Higher Order Expectations in Asset Pricing

Philippe Bacchetta\textsuperscript{2} \hspace{1cm} Eric van Wincoop\textsuperscript{3}
Study Center Gerzensee \hspace{1cm} University of Virginia
University of Lausanne \hspace{1cm} NBER
Swiss Finance Institute and CEPR

October 17, 2006

\textsuperscript{1} We are grateful to John Campbell, Bernard Dumas, Hyun Shin, Narayana Kocherlakota, and two anonymous referees for comments on a previous draft. Financial support from the Bankard Fund for Political Economy and the National Centre of Competence in Research "Financial Valuation and Risk Management" (NCCR FINRISK).

\textsuperscript{2} Corresponding author: Philippe Bacchetta, Study Center Gerzensee, PO Box 21, 3115 Gerzensee, Switzerland. philippe.bacchetta@szgerzensee.ch.

\textsuperscript{3} Department of Economics, University of Virginia, P.O. Box 400182, Charlottesville, VA 22904-4182, USA. vanwincoop@virginia.edu.
Abstract

We examine formally Keynes’ idea that higher order beliefs can drive a wedge between an asset price and its fundamental value based on expected future payoffs. Higher order expectations add an additional term to a standard asset pricing equation. We call this the higher order wedge, which depends on the difference between higher and first order expectations of future payoffs. We analyze the determinants of this wedge and its impact on the equilibrium price. In the context of a dynamic noisy rational expectations model, we show that the higher order wedge depends on first order expectational errors about the mean set of private signals. This in turn depends on expectational errors about future asset payoffs based on errors in public signals. We show that the higher order wedge reduces asset price volatility and disconnects the price from the present value of future payoffs. The impact of the higher order wedge on the equilibrium price can be quantitatively large.

JEL: G0,G1,D8
Keywords: higher order beliefs, beauty contest, asset pricing
1 Introduction

In his General Theory, Keynes (1936) devotes significant attention to factors that can drive a wedge between an asset price and its fundamental value based on expected future payoffs.¹ He emphasizes in particular two factors, mass psychology and higher order opinions. Although market psychology had largely been neglected for decades, it is now receiving significant attention in the growing field of behavioral finance.² On the other hand, the impact of higher order expectations (henceforth HOE) on the equilibrium asset price has received little attention and is not well understood. HOE refer to expectations that investors form of other investors’ expectations of an asset’s subsequent payoffs. In the words of Keynes, investors “are concerned, not with what an investment is really worth to a man who buys it for keeps, but with what the market will value it at ... three months or a year hence”.³

HOE naturally play an important role in dynamic models with heterogeneous information. While this is well known, the role of HOE in asset pricing has not been formally analyzed until the recent paper by Allen, Morris and Shin (2006). These authors show that in general the law of iterated expectations does not hold for average expectations, so that HOE differ from first order average expectations of the asset’s payoff. Moreover, they explicitly solve an equilibrium asset price as a function of HOE of the asset’s payoff in the context of a model with an asset that has a single terminal payoff. They show that the farther we are from the terminal date, the higher the order of expectations. The key implication of HOE emphasized in their paper is that more weight is given to public information as a result of HOE.

In this paper we further explore the role of HOE by analyzing the “higher order wedge,” which captures the impact of HOE on the equilibrium price. It is equal to the difference between the equilibrium price and what it would be if higher

¹See Chapter 12, section 5.
²See Barberis and Thaler (2003) and Hirshleifer (2001) for surveys of the field.
³In a well-known paragraph he compares asset markets to a beauty contest, where contestants have to pick the faces that other competitors find the most beautiful. Keynes argues that third and higher order expectations matter as well: “We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practice the fourth, fifth and higher degrees.”
order expectations were replaced by first order expectations. The latter yields the
standard asset pricing formula as the expected present value of future payoffs and
risk premia. The higher order wedge therefore depends on the difference between
higher order and first order expectations of future payoffs and risk premia. This
wedge adds a third asset pricing component to standard models, where the price
depends on expected payoffs and discount rates. The goal of the paper is to analyze
the determinants of the higher order wedge, and its impact on the equilibrium
price, in the context of a dynamic noisy rational expectations model. In contrast
to Allen, Morris and Shin, we consider an infinitely-lived asset.

The higher order wedge can be expressed as the sum of first and higher or-
der expectations of future market expectational errors about the present value of
subsequent asset payoffs. However, we show that this expression can be reduced
to first order average expectational errors about the mean set of private signals.
We can also show that the higher order wedge depends linearly on expectational
errors about future asset payoffs based on errors in public signals. We find that
the wedge is largest for intermediate levels of the quality of private signals.

Regarding the impact of the higher order wedge on the equilibrium price, we
first show that it reduces asset price volatility. Second, it tends to reduce the
impact of future asset payoff innovations on the equilibrium price and amplify
the impact of unobserved supply or noise trading shocks on the equilibrium price.
Third, it disconnects the equilibrium price from the present value of future as-
et payoffs. Fourth, we show that the impact of the higher order wedge on the
equilibrium price can be quantitatively large.

The finding that the higher order wedge depends on first order expectational
errors about the mean set of private signals is a key result from which many
other results are derived. Intuitively it can be understood in two steps. First,
portfolio holdings of investors will depend on expectational errors that investors
expect the market to make next period about future payoffs. This is in line with
Keynes’ reasoning discussed above. If investors expect that the market will value
the asset too high next period, they will buy the asset, pushing up its price. Second,
investors expect the market to make expectational errors to the extent that they
expect average private signals to differ from their own. When this is systematic
there is an average expectational error about the mean set of private signals.

