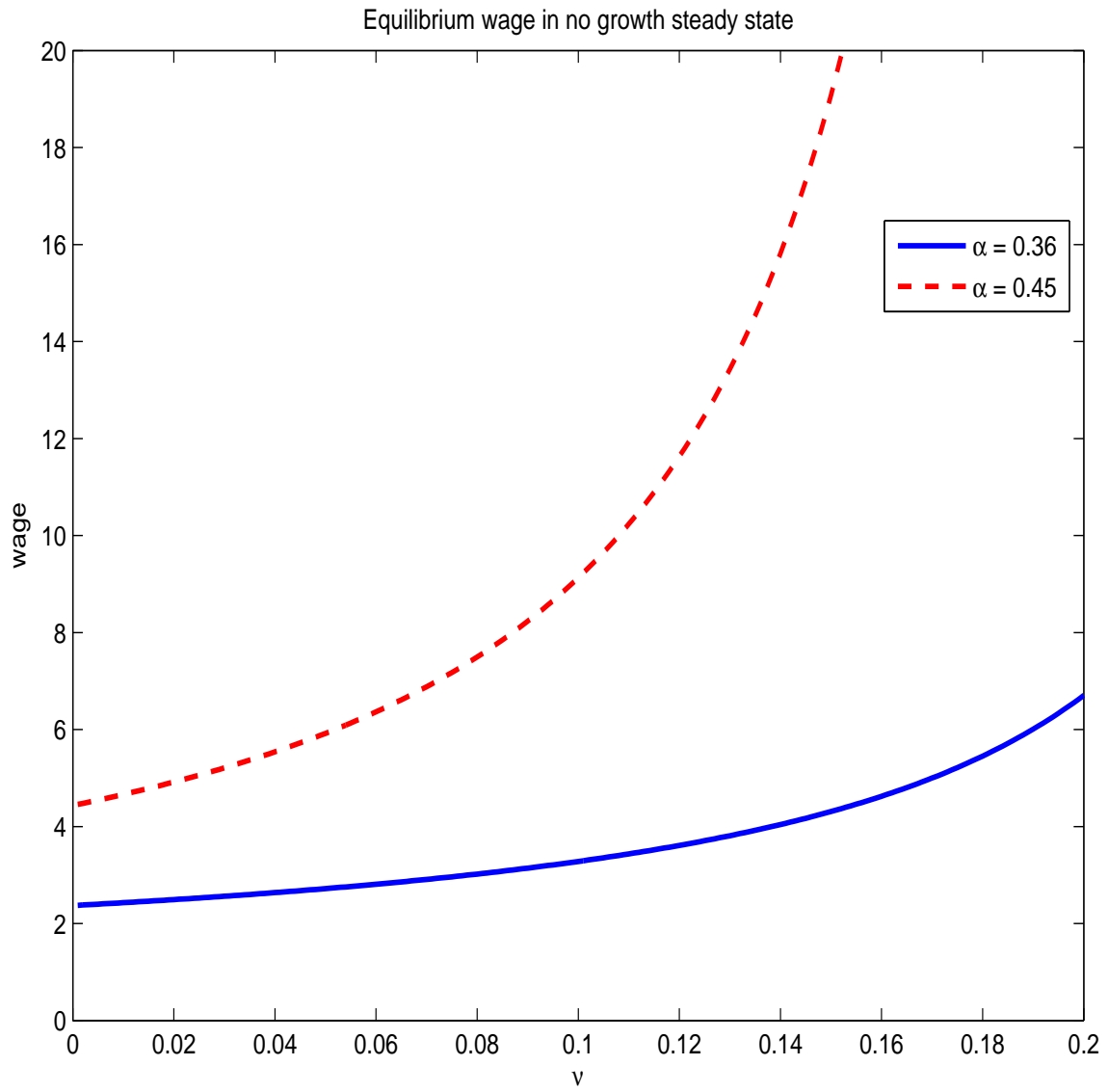


Figure 2: Steady State Inequality

Steady state income inequality

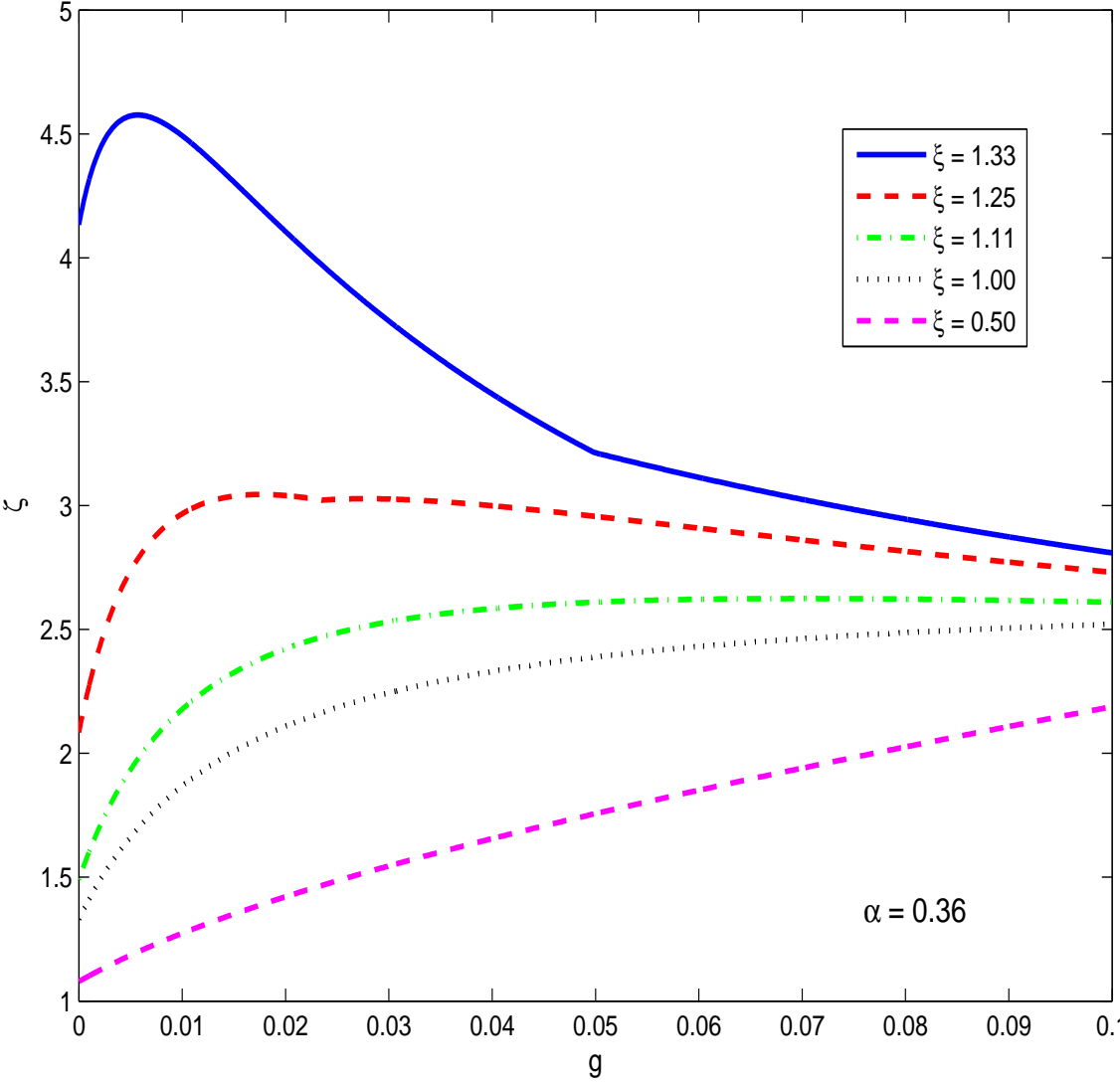


Figure 3: Steady State Inequality

Steady state income inequality

