

# 1 Existence of Recursive Competitive Equilibrium

In this section we want to define a recursive competitive equilibrium and establish some conditions under which RCE exist. One method is to prove that the welfare theorems apply and then use Negishi's approach – you have probably already done this for a homogeneous agent economy where the planning weights are trivial, and we've already seen how we can "decentralize" a social optimum using sequences of prices in the growth model. If the welfare theorems don't apply we need to take a different approach.

Before we get into that, we'll need to carefully define a recursive competitive equilibrium. Take the following economy. The household problem is given by

$$v(k, K) = \max_{F(k, K) \geq k' \geq 0} \{u(F(k, K) - k') + \beta v(k', K')\}$$

where

$$F(k, K) = k + r(K)k + w(K)$$

is total household income. The firm problem is

$$\max_{K, N} \{Y(K, N) - r(K, N)K - w(K, N)N\}$$

where  $Y$  is constant returns to scale. Finally, the market clearing conditions for capital, labor, and goods:

$$\begin{aligned} k &= K \\ 1 &= N \\ C + I &= Y(K, N) \end{aligned}$$

where  $I = K' - K$  is net investment (assume depreciation is included in  $Y$  for simplicity). Note that if  $c = C$  and  $k = K$  and  $N = 1$  we must have  $k' = K'$  (just use the budget constraint).

**Definition 1** *A recursive competitive equilibrium for this economy is a value function  $v : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{R}$ , policy functions  $\pi_k : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$  and  $\pi_c : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{C}$ , pricing functions  $r : \mathcal{K} \rightarrow \mathcal{R}$  and  $w : \mathcal{K} \rightarrow \mathcal{R}$ , and aggregate functions  $\Pi_K : \mathcal{K} \rightarrow \mathcal{K}$  and  $\Pi_C : \mathcal{K} \rightarrow \mathcal{C}$  such that*

1. *Given  $(r, w, \Pi_K, \Pi_C)$ ,  $(v, \pi_k, \pi_c)$  solve the household dynamic program;*
2. *The firm optimally chooses  $K$  and  $N = 1$  given  $(r, w)$  and earns zero profit;*
3. *Markets clear, which means  $\pi_k(K, K) = \Pi_K(K)$  and  $\pi_c(K, K) = \Pi_C(K)$ .*

Before proceeding to a (somewhat) general existence proof, we'll calculate a couple of examples by hand.

## 1.1 Examples

There exists (at least) one nontrivial case where we can solve for the RCE by hand (there are some trivial examples, such as linear technologies, where the prices are effectively exogenous, rendering the problem essentially a planning problem). Suppose  $F(k, K) = \alpha AK^{\alpha-1}k + (1 - \alpha) AK^\alpha$  and  $u(c) = \log(c)$ ; note that this implies  $F(k, K) = r(K)k + w(K)$  for Cobb-Douglas production with inelastic labor. The Bellman equation is

$$v(k, K) = \max_{k'} \{ \log(\alpha AK^{\alpha-1}k + (1 - \alpha) AK^\alpha - k') + \beta v(k', K') \}.$$

We need to guess at the value function and the law of motion for  $K$ :

$$\begin{aligned} v(k, K) &= \pi_0 + \pi_1 \log(K) + \pi_2 \log(k + \pi_3 K) \\ K' &= s\alpha AK^\alpha. \end{aligned}$$

Inserting these guesses into our Bellman equation and taking the first-order condition yields

$$\frac{1}{\alpha AK^{\alpha-1}k + (1 - \alpha) AK^\alpha - k'} = \frac{\beta\pi_2}{k' + \pi_3 s\alpha AK^\alpha}.$$

This implies

$$k' = \frac{\beta\pi_2}{1 + \beta\pi_2} (\alpha AK^{\alpha-1}k + (1 - \alpha) AK^\alpha) - \frac{\pi_3 s\alpha AK^\alpha}{1 + \beta\pi_2}.$$