Public information plays a key role. Assume that public signals are overly fa-
vorable about future payoffs, so that there would be positive expectational errors about future payoffs based on public information alone. Since public information is more favorable than the average private signal, the majority of investors believe that their own private signal is relatively weak and others have more favorable private signals. In other words, there is an average expectational error about average private signals. When a majority of investors expect others to have more favorable private signals, and these signals are still relevant tomorrow, these investors expect the outlook of the market to be too favorable tomorrow. Investors buy the asset in anticipation of this, pushing up the price. Errors in public signals (in this case too favorable) therefore affect the equilibrium price by changing the expectations that agents have about private signals of others and therefore the expectations of others. This is captured by the higher order wedge.

The remainder of the paper proceeds as follows. In the next section we review the related literature. In Section 3 we develop a simple asset price equation that relates the price to first and higher order expectations of future dividends and risk premia. We show that the equilibrium price is driven by three factors: expected payoffs, current and expected future risk premia and the higher order wedge. In Section 4 we introduce a specific information structure in the context of a dynamic noisy rational expectations model and use it to analyze the determinants of the higher order wedge. In Section 5 we discuss the impact of the higher order wedge on the equilibrium price. A specific example of the general information structure is discussed in Section 6, which also provides a numerical illustration of the findings. Section 7 concludes.

2 Related Literature

While higher-order beliefs have been studied in a wide range of contexts, two features make them of special interest in the context of financial markets. First, in financial markets the price today depends on the price tomorrow, so that investors naturally need to form expectations of future market expectations. This dynamic perspective differs from the analysis of 'static' HOE, i.e., expectations of expectations within a period. This is the case when agents interact strategically, e.g.,
as in Morris and Shin (2002), Woodford (2003) or Amato and Shin (2003).\textsuperscript{4} We abstract from strategic interactions by assuming atomistic investors. Second, in financial markets the price provides a mechanism through which idiosyncratic information is aggregated. In forming expectations of other investors’ expectations, special attention is paid to the asset price as it is informative about the private information of others. This additional feature is often not present in the analysis of games with incomplete information, e.g., in global games.

HOE play a role in two types of asset pricing models. The first are models with short sales constraints. Harrison and Kreps (1978) first showed that the price of an asset is generally higher than its “fundamental value” when arbitrage is limited by short sales constraints.\textsuperscript{5} The difference is equal to an option value to resell the asset at a future date to investors with a higher valuation. HOE play a role in this context since the option value depends on the opinions of other investors’ expectations at future dates. However, in this literature the price is equal to its fundamental value when the short sales constraints are removed.

The second type of models featuring HOE are dynamic noisy rational expectation models without short sales constraints. However, these models are usually analyzed without any reference to HOE. This can be done because these models can be solved using a reduced form where HOE are not explicit. This was first shown by Townsend (1983) in the context of a dynamic business cycle model that features dynamic HOE.\textsuperscript{6}

Most of this literature considers a special model where an asset has only one payoff at a terminal date.\textsuperscript{7} Investors receive private information on the final payoff either at an initial date or every period. They trade every period and progressively learn about the final payoff by observing the price. Such a model is studied in par-

\textsuperscript{4}Hellwig (2003) characterizes explicitly HOE in the model proposed by Woodford (2003). It is therefore related in spirit to our approach.

\textsuperscript{5}See also Scheinkman and Xiong (2003), Allen, Morris and Postlewaite (1993) and Biais and Bossaerts (1998).

\textsuperscript{6}The general approach is the method of undetermined coefficients. In the context of asset pricing one first assumes some equilibrium asset price as a linear function of current and past innovations. Investors make decisions based on this conjectured price equation. The resulting equilibrium price equation is then equated to the conjectured one in order to solve for the coefficients.

\textsuperscript{7}See Brunnermeier (2001) for a nice survey of the literature.
ticular by He and Wang (1995), Vives (1995), Foster and Viswanathan (1996), Brennan and Cao (1997), and Allen, Morris, and Shin (2006).\textsuperscript{8} Among the issues analyzed are trading volume and intensity, market depth and liquidity, the informativeness of prices, as well as important aspects of the solution procedure. As mentioned in the introduction, only Allen, Morris and Shin (2006) explicitly analyze the role of HOE in a terminal payoff model.

Although they do not explicitly study the asset pricing implications of HOE, He and Wang (1995) and Foster and Viswanathan (1996) do make some comments on static HOE within their model (the average expectation at time $t$ of the average expectation at time $t$). While this does not correspond to the dynamic form of HOE that affect the equilibrium asset price\textsuperscript{9}, these authors do make the important point that HOE can be reduced to first order expectations. In this paper we show that this remains true for the relevant dynamic HOE. The higher order wedge depends on the average first order expectational error about the mean set of private signals. It is important to stress that the ability to reduce higher order to first order expectations does not imply that they do not matter. The wedge created by HOE is an additional determinant of the asset price, separate from expected dividends and risk premia, and can be quantitatively very large.\textsuperscript{10}

While the terminal payoff model is technically convenient, it is not very realistic. A few papers have analyzed asset pricing models with HOE in a more realistic dynamic environment with an infinitely-lived asset yielding dividends each period and with a constant flow of information. Such models lead to time-independent second order moments and are more in the tradition of stochastic dynamic macroeconomic models. HOE in an infinite horizon framework were indeed first analyzed in macroeconomics, in the business cycle model of Townsend (1983). The first pa-

\textsuperscript{8}Foster and Viswanathan (1996) consider a model with strategic trading, while the other papers consider competitive investors.