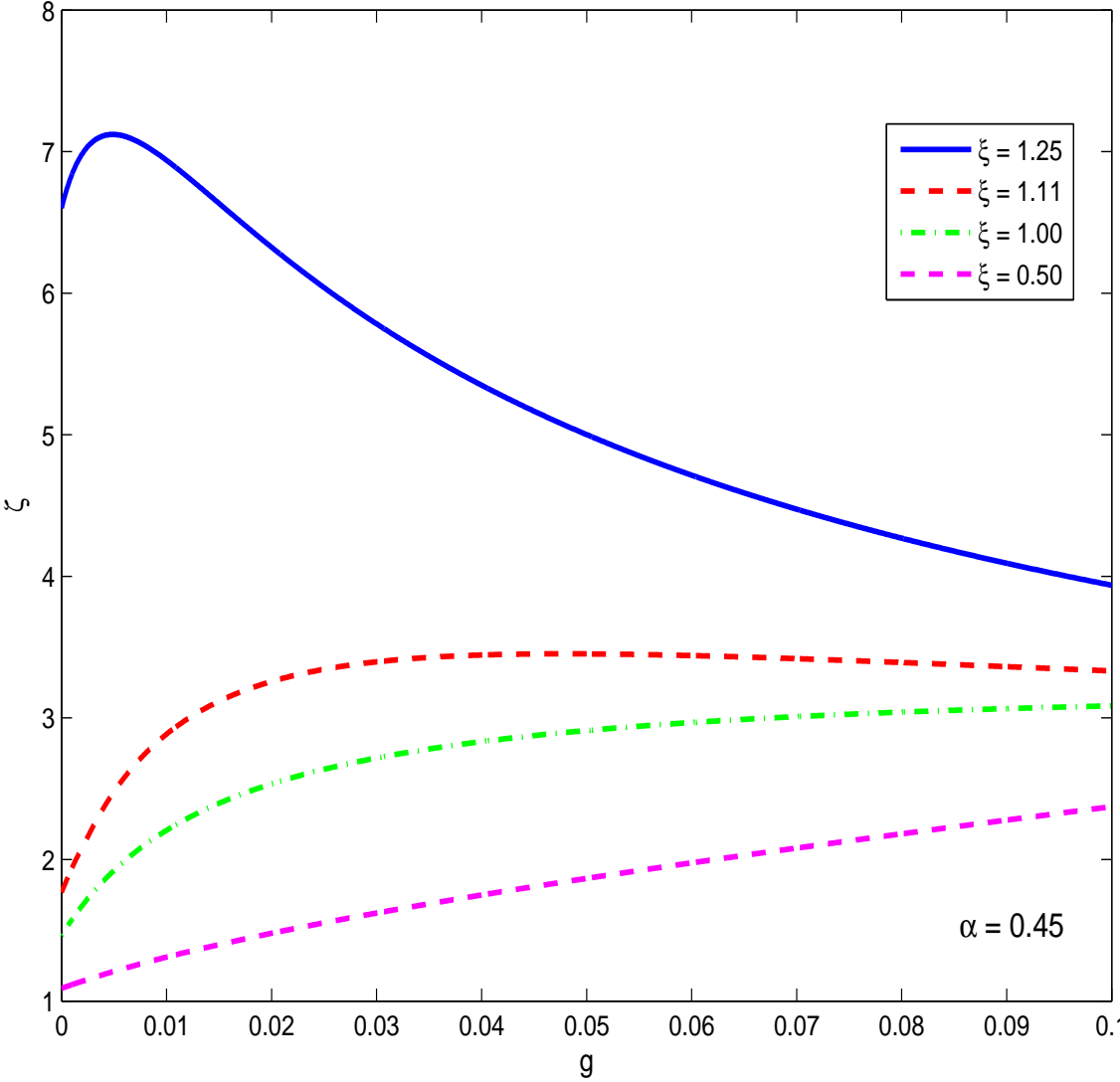


Figure 4: Steady State Wealth

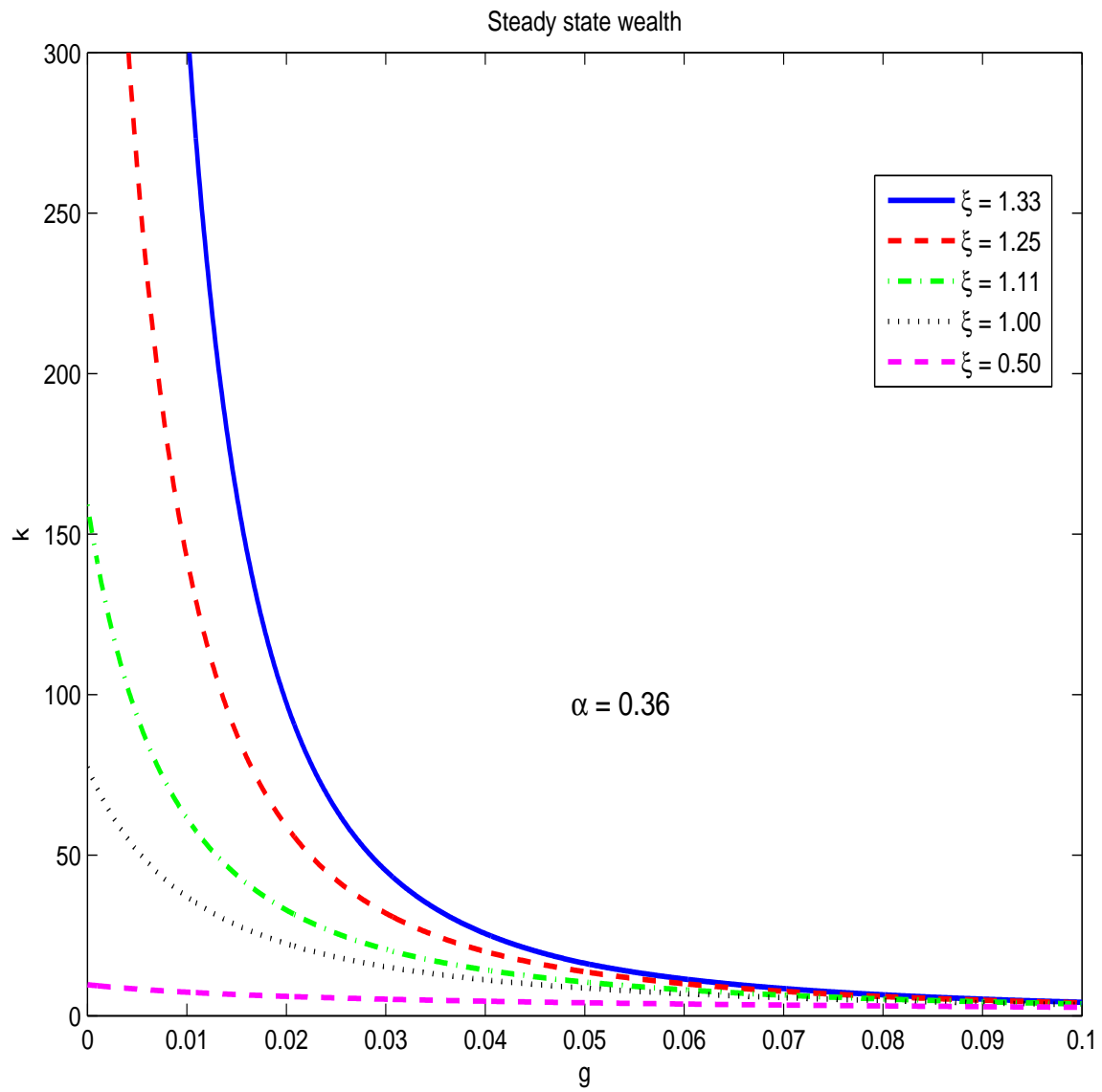


Figure 5: Steady State Wealth

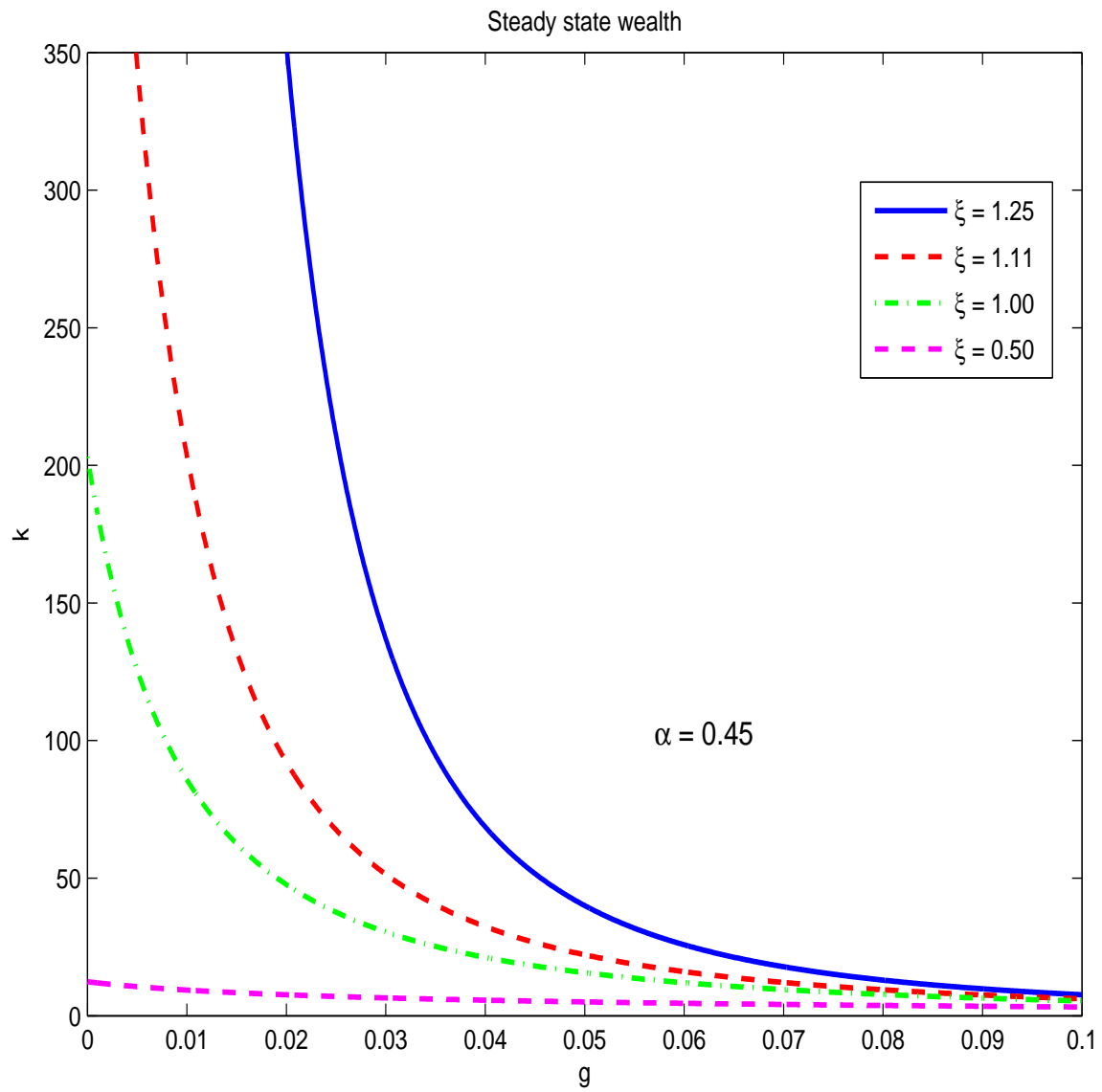