Inserting this decision rule into the Bellman equation and matching coefficients yields

$$\begin{aligned} \pi_0 + \pi_1 \log(K) + \pi_2 \log(k + \pi_3 K) &= \log\left(\frac{1}{1 + \beta\pi_2} (1 - \alpha) AK^\alpha + \frac{1}{1 + \beta\pi_2} \alpha AK^{\alpha-1}k + \frac{1}{1 + \beta\pi_2} \pi_3 s\alpha AK^\alpha\right) + \\ &\quad \beta\pi_0 + \beta\pi_1 \log(s\alpha K^\alpha) + \\ \beta\pi_2 \log\left(\frac{\beta\pi_2}{1 + \beta\pi_2} \alpha AK^{\alpha-1}k + \frac{\beta\pi_2}{1 + \beta\pi_2} (1 - \alpha) AK^\alpha + \frac{\beta\pi_2}{1 + \beta\pi_2} \pi_3 s\alpha AK^\alpha\right) &= \\ = \log\left(\frac{1}{1 + \beta\pi_2} AK^{\alpha-1}\right) + \log(\alpha) + \log\left(k + \frac{(1 - \alpha) + \pi_3 s\alpha}{\alpha} K\right) + & \\ \beta\pi_0 + \beta\pi_1 \log(s\alpha K^\alpha) + \beta\pi_2 \log\left(\frac{\beta\pi_2}{1 + \beta\pi_2} AK^{\alpha-1}\right) + & \\ \beta\pi_2 \log(\alpha) + \beta\pi_2 \log\left(k + \frac{(1 - \alpha) + \pi_3 s\alpha}{\alpha} K\right). & \end{aligned}$$

Matching the coefficients on  $\log(K)$  and  $\log(k + \pi_3 K)$  as well as the constant inside the log term we get

$$\begin{aligned} \pi_1 &= (\alpha - 1) + \beta\pi_1\alpha + \beta\pi_2(\alpha - 1) \\ \pi_2 &= 1 + \beta\pi_2 \\ \pi_3 &= \frac{(1 - \alpha) + \pi_3 s\alpha}{\alpha} \end{aligned}$$

which has solution

$$\begin{aligned}\pi_2 &= \frac{1}{1-\beta} \\ \pi_3 &= \frac{1-\alpha}{\alpha(1-s)} \\ \pi_1 &= \frac{\alpha-1}{(1-\beta)(1-\alpha\beta)}.\end{aligned}$$

We do not yet know  $s$ . But we know that the decision rule for  $k'$  must also hold in the aggregate:

$$s\alpha AK^\alpha = \frac{\beta\pi_2}{1+\beta\pi_2}\alpha AK^\alpha + \frac{\beta\pi_2}{1+\beta\pi_2}(1-\alpha)AK^\alpha - \frac{1}{1+\beta\pi_2}\pi_3 s\alpha AK^\alpha$$

which implies

$$s\alpha = \frac{\beta\frac{1}{1-\beta} - \frac{1-\alpha}{\alpha(1-s)}s\alpha}{1+\beta\frac{1}{1-\beta}}.$$

One root of this equation is  $s = \beta$  and we ignore the other root  $s = \frac{1}{\alpha}$  because it implies an aggregate saving rate of 1 and that is never individually optimal (we will see this issue again). The implied value for  $\pi_3$  is then

$$\pi_3 = \frac{1-\alpha}{\alpha(1-\beta)}.$$

The constant  $\pi_0$  solves

$$\pi_0 = \log\left(\frac{1}{1+\beta c}A\right) + \log(\alpha) + \beta\pi_1 + \beta\pi_1 \log(s\alpha) + \beta\pi_2 \log\left(\frac{\beta\pi_2}{1+\beta\pi_2}A\right) + \beta\pi_2 \log(\alpha)$$

which implies

$$\begin{aligned}\pi_0 &= \frac{1}{1-\beta} \left( \log((1-\beta)A) + \log(\alpha) + \frac{(\alpha-1)\beta}{(1-\beta)(1-\alpha\beta)} \log(\beta\alpha) + \frac{\beta}{1-\beta} \log(\beta A) + \frac{\beta}{1-\beta} \log(\alpha) \right) \\ &= \frac{1}{1-\beta} \left( \log((1-\alpha\beta)A) + \frac{\alpha\beta}{1-\alpha\beta} \log(\alpha\beta A) \right).\end{aligned}$$