\textsuperscript{9}For example, the average expectation at $t$ of the average expectation at $t+1$ of the dividend at $t+2$.

\textsuperscript{10}He and Wang (1995) and Foster and Viswanathan (1996) argue that the ability to reduce higher order to first order expectations helps solve the model since the infinite space of mean beliefs that Townsend (1983) alluded to is reduced to a space of only first order beliefs. However, the method of undetermined coefficients used by Townsend to solve the model does not make any reference to the space of mean beliefs and the solution methods in these two papers also make no use of the fact that higher order expectations can be reduced to first order expectations.
per in finance to analyze such a model is Singleton (1987). He focuses on the time series properties of the equilibrium price, without considering the role of HOE. In Bacchetta and van Wincoop (2006), we solve an infinite horizon model of exchange rate determination in which HOE arise. Using the results from the present paper, we show that HOE can help contribute to the puzzling disconnect between the exchange rate and observed macroeconomic aggregates.

The above references analyze models with heterogeneous beliefs leading to HOE. However, the presence of information heterogeneity does not necessarily lead to HOE, even in a dynamic asset pricing context. Kasa et al. (2006) and Walker (2005) present examples where the asset price is fully revealing,\footnote{These two papers closely follow Futia’s (1981) setup and use frequency domain techniques to derive analytic solutions. Kasa et al. (2006) show that whether the price is fully revealing depends on model parameters. If it is not fully revealing, then HOE do matter.} so that heterogeneous information has no impact in equilibrium.\footnote{As an illustration, consider the case where there are two types of investors, each receiving a different private signal. If the equilibrium price is only affected by these two private signals, each type of investor can derive exactly the signal of the other type. The two-type case is also considered in the first model of Townsend (1983) and analyzed by Pearlman and Sargent (2005).} In these examples, there is no need to guess other investors’ expectations and HOE obviously do not arise. Alternatively, there are models where heterogeneity matters for equilibrium prices, but the information structure is such that HOE collapse to first order expectations. This is the case, for example, in models with a hierarchical information structure as in Wang (1993, 1994). He considers two types of investors in a noisy rational expectations framework where the price is not fully revealing. However, he assumes that one type of agents is fully informed about the current fundamental and can exactly derive the expectations of uninformed investors. There is no HOE in that context. In the model presented below, the asset price is never fully revealing and we will give conditions under which HOE arise.

3 A Simple Asset Pricing Equation

3.1 Assumptions and Equilibrium Price

In this section we derive a simple asset price equation that relates the asset price to HOE of future payoffs. We adopt a share economy that is standard in the
noisy rational expectations literature and allows for an exact solution without using linearization methods. The basic assumptions are: (i) constant absolute risk aversion; (ii) investors invest for one period only (overlapping-generations of two-period lived investors); (iii) an excess return that is normally distributed with the same conditional variance for all agents; (iv) a constant risk-free interest rate; (v) a share economy with a stochastic supply of shares; (vi) a competitive market with a countable infinite set of agents $N = 1, 2, \ldots$ (the set of natural numbers).

These assumptions are commonly made in the noisy rational expectations literature, but deserve some comments. First, assumption (i) leads to a simple optimal portfolio allocation without the need for any approximations. Assumption (ii) significantly simplifies the portfolio choice problem of investors. If agents have longer horizons the optimal portfolio includes a hedge against possible changes in expected returns, which unnecessarily complicates matters. Assumption (iii) is made exogenously for now, but in Section 4 it will be the endogenous outcome of the assumed information structure. The stochastic supply of shares in assumption (v) is important in that it prevents the equilibrium asset price from completely revealing the average of private information.\textsuperscript{13} The per capita random supply of shares is $X_t$ and is not observable.

The countability of agents in assumption (vi) is frequently adopted as well (e.g., Hellwig, 1980, Brown and Jennings, 1989, He and Wang, 1995). The problem with assuming a continuum of agents is that integrals of a continuum of random variables are generally not well-defined as the law of large numbers does not necessarily hold. For a nice discussion of these issues see Vives (2006), who suggest the solution adopted in Feldman and Gilles (1985) and He and Wang (1995) with a countable infinite number of agents. Integrals of a countable infinite number of random variables are well defined with respect to the following measure $\mu$ on subsets $A$ of natural numbers:

\begin{equation}
\mu(A) = \lim_{I \to \infty} \frac{1}{I} \#A(1, \ldots, I)
\end{equation}

where $\#A(1, \ldots, I)$ denotes the number of elements in $A$ between 1 and $I$. The integral of any set $q_i$ of random variables over all $i \in N$ with respect to the

\textsuperscript{13} A typical justification for this assumption is that the net supply of shares is random. For example, He and Wang (1995) assume that the total number of shares is constant but changes in demand by exogenous liquidity traders makes the residual supply stochastic. Such exogenous traders are also commonly referred to as noise traders.
measure \( \mu \) is then
\[
\int q_i = \int_{i \in N} q_i d\mu(i) = \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{I} q_i
\]
We analogously define the average expectation as
\[
\overline{E}(\cdot) = \int E_i(\cdot) d\mu(i) = \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{I} E_i(\cdot)
\]
where \( E_i(\cdot) \) is the expectation of investor \( i \).