Figure 6: Capital to Income Ratio in a Zero Growth Steady State

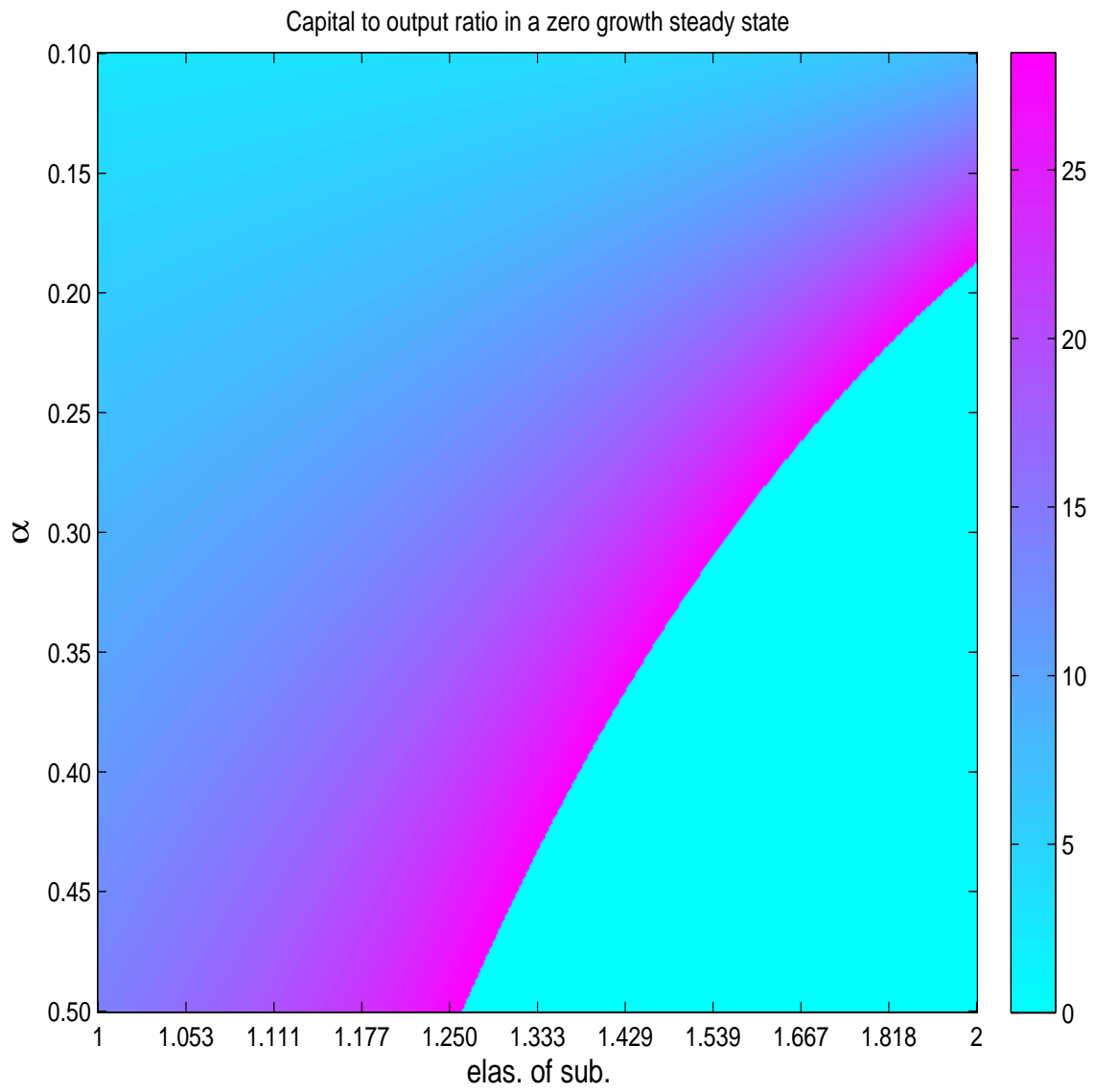


Figure 7: Capital Share of Income in a Zero Growth Steady State

Income inequality in a zero growth steady state (maximum 12)