The implied decision rules are

$$k' = \alpha\beta AK^{\alpha-1}k$$

and

$$K' = \alpha\beta AK^\alpha.$$

If you remember back to the social planning problem we obtained the same aggregate law of motion, meaning that the Second Welfare Theorem holds for this economy.

If the utility function takes the form

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

we cannot solve exactly for the decision rule, but we can solve for a general functional form, even for  $\delta \neq 1$ . The Euler equation for the household is

$$(R(K)k + w(K) - k')^{-\sigma} = \beta(R(K')k' + w(K') - k'')^{-\sigma} R(K')$$

where

$$\begin{aligned} R(K) &= r(K) + 1 - \delta \\ &= f'(K) + 1 - \delta. \end{aligned}$$

and

$$w(K) = f(K) - f'(K)K.$$

We guess that the decision rule takes the form

$$k' = \lambda(K)k + \mu(K)$$

for some unknown functions  $\lambda(K)$  and  $\mu(K)$ . That is, the household's decision is linear in individual capital. Inserting this guess into our Euler equation yields

$$\frac{(R(K)k + w(K) - \lambda(K)k - \mu(K))^{-\sigma}}{(R(K')k' + w(K') - \lambda(K')k' - \mu(K'))^{-\sigma}} = \beta R(K').$$

More algebraic manipulation yields

$$\left[ \left( \frac{R(K) - \lambda(K)}{(R(K') - \lambda(K'))\lambda(K)} \right) \left( \frac{k + \frac{w(K) - \mu(K)}{R(K) - \lambda(K)}}{k + \frac{\mu(K)}{\lambda(K)} + \frac{w(K') - \mu(K')}{\lambda(K)(R(K') - \lambda(K'))}} \right) \right]^{-\sigma} = \beta R(K').$$

This equation must hold for every  $k$ . Therefore, it must be the case that

$$\frac{w(K) - \mu(K)}{R(K) - \lambda(K)} = \frac{\mu(K)}{\lambda(K)} + \frac{w(K') - \mu(K')}{\lambda(K)(R(K') - \lambda(K'))}.$$

Inserting this equation back into the Euler equation we get

$$\left( \frac{R(K) - \lambda(K)}{(R(K') - \lambda(K'))\lambda(K)} \right)^{-\sigma} = \beta R(K').$$

These two equations determine  $\lambda(K)$  and  $\mu(K)$  for each value of  $K$ , given that

$$K'(K) = \lambda(K)K + \mu(K).$$

One could solve the system numerically using a number of approaches.

## 1.2 Existence

We know how to handle the existence of components (1) and (2); it is the third requirement that takes a bit of work. The idea will be to construct the equilibrium "Euler equation"

$$u'(F(K, K) - K') = \beta u'(F(K', K') - K'') F_1(K', K')$$

and a sequence of functions that converge to the solution of this equation. We then prove that any such sequence also converges to a function that satisfies the individual Euler equation

$$u'(F(k, K) - k') = \beta u'(F(k', K') - k'') F_1(k', K')$$

and satisfies  $k' = K'$ .