Based on these preliminaries, we can examine the investors’ decisions and the equilibrium asset price. Investors allocate optimally their wealth between a risky stock and a safe asset. Let \( P_t \) be the ex-dividend share price, \( D_t \) the dividend, and \( R \) the constant gross interest rate. The dollar excess return on one share is
\[
Q_{t+1} = P_{t+1} + D_{t+1} - RP_t
\]
This leads to the standard asset demand equation
\[
x_i^t = \frac{E_i^t(P_{t+1} + D_{t+1}) - RP_t}{\gamma \sigma_t^2}
\]
where \( \gamma \) is the rate of absolute risk aversion and \( \sigma_t^2 \) is the conditional variance of next period’s excess return.

The market equilibrium condition is
\[
\int x_i^t = X_t
\]
If we define the risk premium term as \( \phi_t = \gamma \sigma_t^2 X_t / R \), and using the definition \( \overline{E}(\cdot) \), the market clearing condition gives:\( ^{14} \)
\[
P_t = \frac{1}{R} \overline{E}_t(P_{t+1} + D_{t+1}) - \phi_t.
\]
To compute the equilibrium price, we need to integrate (4) forward. In typical asset pricing formulas, this is done by applying the law of iterated expectations. While this law always holds for individual expectations, it may not hold for market expectations when investors have different information sets. For example, \( \overline{E}_t \bar{E}_{t+1} D_{t+2} \neq E_t D_{t+2} \). Thus, we define the average expectation of order \( k \) as
\[
\overline{E}_t^k = \overline{E}_t \bar{E}_{t+1} \ldots \bar{E}_{t+k-1}
\]
\(^{14}\)Notice that, despite heterogeneity, we could express the price in terms of a stochastic discount factor. However, we do not follow this route.
for $k > 1$. Moreover, $E^0_t x = x$, $E^1_t x = E_t x$. The equilibrium price is then (ruling out bubbles):

$$P_t = \sum_{s=1}^{\infty} \frac{1}{R^s} E_t^s D_{t+s} - \sum_{s=1}^{\infty} \frac{1}{R^s} E_t^s \phi_{t+s} - \phi_t$$  \hspace{1cm} (6)

The stock price is equal to the present discounted value of expected dividends minus risk premia. The difference with a standard asset pricing equation is that first order expectations are replaced by HOE. A dividend accruing $s$ periods ahead has an expectation of order $s$. For example, if $s = 2$, we need to compute the market expectation at time $t$ of the market expectation at $t+1$ of $D_{t+2}$ rather than the first-order expectation of $D_{t+2}$. This implies that investors have to predict the future market expectation of the dividend rather than the dividend itself. This is the ’beauty contest’ phenomenon described by Keynes. Moreover, with an infinite horizon, the order of expectation can obviously go to infinity.

### 3.2 The Higher Order Wedge

Define $P^*_t$ as the price that emerges when higher order expectations in (6) are replaced by first order expectations:

$$P^*_t = \sum_{s=1}^{\infty} \frac{1}{R^s} E_t D_{t+s} - \sum_{s=1}^{\infty} \frac{1}{R^s} E_t \phi_{t+s} - \phi_t$$  \hspace{1cm} (7)

To examine the impact of HOE on asset prices, we define the higher order wedge as the difference between $P_t$ and $P^*_t$:

$$\Delta_t = P_t - P^*_t = \sum_{s=1}^{\infty} \frac{1}{R^s} [E_t^s D_{t+s} - E_t D_{t+s}] - \sum_{s=1}^{\infty} \frac{1}{R^s} [E_t^s \phi_{t+s} - E_t \phi_{t+s}]$$  \hspace{1cm} (8)

It depends on the present value of deviations between higher order and first order expectations of dividends minus risk premia. The higher order wedge $\Delta_t$ adds a third element to the standard asset pricing equation:

$$P_t = \sum_{s=1}^{\infty} \frac{1}{R^s} E_t D_{t+s} - \left( \sum_{s=1}^{\infty} \frac{1}{R^s} E_t \phi_{t+s} + \phi_t \right) + \Delta_t$$  \hspace{1cm} (9)

The first term is associated with expected payoffs; the second term captures current and expected future risk premia (affecting discount rates); the last term is the higher order wedge.

---

\(^{15}\)We adopt the same notation as Allen, Morris, and Shin (2006). Notice that the time horizon changes with the order of expectation.
For expositional purposes, we will assume that the second term in (8) is zero, so that the higher order wedge is only associated with the difference between higher and first order expectations of dividends. This will for example be the case when there is only public information about future risk premia. However, all the results derived below hold for the general case where HOE of both future dividends and risk premia differ from first order expectations. One simply needs to replace the word “dividend” by “dividend minus risk-premium” in what follows.

Before introducing more specific assumptions in the next section, we show that the difference between higher and first order expectations in (8) can be written in terms of expectations of market expectational errors. This makes concrete the conjecture by Keynes (1936) that investors do not just make decisions based on their own perception of the “prospective yield” (expected future dividends), but worry about market expectations. It also allows us to adopt an iterative procedure in Section 4.2 to convert the wedge into an expression that depends on first order expectational errors about average private signals.

First consider $s = 2$. The difference between the second and first order expectation is equal to the average expectation at time $t$ of the average expectational error at $t + 1$ about $D_{t+2}$:

$$E_t^2 D_{t+2} - E_t D_{t+2} = E_t(E_{t+1} D_{t+2} - D_{t+2})$$

The intuition behind this term is as follows. Investment decisions at time $t$ are based on the expected price at $t+1$. This price will reflect the market expectation of subsequent dividends. An investor at time $t$ therefore makes investment decisions not just based on what he believes the dividend at $t + 2$ to be, but also on whether he believes the market to make an expectational error at $t + 1$ about the dividend at $t + 2$. When investors have common information, they expect no future market expectational errors. But as we show below, this is no longer the case when information is heterogeneous.