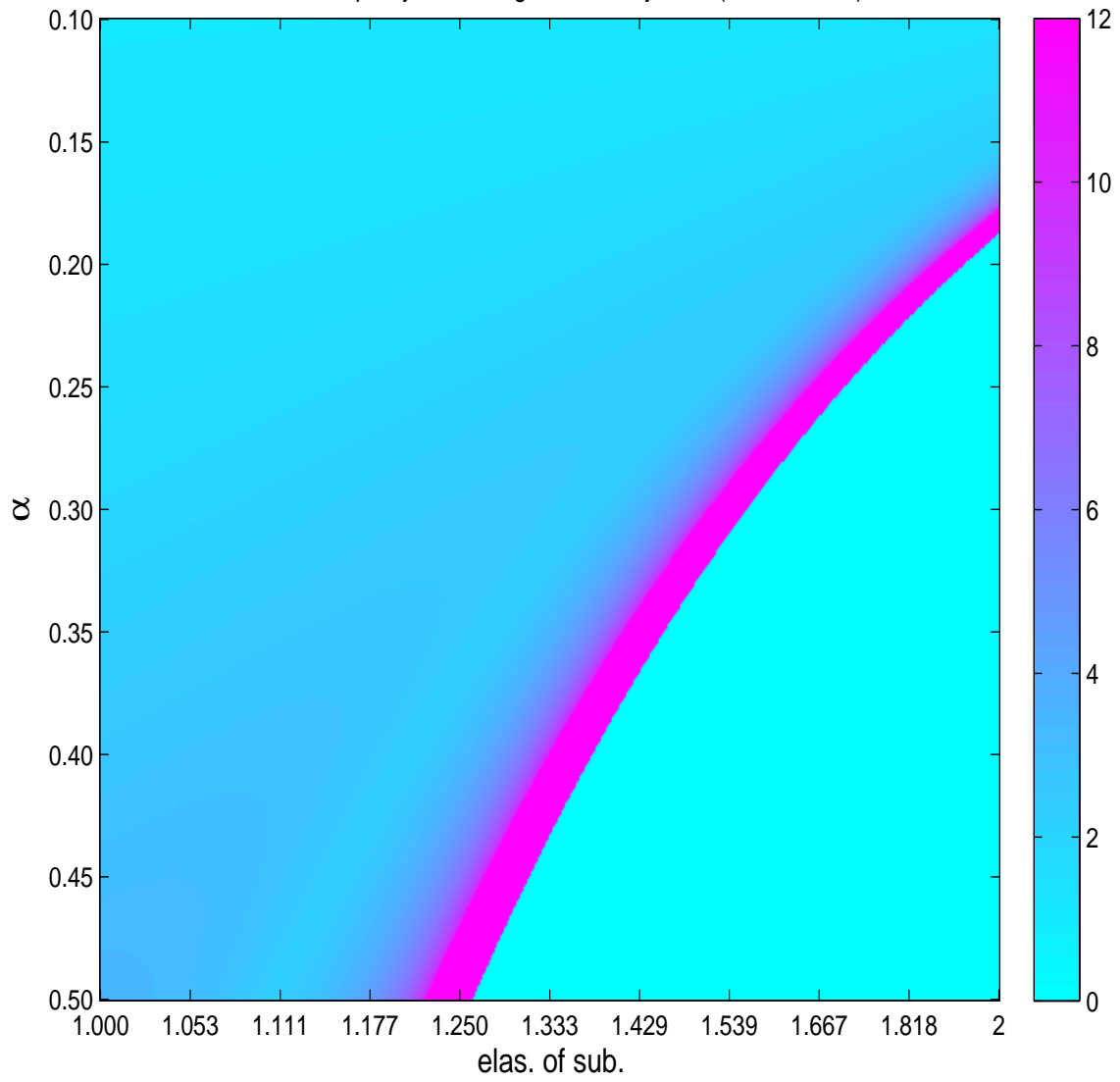


Figure 8: Steady State Inequality

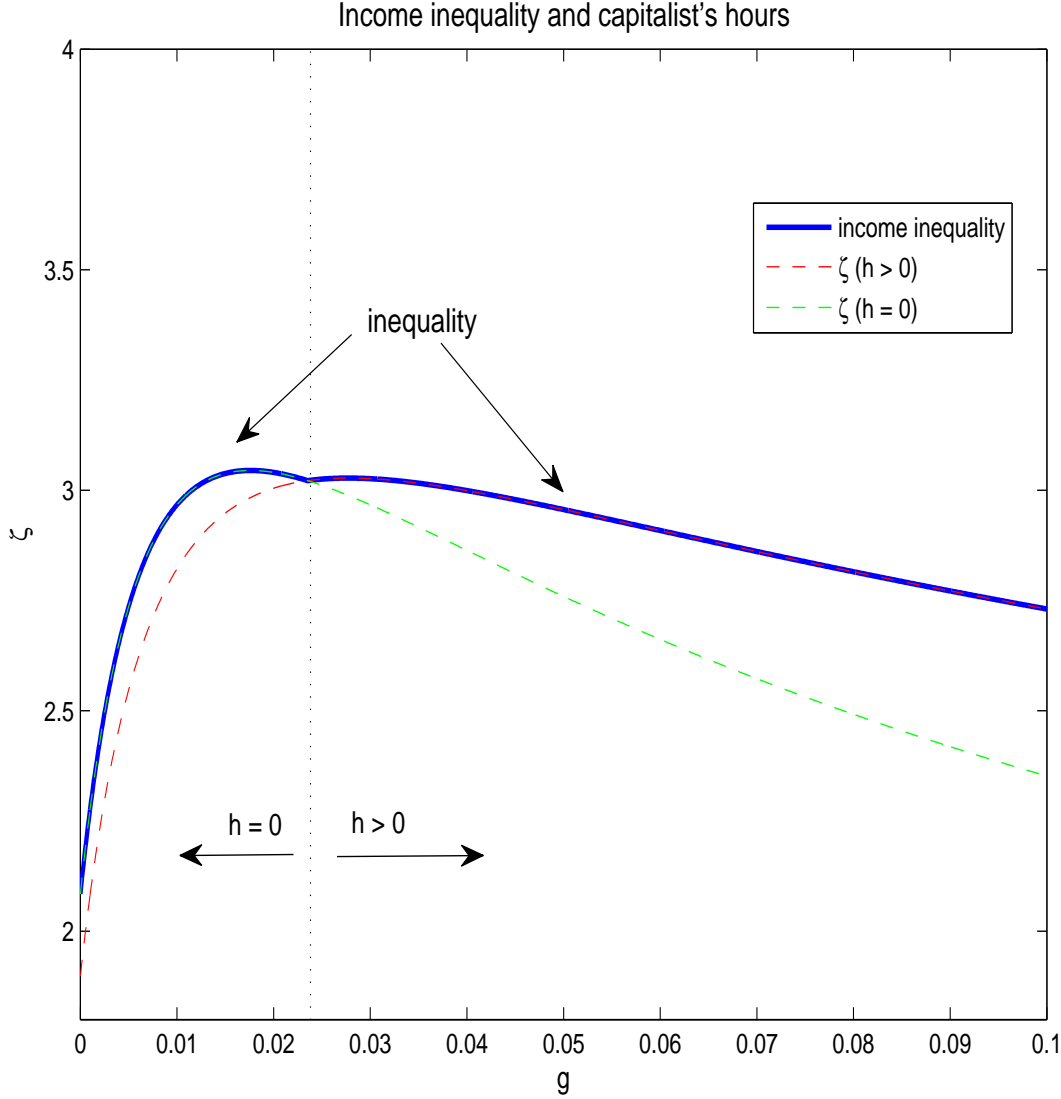


Figure 9: Dynamics of Inequality

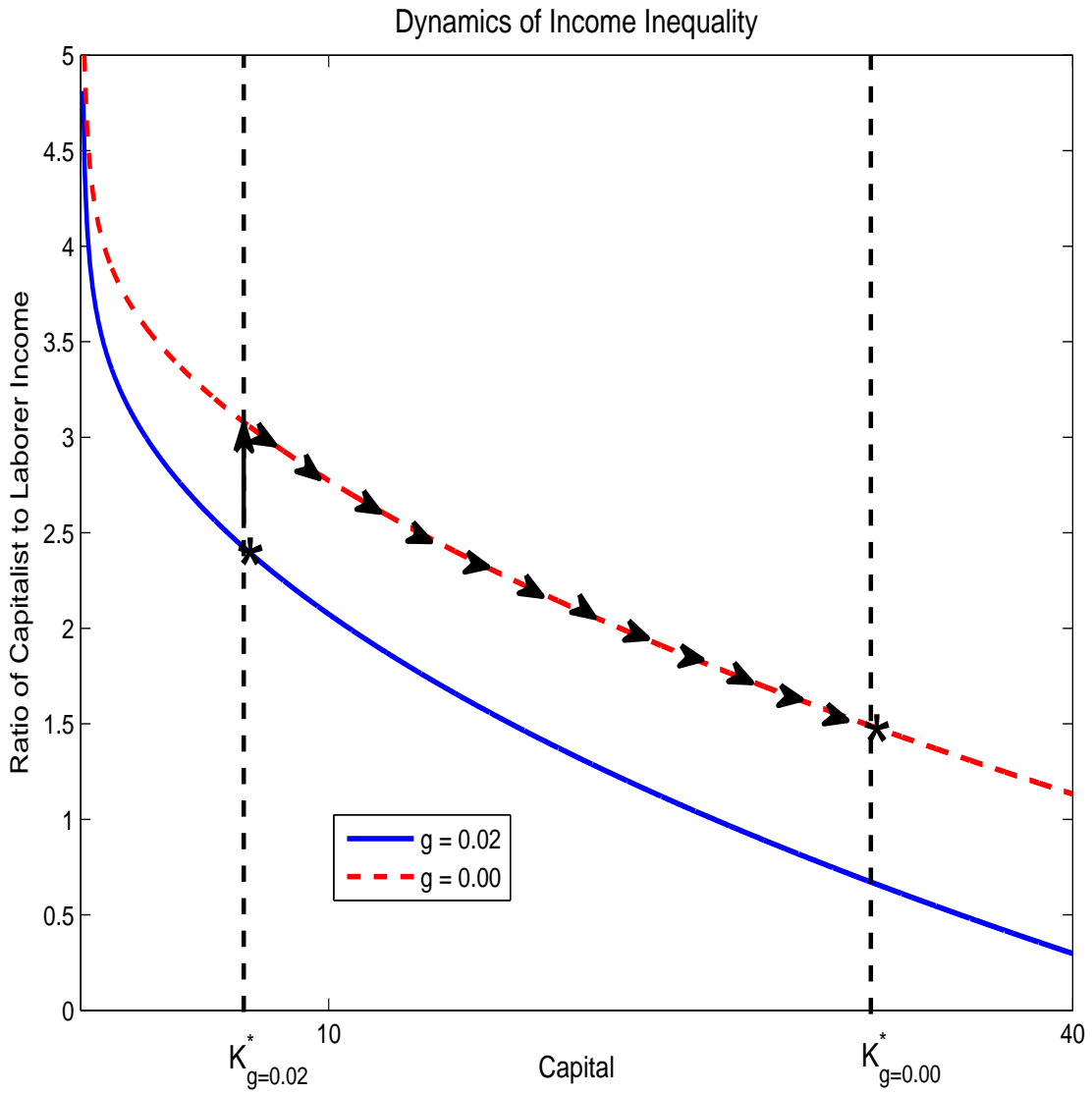


Figure 10: Laffer Curve

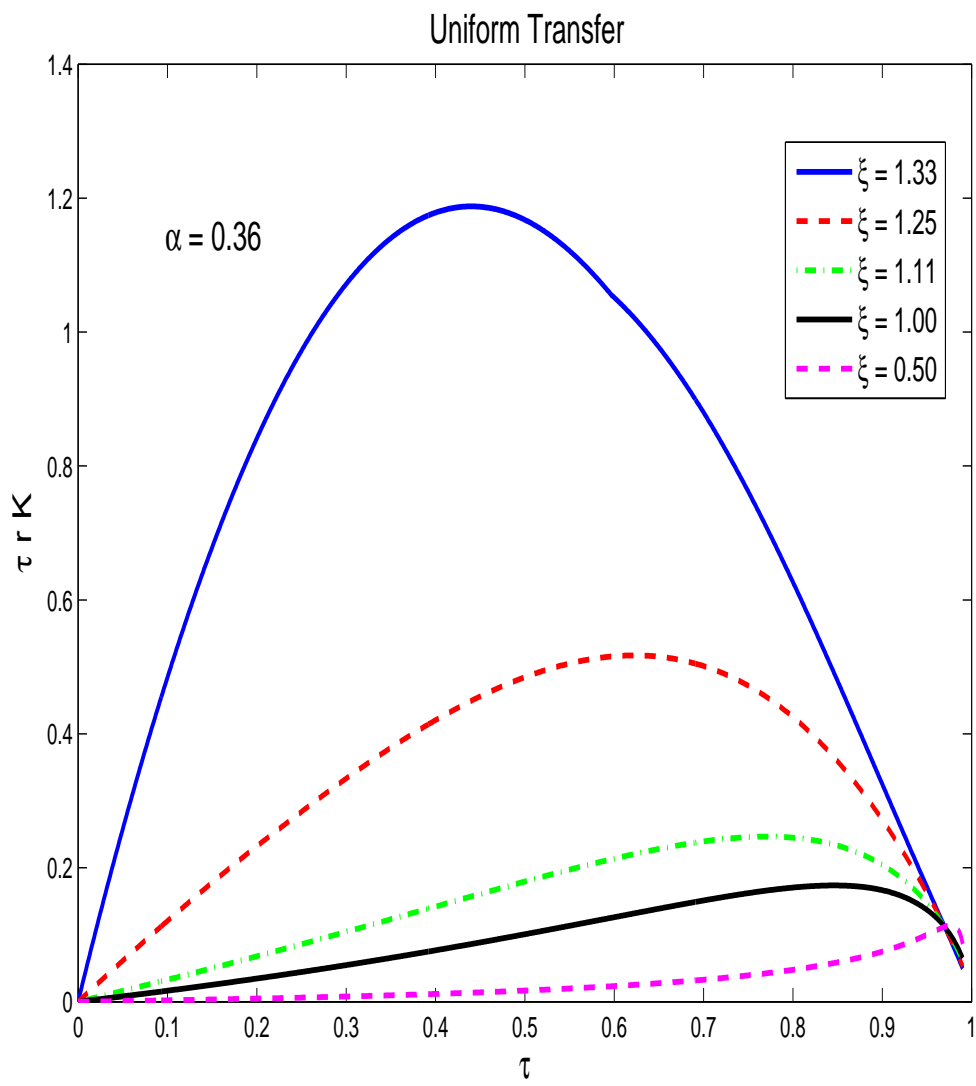


Figure 11: Laffer Curve