The power of the approach we will look at shows up when the market economy is distorted by taxes, market power, or externalities. For example, the household "production function" with income taxes would be

$$\begin{aligned} F(k, K) &= k + (1 - \tau) r(K) k + (1 - \tau) w(K) \\ &= k + (1 - \tau) Y_K(K) k + (1 - \tau) (Y(K) - Y_K(K) K). \end{aligned}$$

With market power, the factor prices are not equal to the marginal products; for example, with monopolistic competition a markup of  $\mu \geq 1$  is applied to all goods, meaning that

$$\begin{aligned} F(k, K) &= k + r(K) k + w(K) \\ &= k + \frac{Y_K(K)}{\mu} k + \frac{Y(K) - Y_K(K) K}{\mu}. \end{aligned}$$

There could also be external effects, where the aggregate values enter directly into the optimization problem of the firm. For example, let  $\bar{K}$  denote average capital and assume that there are knowledge spillovers proportional to  $\bar{K}$ ; the firm production function is then

$$Y = K^\alpha \bar{K}^\theta N^{1-\alpha-\theta}$$

where  $\alpha + \theta < 1$ . The wage and rental rates in equilibrium (where  $K = \bar{K}$  and  $N = 1$ ) are

$$\begin{aligned} r &= \alpha K^{\alpha+\theta-1} \\ w &= (1 - \alpha) K^{\alpha+\theta}. \end{aligned}$$

We will follow Greenwood and Huffman's proof of existence. Let the household's dynamic programming problem be given by

$$v(k, K) = \max_{k'} \{u(F(k, K) - k') + \beta v(k', K')\} \quad (1)$$

where

$$F(k, K) = r(K)k + w(K) \quad (2)$$

and the aggregate state evolves according to the nontrivial law of motion

$$K' = Q(K). \quad (3)$$

This law of motion is "nontrivial" in the sense that  $Q(K) > 0$  if  $K > 0$ . We will assume that the following conditions hold:

$$F(k, K) > 0 \quad \forall k, K > 0 \quad (4)$$

$$\lim_{K \rightarrow 0} D_1 F(K, K) = \infty \quad (5)$$

$$\exists \bar{K} \text{ such that } F(\bar{K}, \bar{K}) \leq \bar{K} \quad (6)$$

$$\forall K \in [0, \bar{K}] \quad D_1 F(K, K) + D_2 F(K, K) \geq 0 \text{ and } D_{11} F(K, K) + D_{21} F(K, K) < 0. \quad (7)$$

These assumptions are not terribly controversial – the first says that production and income are positive if capital is positive, the second is an Inada condition on the production function at the equilibrium capital stock, the third is a maximum sustainable stock condition, and the last says that the aggregate marginal product of capital is positive and diminishing in the aggregate capital stock in equilibrium. From the firm's first-order conditions we have

$$r(K) = D_1 Y(K, 1) \quad (8)$$

and

$$w(K) = Y(K, 1) - r(K)K \quad (9)$$

where  $Y(K)$  is output. We will also assume that  $u$  is twice continuously differentiable, strictly increasing, strictly concave, and satisfies the Inada conditions – all of these ensure an interior solution to the household problem. It is important to note that the approach applied here may not work for an economy with elastic labor supply, but does work with shocks to technology.

Let the policy function be given by

$$k' = q(k, K). \quad (10)$$

Importantly, it should be clear that

$$q(k, K) < F(k, K) \quad (11)$$

if  $k, K > 0$  – the household would never find it optimal to consume nothing in a period (due to the Inada conditions). The Euler equation associated with this problem is

$$Du(F(k, K) - k') = \beta Du(F(k', K') - k'') D_1 F(k', K'). \quad (12)$$

We know that, in equilibrium, we must have

$$k = K \quad (13)$$

and

$$q(K, K) = Q(K). \quad (14)$$

Using this fact we arrive at the "aggregate Euler equation" which describes the equilibrium:

$$Du(F(K, K) - K') = \beta Du(F(K', K') - K'') D_1 F(K', K'). \quad (15)$$

We now begin the process of proving that an equilibrium exists – we will iterate on the above equation from an initial guess for the aggregate policy function  $Q(K)$ .