Next consider $s = 3$. The difference between the third and first order expectation is equal to the difference between the first and second order expectation plus the difference between the second and third order expectation. This can be written as the average expectation at $t$ of the average expectational error at $t + 1$ plus the second order expectation at $t$ of the average expectational error at $t + 2$:

$$E_t^3 D_{t+3} - E_t D_{t+3} = E_t(E_{t+1} D_{t+3} - D_{t+3}) + E_t^2(E_{t+2} D_{t+3} - D_{t+3})$$
The last term can be understood as follows. Just as the price at time $t$ depends on expected average expectational error at $t + 1$, so does the price at $t + 1$ depend on expected average expectational error at $t + 2$. The expected return from $t$ to $t + 1$ then depends on the expectation at time $t$ of the market’s expectation at $t + 1$ of the market’s expectational error at $t + 2$. In other words, investment decisions at time $t$ depend on the second order expectation at $t$ of the market’s expectational error at $t + 2$.

Proceeding along this line for expectations of even higher order, and defining the present value of future dividends as $PV_t = \sum_{s=1}^{\infty} \frac{1}{R^s}D_{t+s}$, Appendix A shows that we can rewrite (8) as follows:

$$\Delta_t = \sum_{s=1}^{\infty} \frac{1}{R^s} E_t^s (E_{t+s} PV_{t+s} - PV_{t+s})$$  \hspace{1cm} (10)

The higher order wedge therefore depends on first and higher order expectations of future expectational errors of the subsequent present value of dividends: the market expectation at $t$ of the market’s expectational error at $t + 1$ of $PV_{t+1}$, the second order expectation at $t$ of the expectational error at $t + 2$ of $PV_{t+2}$, and so on. Investors make decisions not just based on what they expect future dividends to be, but also on what they expect the market’s expectational error next period to be about those future dividends, and what they expect next period’s market expectation of the expectational error in the subsequent period to be. In the rest of our analysis we will use (10) instead of (8) to interpret the wedge.

4 A Dynamic Noisy Rational Expectations Model

4.1 Basic Setup

In order to describe what determines the expectations of future expectational errors, as expressed in (10), we need to be more precise about the information structure and the process of dividends. We develop an infinite horizon noisy rational expectations framework in which there is a constant flow of information, specified below, leading to an equilibrium asset price that is a time-invariant function of shocks.

Dividends are observable and the process of dividends is known by all agents.
We assume a general process:

\[ D_t = \bar{D} + C(L)\varepsilon_t^d \]  

(11)

where \( C(L) = c_1 + c_2 L + \ldots \) is an infinitely lagged polynomial with \( c_1 \neq 0, c_s \) approaching a constant as \( s \to \infty \) and \( \varepsilon_t^d \sim N(0, \sigma_d^2) \). Asset supply is not observable, but its process is known by all agents. We again assume a general process:

\[ X_t = F(L)\varepsilon_t^x \]  

(12)

with \( F(L) = f_1 + f_2 L + \ldots \) an infinitely lagged polynomial with \( f_1 \neq 0 \) and \( \varepsilon_t^x \sim N(0, \sigma_x^2) \).

Each period investors obtain a vector of \( J \) independent (orthogonal) private signals about a linear combination of dividends over the next \( T \) periods:

\[ \hat{v}_t^i = \Omega \begin{pmatrix} D_{t+1} \\ D_{t+2} \\ \vdots \\ D_{t+T} \end{pmatrix} + \varepsilon_t^{vi} \equiv \Omega D_t^T + \varepsilon_t^{vi} \]  

(13)

with \( D_t^T \) being the set of future dividends on which there are signals at time \( t \), \( \Omega \) a \( J \times T \) matrix and the \( J \) errors in \( \varepsilon_t^{vi} \) uncorrelated (independent signals) and each distributed \( N(0, \sigma_v^2) \). It is assumed that the last column of \( \Omega \) is non-zero, so that at least one private signal provides information about the dividend payment \( T \) periods later.

Since future dividends depend partly on past dividend innovations, private signals are related to innovations that have taken place before time \( t \). A component of the private signals is therefore public information. It is useful to define the really informative measure of private signals by subtracting the publicly known components of the raw private signals. This gives

\[ v_t^i = \Omega \varepsilon_t^d + \varepsilon_t^{vi} \]  

(14)

where \( \Omega \) is a \( T \times T \) matrix and \( \varepsilon_t^{d'} = (\varepsilon_{t+1}^d, \varepsilon_{t+2}^d, \ldots, \varepsilon_{t+T}^d) \). \( \varepsilon_t^d \) is the component of the future dividend vector \( D_t^T \) that depends on future dividend innovations that are not yet known at time \( t \).

When \( T > 1 \), there is by assumption at least one private signal for each period from \( t - T + 1 \) to \( t - 1 \) that contains information about future dividend innovations.
at time $t$. Now consider the entire set of signals received from $t-T+1$ to $t-1$ that contain information about future dividends at time $t$. After subtracting the components of these signals that are known at time $t$, we can summarize these past private signals as

$$V_{t-1}^{i} = \Gamma \epsilon_t^{d} + \epsilon_{t-1}^{V}$$

(15)

where $\Gamma$ contains all the non-zero rows of the matrix $\Omega^V C$ in which $\Omega^V$ is a $T J \times T$ matrix with rows $(j-1)J + 1$ to $jJ$ equal to $[\Omega_{[j-1:j]} \; 0_{J \times J}]$. The vectors $v_t^i$ and $V_{t-1}^{i}$ jointly contain all current and past private signals available to agent $i$ at time $t$ that are informative about future dividend innovations.