Uniform Transfer

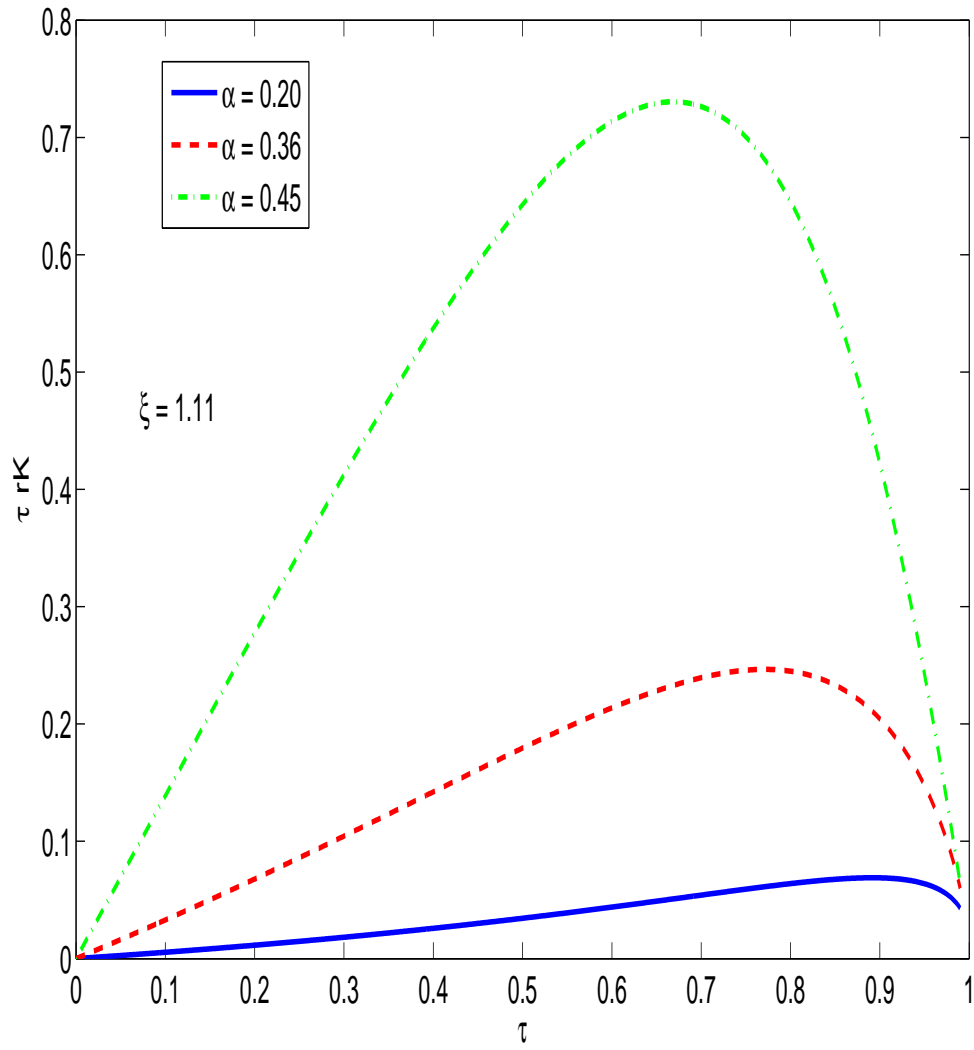


Figure 12: Long run inequality under capital income taxation and zero growth

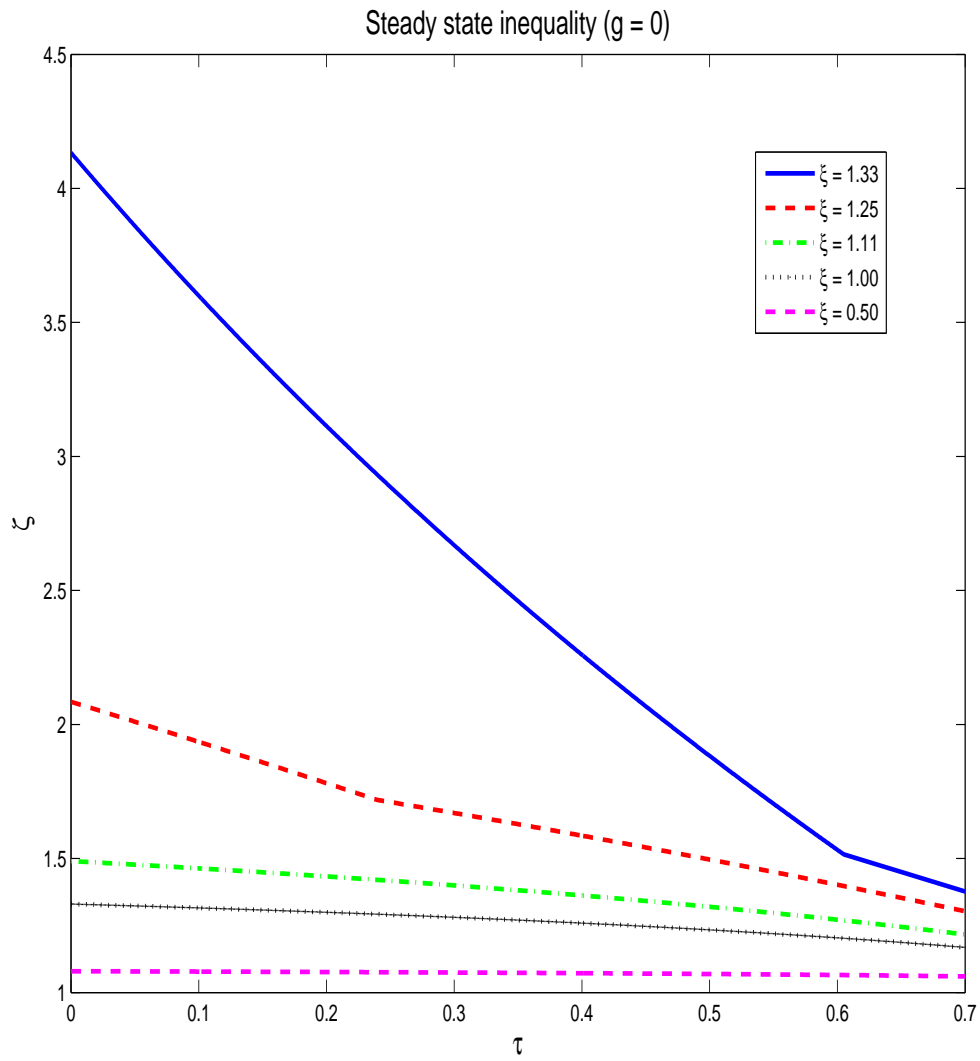


Figure 13: Necessary elasticity of substitution to cause higher inequality with zero growth