Define this initial guess  $H^0(K) \equiv 0$ . Then, given a current value  $H^j(K)$  we update our guess by solving for  $x = H^{j+1}(K)$  in the equation

$$Du(F(K, K) - x) = \beta Du(F(x, x) - H^j(x)) D_1 F(x, x). \quad (16)$$

Obviously we are unlikely to have a closed-form solution for this function, but we can be sure one exists using the implicit function theorem. Thus, we have implicitly defined an operator  $G$  which maps  $H^j$  into  $H^{j+1}$ . Under the assumption

$$0 \leq \frac{\partial H^j(K)}{\partial K} \leq [D_1 F(K, K) + D_2 F(K, K)] \quad (17)$$

we have that the LHS is strictly increasing in  $x$  and the RHS is strictly decreasing (obviously this bound holds for  $H^0(K) = 0$ ):

$$\begin{aligned} \frac{dLHS}{dx} &= -D^2 u > 0 \\ \frac{dRHS}{dx} &= \beta (D^2 u) (D_1 F) \left( D_1 F + D_2 F - \frac{\partial H^j}{\partial x} \right) + \beta (Du) (D_{11} F + D_{12} F) < 0 \end{aligned}$$

where the second term is negative by assumption above and the first is negative if the bounds on the derivative hold. Hence there must be a unique solution, since they cannot cross more than once. Furthermore, we know that

$$0 \leq \frac{\partial H^{j+1}(K)}{\partial K} \leq [D_1 F(K, K) + D_2 F(K, K)] \quad (18)$$

and that replacing  $H^j(K)$  by a function that is greater than it everywhere results in a larger solution for  $x$  (because the LHS is independent of  $H^j(K)$  and the RHS is increasing in  $H^j(K)$  and decreasing in  $x$ ); in other words we have created a sequence of monotonically increasing functions  $\{H^j(K)\}_{j=0}^{\infty}$  defined on a compact set  $[0, \bar{K}]$ . This sequence must be bounded above by  $\bar{K}$ . Thus this sequence must possess a limit point – it is a monotone sequence on a compact set. Denote this limit by

$$Q(K) = \lim_{j \rightarrow \infty} H^j(K), \quad (19)$$

where the limit is taken pointwise. With our bounds on the derivative of our candidate functions  $H^j(K)$  not depending on  $j$  we have created a family of

functions which are equicontinuous. Thus, by the Arzela-Ascoli theorem (see Rudin's textbook) this family is compact in the sense of uniform convergence; our limiting function  $Q(K)$  is therefore also continuous and satisfies the same bounds on its derivative:

$$0 \leq \frac{\partial Q(K)}{\partial K} \leq [D_1 F(K, K) + D_2 F(K, K)]; \quad (20)$$

furthermore, we obtain uniform convergence of  $H^j$  to  $Q$ . Since by construction  $Q(K)$  satisfies the aggregate Euler equation it constitutes a candidate for the equilibrium law of motion for capital. All we need to do is show that

$$Q(K) < F(K, K) \quad (21)$$

and we are done – this is the aggregate counterpart to

$$q(k, K) < F(k, K) \quad (22)$$

and implies that the sequence of functions converges to a function that also solves the household problem.

Take our candidate guess for the value function  $v^j(k, K)$  and our candidate law of motion  $H^j(K)$ . The Bellman equation defined by these guesses is

$$v^{j+1}(k, K) = \max_{k'} \{u(F(k, K) - k') + \beta v^j(k', H^j(K))\}. \quad (23)$$

Since the Bellman operator is a contraction, for a fixed function  $H^j(K)$  there exists a unique fixed point that solves this equation  $v(k, K)$  and it solves the household problem stated above. Let  $\{Q^j\}$  denote a sequence of functions that converge uniformly to  $Q$  on  $[0, \bar{K}]$  and consider the sequence of functions  $\{\hat{v}^n\}$  generated by

$$\hat{v}^{n+1}(k, K) = \max_{k'} \{u(F(k, K) - k') + \beta \hat{v}^n(k', Q^n(K))\}$$

with associated optimal policy function  $k' = q^n(k, K)$ ; suppose that this sequence is initialized with  $\hat{v}^0(k, K) = 0$ . We need to show that  $\lim_{n \rightarrow \infty} \hat{v}^n = v$ , where

$$v(k, K) = \max_{k'} \{u(F(k, K) - k') + \beta v(k', Q(K))\}.$$

First, three facts are established.