Using the definition (1) of average random variables, together with the assumption of uncorrelated errors in private signals, the average of current and past private signals are $v_t = \Omega C \epsilon_t^{d}$ and $V_{t-1} = \Gamma \epsilon_t^{d}$. Note that the last column of $\Gamma$ is zero because private signals only contain information about dividend innovations up to $T$ periods later. Therefore we will also write $V_{t-1} = G \epsilon_t^{d}$, where $G$ has zeros in the first column and the remainder consists of the first $T - 1$ columns of $\Gamma$.

### 4.2 Expectation of Future Dividends

Investors have no private signals on dividends more than $T$ periods from now, so that based on their prior $E_i^t(\epsilon_{t+j}^{d}) = 0$ for $j > T$. Investors form expectations of dividend innovations in the following $T$ periods, $\epsilon_t^{d}$, by using a combination of private and public signals. Public information takes two forms. First, the $N(0, \sigma_2^d)$ distribution of future dividend innovations is public information. One can summarize these public signals with the vector $o^T$ of $T$ zeros. The dividend innovations themselves are the errors in these zero-signals. The other pieces of public information are current and past prices. We show in Appendix D that the equilibrium price can be written as:

$$P_t = \frac{D}{R - 1} + A(L) \epsilon_{t+T}^{d} + B(L) \epsilon_{t}^{r}$$

(16)

with $A(L) = a_1 + a_2 L + a_3 L^2 + ...$ and $B(L) = b_1 + b_2 L + b_3 L^2 + .....$

\(^{16}\)This follows from the assumption that the last column of $\Omega$ is non-zero.
Since investors use their private signals to form expectations, and the average of the private signals depends on future dividend innovations, the price at time $t$ depends on dividend innovations over the next $T$ periods. The price therefore contains information about future dividend innovations. This information is imperfect since the price also depends on asset supply innovations at time $t$ and earlier that cannot be observed. These are the errors in the public price signals. At time $t$ only the supply innovations $\varepsilon_t^x = (\varepsilon_{t-T+1}^x, \ldots, \varepsilon_{t}^x)$ are unknown since supply innovations at $t-T$ and earlier can be extracted from prices at time $t-T$ and earlier.\(^{17}\)

It is useful to again subtract the components of the price signal that depend on known current and past dividend innovations. This leads to the following adjusted time $t$ price signal:

$$P_t^a = a_1 \varepsilon_{t+T}^d + a_2 \varepsilon_{t+T-1}^d + \ldots + a_T \varepsilon_{t}^d + b_1 \varepsilon_{t-1}^x + b_2 \varepsilon_{t-2}^x + \ldots + b_T \varepsilon_{t-T+1}^x$$

$$\equiv a \varepsilon_t^d + b \varepsilon_t^x$$  \hspace{1cm} (17)

All prices between $t-T+1$ and time $t$ contain information about future dividend innovations. Subtracting the known components of the entire set of price signals that depend on current and past dividend innovations, the set of price signals $p_t' = (P_t^a, P_{t-1}^a, \ldots, P_{t-T+1}^a)$ can be written as

$$p_t = A \varepsilon_t^d + B \varepsilon_t^x$$  \hspace{1cm} (18)

where row $j$ of $A$ contains $j-1$ zeros, followed by the first $T-j+1$ elements of the vector $a$, and similarly for $B$. Notice that prices do not fully reveal any linear combination of future dividend innovations since the matrix $B$ is invertible.\(^{18}\)

Other than the zero prior signals, the signals of future dividend innovations can be written as

$$\begin{pmatrix} p_t \\ v_t^i \\ v_{t-1}^i \end{pmatrix} = H \varepsilon_t^d + \begin{pmatrix} B \varepsilon_t^x \\ \varepsilon_t^v \\ \varepsilon_t^{vi} \end{pmatrix}$$  \hspace{1cm} (19)

where $H$ stacks the matrices $A$, $\Omega C$ and $\Gamma$.

\(^{17}\)This assumes that the polynomial $B(L)$ is invertible (the roots of $B(L) = 0$ are outside the unit circle), as is usually the case in applications (e.g. see Bacchetta and van Wincoop (2006)). The dividend innovations entering prices at $t-T$ and earlier are all known at time $t$.

\(^{18}\)This is because the supply innovation at $t-s$ affects the price at $t-s$, but supply innovations after $t-s$ do not affect the price at $t-s$. 
Using that the prior signal of \( \epsilon_t^d \) has mean \( \mathbf{0}^T \) and variance \( \sigma_d^2 \mathbf{I} \), with \( \mathbf{I} \) an identity matrix of size \( T \), and writing the variance of the errors of the signals in (19) as \( \mathbf{R} \), the standard signal extraction formula implies

\[
E_t^i(\epsilon_t^d) = (\mathbf{I} - \mathbf{M} \mathbf{H}) \mathbf{0}^T + \mathbf{M} \begin{pmatrix} \mathbf{p}_t \\ \mathbf{v}_t \\ \mathbf{V}_{t-1} \end{pmatrix}
\]  

(20)

where \( \mathbf{M} = \sigma_d^2 \mathbf{H}' [\sigma_d^2 \mathbf{H} \mathbf{H}' + \mathbf{R}]^{-1} \). This can also be written as

\[
E_t^i(\epsilon_t^d) = \mathbf{M}_1 \mathbf{Z}_t + \mathbf{M}_2 \mathbf{v}_t + \mathbf{M}_3 \mathbf{V}_{t-1}
\]  

(21)

where \( \mathbf{Z}_t = (\mathbf{0}^T\mathbf{p}_t) \) contains all the public signals about future dividend innovations.