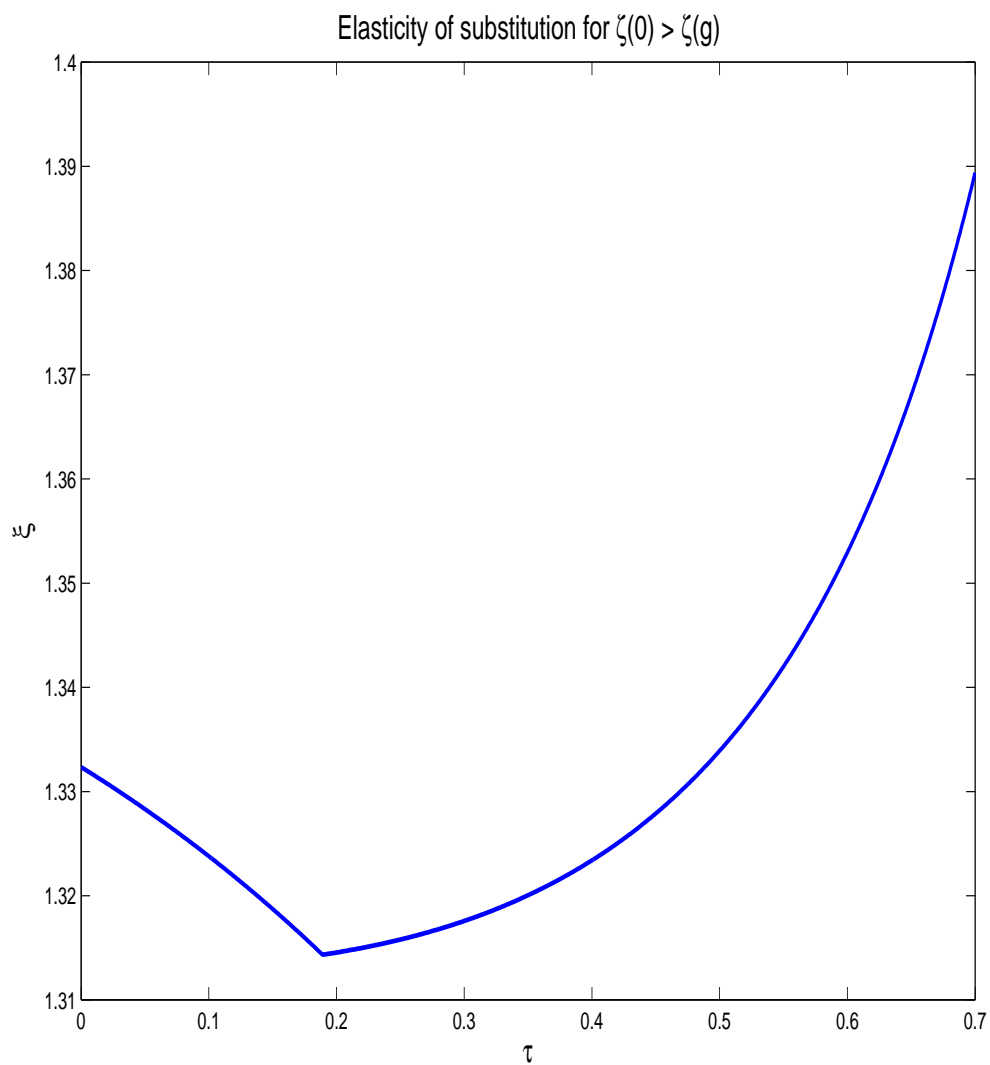


Figure 14: Capitalist Hours

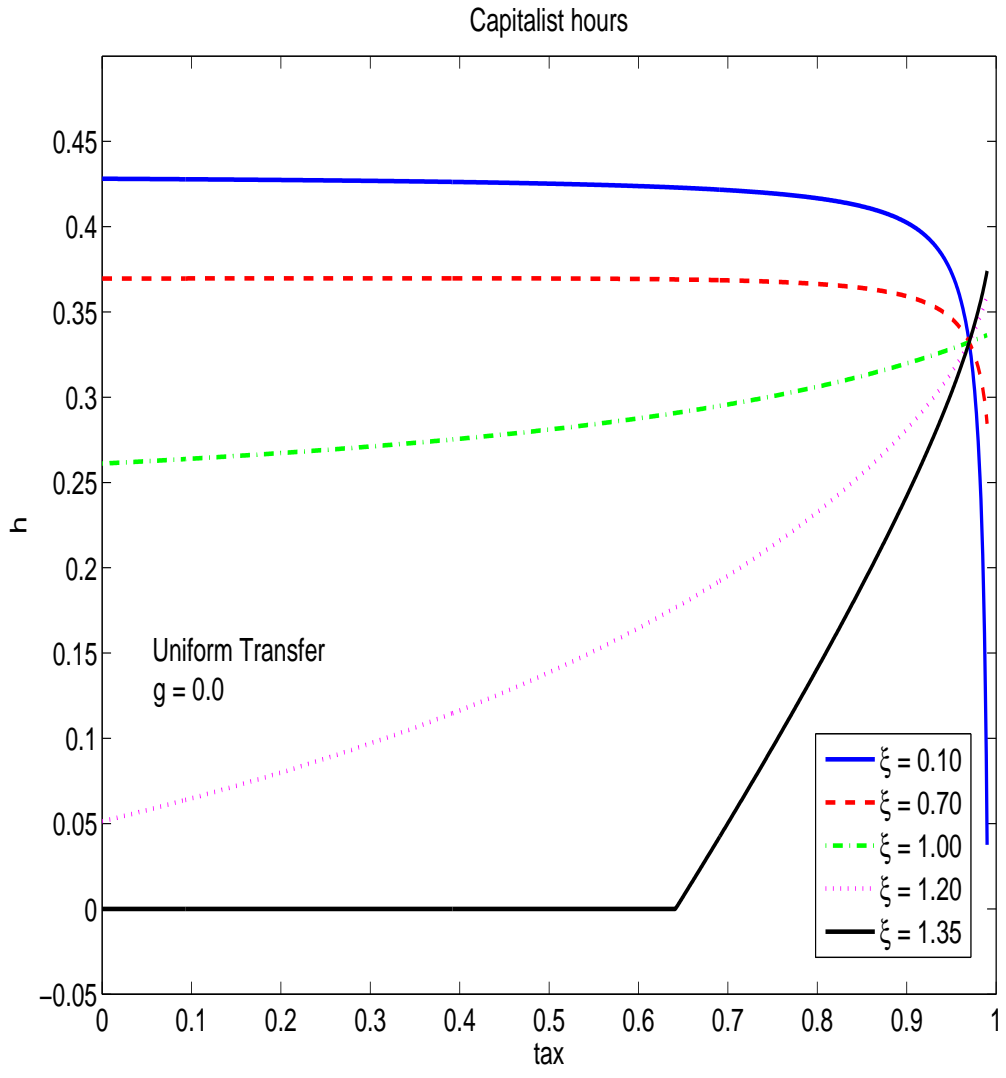


Figure 15: Steady State with Labor Productivity Risk

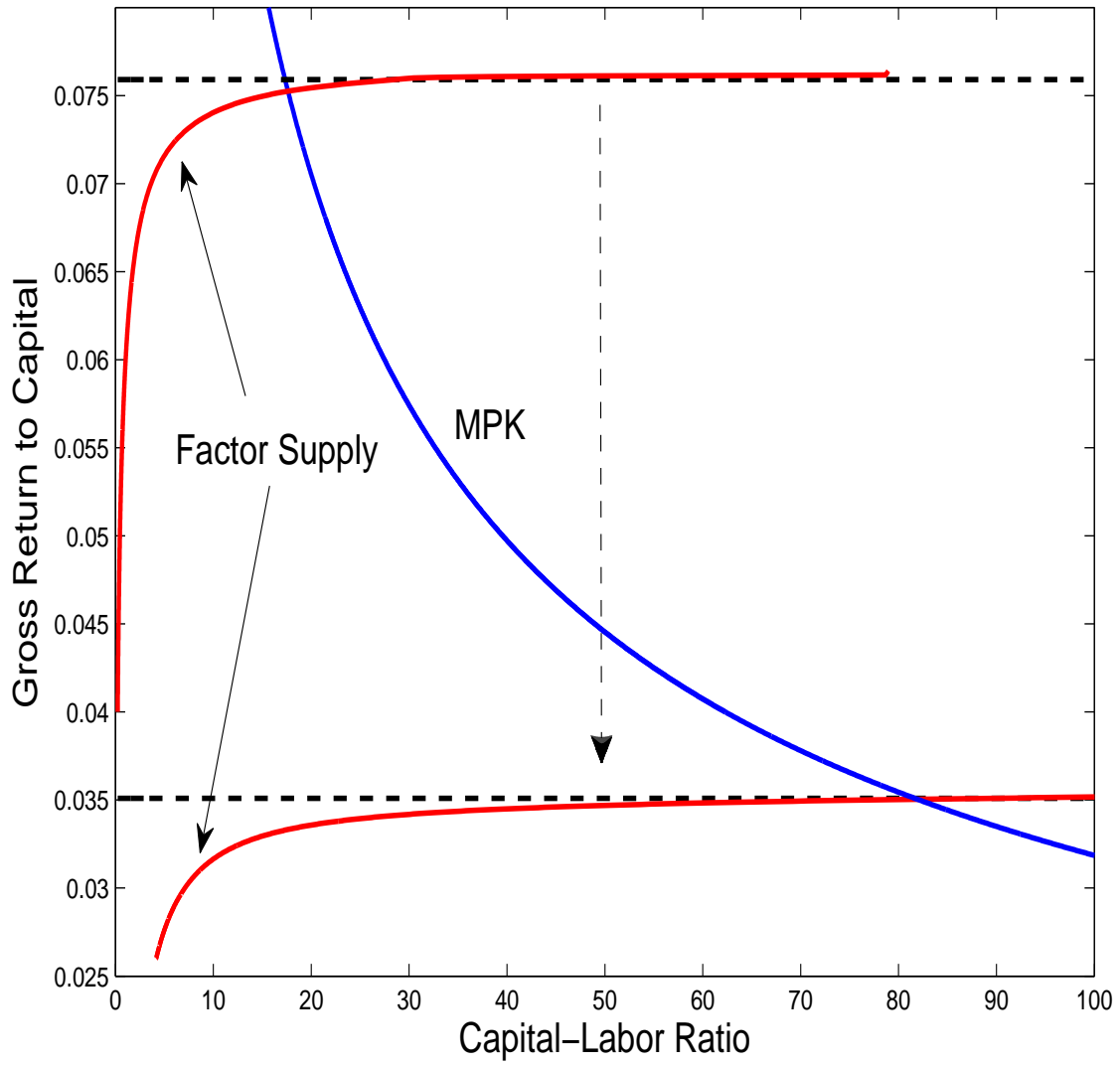