1. Since  $v^n \rightarrow v$  uniformly,  $\forall \epsilon > 0$  there exists  $N$  such that  $\forall n \geq N$

$$|v^n(k, K) - v(k, K)| \leq \frac{\epsilon}{3}$$

$$\forall k, K \in [0, \bar{K}].$$

2. Next, define

$$\pi^q(K_0) = Q(\pi^{q-1}(K_0))$$

which gives the aggregate capital stock in period  $q$  given an initial condition  $K_0$  and

$$\widehat{\pi}^{m,q}(K_0) = Q^{n-m}(\widehat{\pi}^{m,q-1}(K_0)).$$

Since  $Q^j \rightarrow Q$  uniformly we must have  $\lim_{m \rightarrow \infty} \widehat{\pi}^{m,q} = \pi^q \forall q > 0$ . Thus, we have the existence of  $M$  such that  $\forall m \geq M \geq \max\{q, 1\}$  we have

$$|U(F(k, \widehat{\pi}^{m,t}(K_0)) - k') - U(F(k, \pi^t(K_0)) - k')| \leq \frac{(1-\beta)\epsilon}{3}$$

$\forall k, k', K_0 \in [0, \overline{K}]$  and  $0 \leq t \leq q$ .

3. If we define  $B = U(F(\overline{K}, \overline{K}))$  then  $\forall \epsilon > 0$  there exists  $P$  such that  $\forall p \geq P$  we have

$$\left(\frac{\beta^p}{1-\beta}\right) B < \frac{\epsilon}{3}.$$

We are now in a position to finish our existence proof. Pick  $\epsilon > 0$  and choose  $N$  and  $P$  as above. Let  $q = \max\{N, P\}$  and choose  $M \geq \max\{q, 1\}$  large enough to satisfy the above condition. Note that

$$\widehat{v}^m(k_0, K_0) = \max_{\{k_{t+1}\}_{t=0}^{m-1}} \left\{ \sum_{t=0}^{m-1} \beta^t U(F(k_t, \widehat{\pi}^{m,t}(K_0)) - k_{t+1}) \right\}$$

and set  $m \geq M$ . Then

$$|v(k_0, K_0) - \widehat{v}^m(k_0, K_0)| \leq \left| v(k_0, K_0) - \max_{\{k_{t+1}\}_{t=0}^{q-1}} \sum_{t=0}^{q-1} \beta^t U(F(k_t, \widehat{\pi}^{m,t}(K_0)) - k_{t+1}) \right| + \frac{\epsilon}{3},$$

since  $q \geq P$ . We therefore have

$$\begin{aligned} |v(k_0, K_0) - \widehat{v}^m(k_0, K_0)| &\leq \left| v(k_0, K_0) - \max_{\{k_{t+1}\}_{t=0}^{q-1}} \sum_{t=0}^{q-1} \beta^t U(F(k_t, \pi^t(K_0)) - k_{t+1}) \right| + \frac{2\epsilon}{3} \\ &= |v(k_0, K_0) - v^q(k_0, K_0)| + \frac{2\epsilon}{3}. \end{aligned}$$

Since  $q > N$  we therefore have

$$|v(k_0, K_0) - \widehat{v}^m(k_0, K_0)| \leq \epsilon.$$

That proves that the sequence of functions  $H^j$  converges to a function  $Q(K) < F(K, K)$ , implying that the RCE exists.