### 4.3 The Wedge as a First Order Expectational Error

Using (21), Appendix B shows that:

\[
E_{t+1}^i \mathbf{P} \mathbf{V}_{t+1} = \sum_{s=1}^{\infty} \frac{\mathbf{D}_{t+s+1}}{R^s} + \mathbf{\theta}' \mathbf{V}_{t} + \mathbf{\beta}' \mathbf{v}_{t+1} + \mathbf{\gamma}' \mathbf{Z}_{t+1}
\]  

(22)

\[
E_{t+1}^i \mathbf{\nabla}_{t+1} = \mathbf{\Psi}' \mathbf{V}_{t} + \mathbf{\mu}' \mathbf{v}_{t+1} + \mathbf{\lambda}' \mathbf{Z}_{t+1}
\]  

(23)

where \( \mathbf{D}_{t+s+1} \) is the known component of dividend \( D_{t+s+1} \) at \( t + 1 \), e.g. \( \mathbf{D}_{t+2} = \mathbf{D} + c_2 \mathbf{\varepsilon}_{t+1} + c_3 \mathbf{\varepsilon}_t + c_4 \mathbf{\varepsilon}_{t-1} + \ldots \). Using this, in Appendix C we then derive the following Proposition about the higher order wedge.

**Proposition 1** The deviation between higher and first order expectations that affects the equilibrium asset price is

\[
\Delta_t = \mathbf{\Pi}'(E_t \mathbf{\nabla}_t - \mathbf{\nabla}_t)
\]  

(24)

where \( \mathbf{\Pi} = \frac{1}{R}(I - \mathbf{\Psi}\frac{1}{R})^{-1}\mathbf{\theta} \).

The proposition tells us that the higher order wedge depends on the average expectational error at time \( t \) about the vector of average private signals that remain informative about future dividends at \( t + 1 \). The proposition therefore reduces differences between higher and first order expectations to a simple first order expectational error.
In order to provide some intuition behind Proposition 1, it is useful to write

$$\Pi = \frac{1}{R} \theta + \sum_{s=2}^{\infty} \frac{1}{R^s} \Psi^s \theta$$

Consider the first element of $\Pi$, $\theta/R$. It corresponds to the first element of (10), the average expectation at $t$ of the market expectational error at $t+1$ about $PV_{t+1}$.

An investor’s expectation of this error can be written as $E_i^t(E_{t+1}PV_{t+1} - PV_{t+1}) = E_i^t(E_{t+1}PV_{t+1} - E_i^{t+1}PV_{t+1})$. From (22) it follows that:

$$E_{t+1}PV_{t+1} - E_i^{t+1}PV_{t+1} = \theta'(\nabla_t - V_i^t) + \beta'(\nabla_{t+1} - V_{t+1}^i)$$

(25)

An investor expects the market to make expectational errors to the extent that the market is expected to have a different set of private signals. The second term in (25) is expected to be zero. Thus, an investor only expects the market to make expectational errors tomorrow if the average private signals $\nabla_t$ are expected to be different from the investor’s own private signals. Taking the expectation of (25) for investor $i$ at time $t$ yields $\theta'(E_t\nabla_t - V_i^t)$. The average of this across investors is $\theta'(E_t\nabla_t - \nabla_t)$, which corresponds to the first element of $\Pi$.

The second element in $\Pi$ corresponds to the sum of HOE of future expectational errors. Consider for example the second-order expectation of the market’s expectational error at $t + 2$ about $PV_{t+2}$. Corresponding to the discussion above, the average expectation at $t+1$ of the market’s expectational error at $t+2$ about $PV_{t+2}$ is $\theta'(E_{t+1}\nabla_{t+1} - \nabla_{t+2})$. This depends itself on an average expectational error, this time not about future dividends but about average private signals. Using a similar argument as above, but using (23), the average expectation at time $t$ of the market’s expectational error at $t+1$ about average private signals is equal to $\Psi'(E_t\nabla_t - \nabla_t)$. Following an iterative argument one can similarly derive third and HOE of future expectational errors. The critical point is that these all depend on average expectational errors at time $t$ about average private signals.

One can think of first and higher order expectations of future expectational errors as resulting from a chain effect. This explains why current expectational mistakes $E_t\nabla_t - \nabla_t$ affect expectations of all orders. As an illustration consider the case where $V_i^t$ has only one element, so that investors have only one private signal at time $t$ that is still relevant at $t+1$. Assume that a higher private signal $V_i^t$ at time $t$ makes the investor both more optimistic at $t+1$ about future payoffs ($\theta > 0$) and more optimistic at $t+1$ about average private signals ($\Psi > 0$).
Now consider what happens when the average investor at time $t$ expects others to have more favorable, and therefore too optimistic, private signals, i.e., $E_t \nabla_t > \nabla_t$. The average investor then expects that (1) the market is too optimistic at $t+1$ about future dividends and (2) the market is too optimistic at $t+1$ about average private signals. The first leads to first order expectations of positive expectational errors at $t+1$ about $PV_{t+1}$. The second implies a first order expectation of positive expectational errors at $t+1$ about private signals, i.e., a first order expectation at $t$ that $E_{t+1} \nabla_{t+1} > \nabla_{t+1}$. This leads to the next step in the chain. Following the same argument as above, it leads to second order expectations that the market is too optimistic at $t+2$ about future dividends and average private signals. The latter leads to a third step in the chain, and so on.