Figure 16: Lorenz Curves

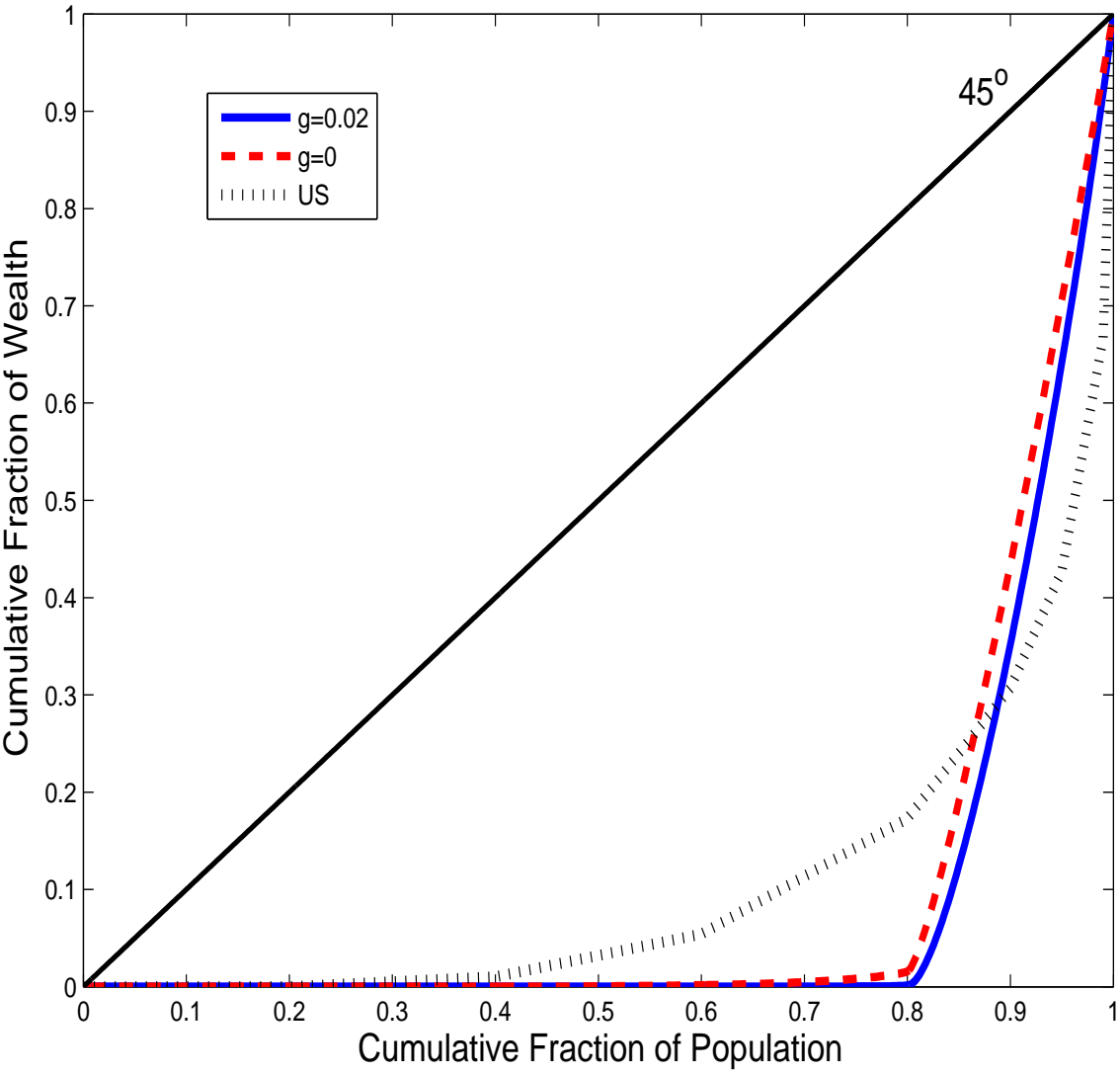


Figure 17: Transitional Dynamics

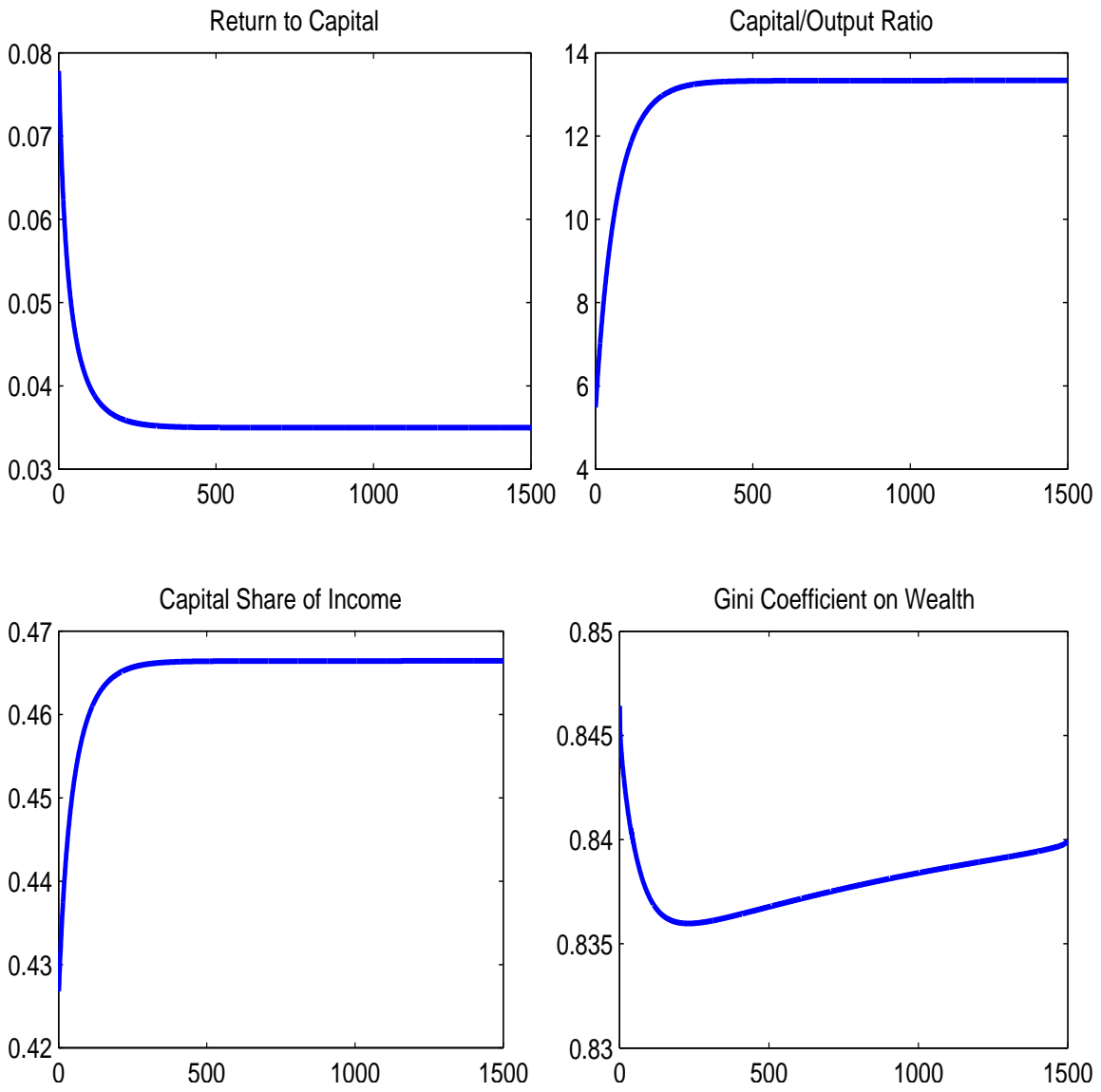


Figure 18: Steady State with Return Risk

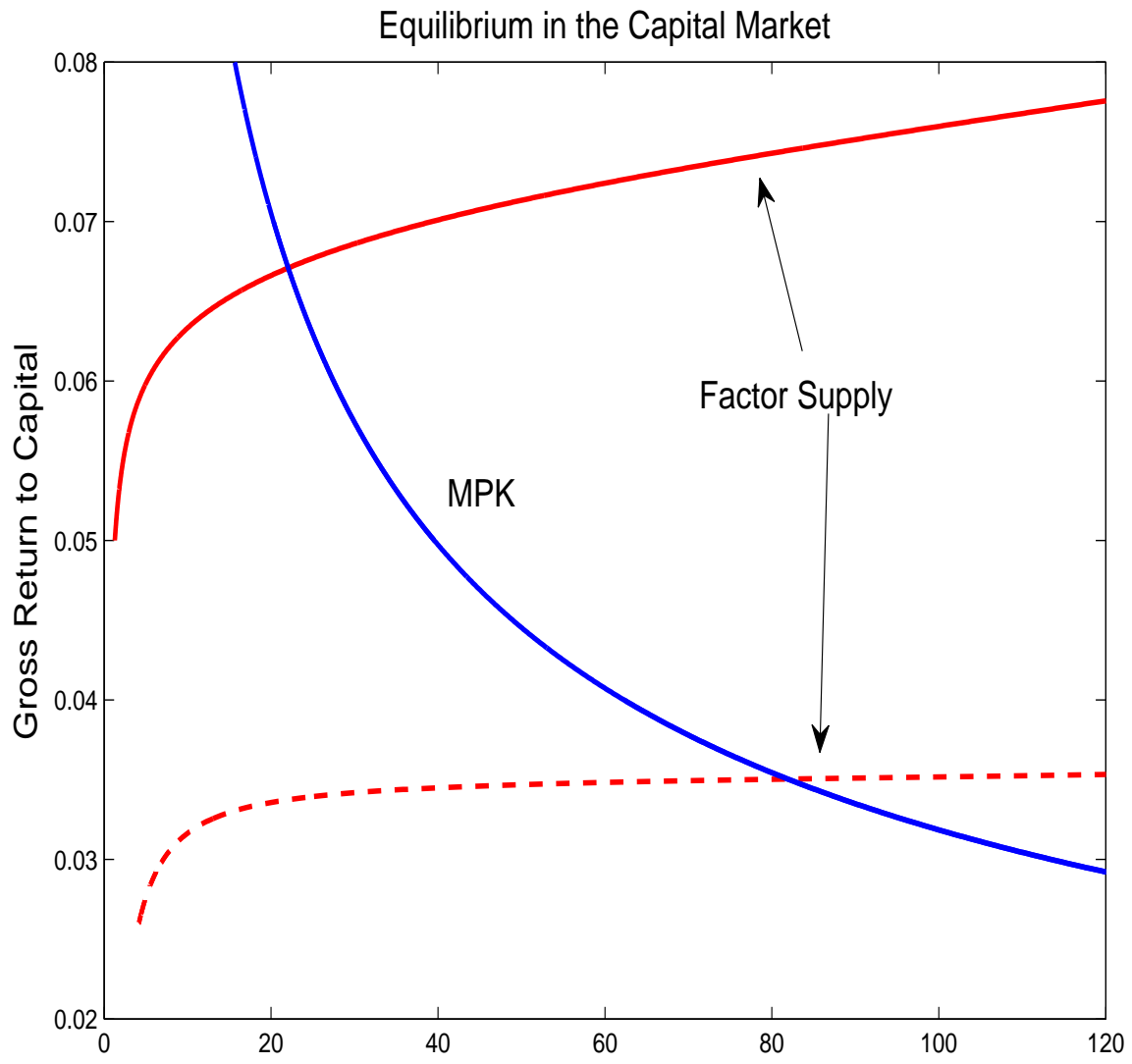