### 1.3 Linear-Quadratic Competitive Equilibrium

Imagine the generic competitive equilibrium growth model formulation

$$v(k, K) = \max_{k'} \{u(r(K)k + w(K) - k') + \beta v(k', K')\} \quad (24)$$

subject to the law of motion for  $K$ :

$$K' = G(K). \quad (25)$$

From the firm's problem we get the conditions

$$r(K) = MPK \quad (26)$$

and

$$w(K) = f(K) - MPK * K \quad (27)$$

(check this for yourself). The following algorithm was suggested by Kydland (1989) to solve these problems:

1. Guess an initial value function  $v^0(k, K)$ .
2. Insert the factor conditions into the consumer's problem.
3. Obtain the first-order condition for the RHS of the Bellman equation and solve for the decision rule  $k' = g(k, K)$ .
4. Aggregate the first-order condition to obtain  $K' = g(K, K) \equiv G(K)$ .
5. Update the guess for the value function by

$$v^1(k, K) = u(r(K)k + w(K) - g(k, K)) + \beta v^0(g(k, K), G(K)). \quad (28)$$

6. Repeat until the value function has converged.

Note that we do not have to guess the form  $K' = G(K)$ ; we obtain it at each step as the aggregation of the individual decision rule  $g(k, K)$ . This algorithm works very nicely in a linear-quadratic environment where all decision rules and transition equations are linear. We will examine this case explicitly as it is instructive on how to solve these problems. The linear-quadratic problem faced by the consumer is

$$[z^T \ S^T \ s^T] \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} z \\ S \\ s \end{bmatrix} = \max_d \left\{ \begin{array}{l} [z^T \ S^T \ s^T \ D^T \ d^T] \begin{bmatrix} Q_{11} & \cdots & Q_{15} \\ \vdots & \ddots & \vdots \\ Q_{51} & \cdots & Q_{55} \end{bmatrix} \begin{bmatrix} z \\ S \\ s \\ D \\ d \end{bmatrix} + \\ \beta [z'^T \ S'^T \ s'^T] \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} z' \\ S' \\ s' \end{bmatrix} \end{array} \right\} \quad (29)$$

subject to the transition equations

$$z' = Az \quad (30)$$

$$s' = B_1z + B_2S + B_3s + B_4D + B_5d \quad (31)$$

and

$$S' = C_1z + C_2S + C_3D. \quad (32)$$

In the notation used here,  $z$  are exogenous variables,  $S$  is the vector of aggregate state variables,  $s$  is the individual vector of state variables,  $d$  is the individual vector of choice variables, and  $D$  is the aggregate version of  $d$ . Equilibrium for this model demands that

$$B_1z + (B_2 + B_3)S + (B_4 + B_5)D = C_1z + C_2S + C_3D. \quad (33)$$

The first-order condition for the household's choice of  $d$  is

$$\begin{aligned} d^* = & - [Q_{55} + \beta B_5^T P_{33} B_5]^{-1} [Q_{51} + \beta B_5^T P_{31} A + \beta B_5^T P_{32} C_1 + \beta B_5^T P_{33} B_1] z \\ & - [Q_{55} + \beta B_5^T P_{33} B_5]^{-1} [Q_{52} + \beta B_5^T P_{32} C_2 + \beta B_5^T P_{33} B_2] S \\ & - [Q_{55} + \beta B_5^T P_{33} B_5]^{-1} [Q_{53} + \beta B_5^T P_{33} B_3] s \\ & - [Q_{55} + \beta B_5^T P_{33} B_5]^{-1} [Q_{54} + \beta B_5^T P_{32} C_3 + \beta B_5^T P_{33} B_4] D. \end{aligned}$$

We can rewrite this more compactly as

$$d^* = F_1z + F_2S + F_3s + F_4D. \quad (34)$$

We now impose that  $s = S$  and  $d = D$  to obtain the equilibrium expression

$$\begin{aligned} D^* &= (I - F_4)^{-1} F_1z + (I - F_5)^{-1} (F_2 + F_3)S \\ &= R_1z + R_2S. \end{aligned}$$

From here we have that

$$\begin{aligned} d^* &= (F_1 + F_4R_1)z + (F_2 + F_4R_2)S + F_3s \\ &= J_1z + J_2S + J_3s. \end{aligned}$$

Also, we can insert this rules into the transition equations to obtain

$$\begin{aligned} S' &= (C_1 + C_3R_1)z + (C_2 + C_3R_2)S \\ &= W_1z + W_2S \end{aligned}$$

and

$$\begin{aligned} s' &= (B_1 + B_4R_1 + B_5J_1)z + (B_2 + B_4R_2 + B_5J_2)S + (B_3 + B_5J_3)s \\ &= X_1z + X_2S + X_3s. \end{aligned}$$

These decision rules are inserted into the Bellman equation to obtain the recursion

$$\begin{aligned}
\begin{bmatrix} P_{11}^{n+1} & P_{12}^{n+1} & P_{13}^{n+1} \\ P_{21}^{n+1} & P_{22}^{n+1} & P_{23}^{n+1} \\ P_{31}^{n+1} & P_{32}^{n+1} & P_{33}^{n+1} \end{bmatrix} &= \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} + \begin{bmatrix} Q_{14} & Q_{15} \\ Q_{24} & Q_{25} \\ Q_{34} & Q_{35} \end{bmatrix} \begin{bmatrix} R_1 & R_2 & 0 \\ J_1 & J_2 & J_3 \end{bmatrix} + \\
&\begin{bmatrix} R_1^T & J_1^T \\ R_2^T & J_2^T \\ 0 & J_3^T \end{bmatrix} \begin{bmatrix} Q_{41} & Q_{42} & Q_{43} \\ Q_{51} & Q_{52} & Q_{53} \end{bmatrix} + \\
&\begin{bmatrix} R_1^T & J_1^T \\ R_2^T & J_2^T \\ 0 & J_3^T \end{bmatrix} \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{54} & Q_{55} \end{bmatrix} \begin{bmatrix} R_1 & R_2 & 0 \\ J_1 & J_2 & J_3 \end{bmatrix} + \\
\beta \begin{bmatrix} A^T & W_1^T & X_1^T \\ 0 & W_2^T & X_2^T \\ 0 & 0 & X_3^T \end{bmatrix} \begin{bmatrix} P_{11}^n & P_{12}^n & P_{13}^n \\ P_{21}^n & P_{22}^n & P_{23}^n \\ P_{31}^n & P_{32}^n & P_{33}^n \end{bmatrix} \begin{bmatrix} A & 0 & 0 \\ W_1 & W_2 & 0 \\ X_1 & X_2 & X_3 \end{bmatrix}.
\end{aligned}$$

We iterate on this equation until it converges as before. Unlike the iterative version of the planning problem there is no guarantee that this process will converge (it is not a contraction in general), but it does seem to work remarkably often. A similar approach can be used to solve dynamic games, either of the Nash or Stackelberg varieties.

## 1.4 Discrete RCE

In a discrete state space environment, solving for an RCE is somewhat computationally-burdensome. For a given value function  $v^0(k, K)$  and law of motion  $K' = G^0(K)$ , we can solve the one-step problem

$$v^1(k, K) = \max_{k'} \{u(F(k, K) - k') + \beta v^0(k', G^0(K))\}$$

to obtain

$$k' = g(k, K).$$

We then update the aggregate law of motion:

$$G^1(K) = g(K, K).$$

The process is repeated until (if) it converges. Note that this approach is generally time-consuming, because the grids for  $k$  and  $K$  must be identical and this typically is wasteful (value functions are generally pretty close to linear in  $K$ ).