Figure 19: Lorenz Curves

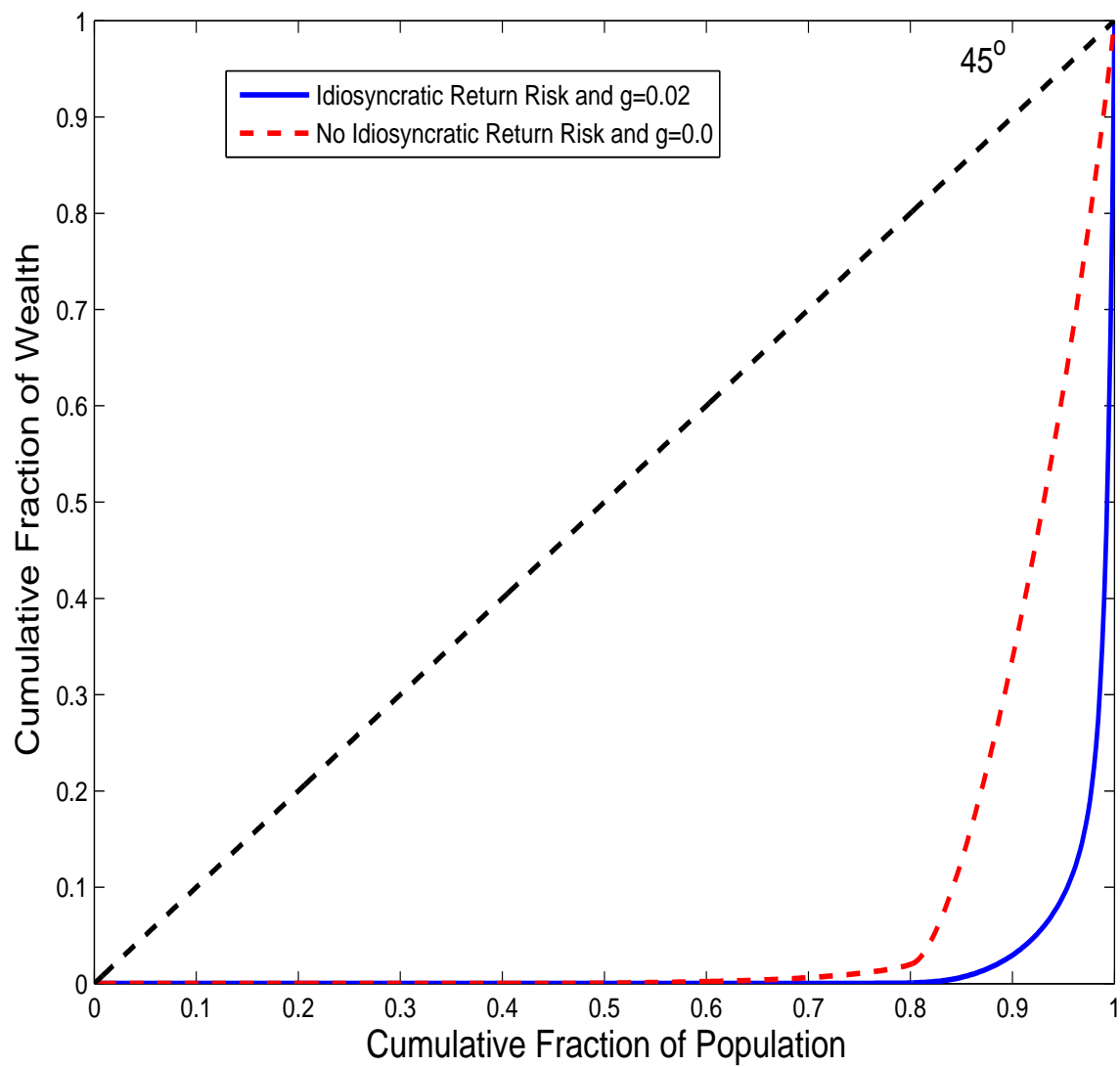


Figure 20: Transitional Dynamics

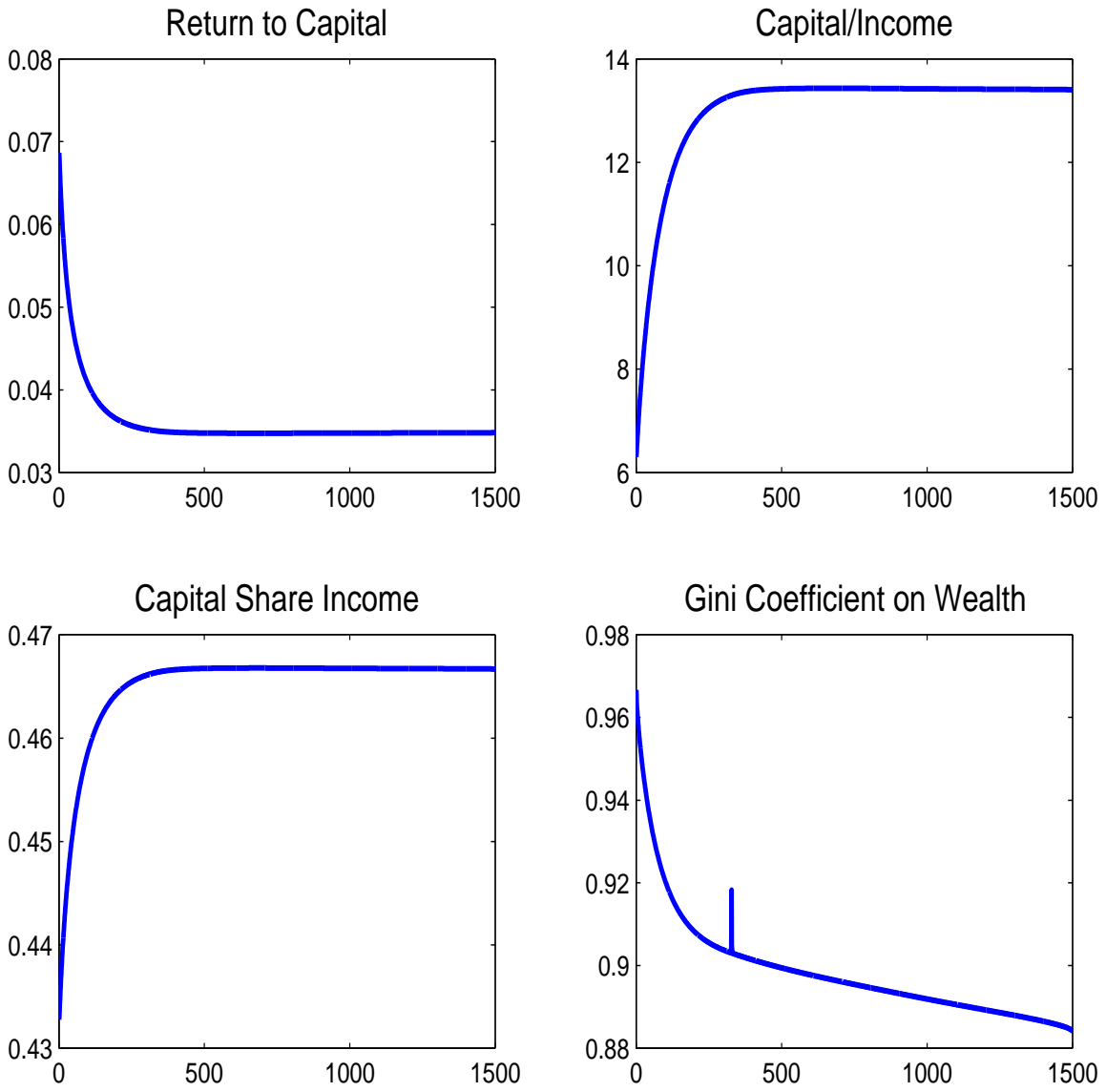


Figure 21: Steady State with Technological Progress