Proposition 1 has several implications.

Corollary 1 The higher order wedge is proportional to expectational errors of future dividends $E_t D_t^T - D_t^T$.

Corollary 1 follows almost immediately from Proposition 1. We have

$$E_t (\nabla_t - \nabla_t) = G \left( E_t \epsilon_t^d - \epsilon_t^d \right) = G C^{-1} \left( E_t D_t^T - D_t^T \right)$$  \hspace{1cm} (26)

The next Corollary follows from the first one.

Corollary 2 The higher order wedge depends on expectational errors of future dividend innovations based on public signals.

There are two public signals in the model that provide information about future dividend innovations: the price signals $p_t$ and the zero-signals $o_T$. The errors in the price signals are summarized by $B \epsilon_t^x$ in (19) and therefore depend on the supply innovations over the past $T$ periods. The errors in the signals $o_T$ are $\epsilon_t^o = o_T - \epsilon_t^d$. Substituting (19) into (20), and taking the average across investors, we have

$$E_t \epsilon_t^d - \epsilon_t^d = (I - MH) \epsilon_t^o + M_P B \epsilon_t^x$$  \hspace{1cm} (27)

where $M_P$ contains the first $T$ columns of $M$. Together with (26) and Proposition 1 it follows that the higher order wedge depends on the errors $\epsilon_t^o$ and $B \epsilon_t^x$ in public signals.
4.4 On the Existence and Magnitude of the Higher Order Wedge

Proposition 1 has two more implications that relate to the existence and magnitude of the wedge.

**Corollary 3** Within the context of the assumed information structure, a necessary and sufficient condition for the higher-order wedge to be non-zero is that $T > 1$.

First consider the necessary part of Corollary 3. When $T = 1$, the set $\nabla_t$ is empty and therefore the higher order wedge is zero. In that case there are no private signals at time $t$ that still contain information about future dividend innovations at $t + 1$. Intuitively, since an investor has no reason to expect that his private signals in future periods differ from the average, there is no reason to expect the market to make expectational errors in the future when predicting future dividends.

Next consider the sufficient part of Corollary 3. It is immediate from Proposition 1 that the higher order wedge is non-zero if both $\theta$ is non-zero and all elements of $E_t \nabla_t - \nabla_t$ are non-zero. First consider $\theta$. We know that $PV_{t+1}$ depends on all future dividend innovations, starting with $\varepsilon^d_{t+2}$. The set $V^i_t$ contains private signals received at time $t$ and earlier that remain informative about future dividend innovations at time $t + 1$. This set is non-zero when $T > 1$. Since the errors in the signals $V^i_t$ are uncorrelated with those of all other signals, and no combination of the other signals can fully reveal $\varepsilon^d_{t+1}$, the expectation of $PV_{t+1}$ will depend on $V^i_t$. This implies that $\theta$ is non-zero.

Next consider $E_t \nabla_t - \nabla_t$. By assumption each element of $\nabla_t$ depends on at least one dividend innovation between $t + 2$ and $t + T$. Since the errors in all signals have a non-zero variance, and there is no linear relation between the errors of the signals, no combination of them fully reveals any of the future dividend

---

19 $\Pi$ is non-zero if and only if $\theta$ is non-zero.

20 If some combination of other signals fully reveals $\epsilon^d_{t+1}$, then $\text{var}_{t+1}(\epsilon^d_{t+1}) = \sigma^2_d(I - MH) = 0$, so that $I = MH$. Multiplying by $H$ and adding $R [\sigma^2_HH' + R]^{-1}H$ to both sides, using the definition of $M$, gives $R [\sigma^2_HH' + R]^{-1}H = 0$. Since $R$ is an invertible matrix (no linear combination of the errors in the signals is zero), it would follow that $H = 0$, which is clearly violated.
innovations. This implies that each of the elements of $E_t \nabla_t - \nabla_t$ are non-zero.

A final implication of Proposition 1 relates to the size of the wedge.

**Corollary 4** The variance of $\Delta_t$ is largest for intermediate levels of the quality of private information as measured by $1/\sigma_v^2$. It vanishes when the variance $\sigma_v^2$ of the errors in private signals approach zero or infinity.

The proof of this Corollary 4 is almost trivial. On the one hand, when $\sigma_v^2 \to 0$, the errors in private signals vanish, so that there are no expectational errors about future dividend innovations and the wedge disappears. On the other hand, when $\sigma_v^2 \to \infty$, private signals contain no information about future dividend innovations, so that $\theta \to 0$ and again the wedge vanishes.

## 5 Some Implications for Equilibrium Asset Prices

So far we have discussed the determinants of the higher order wedge. We now turn to the impact of the wedge on the equilibrium price. A first, and perhaps surprising, result is that the higher order wedge reduces the volatility of asset prices. The asset price without HOE was defined as $P_t^* \equiv P_t - \Delta_t$, so that $P_t^* = E_t PV_t - \phi_t$. Then Proposition 1 implies:

**Corollary 5** Assume that dividends and the asset supply are stationary, such that the asset price $P_t$ is stationary. Then $\text{var}(P_t) < \text{var}(P_t^*)$.

We know from Proposition 1 that the wedge depends on average expectational errors about $\nabla_t$. Since the asset price $P_t$ is in the information set of investors, expectational errors at time $t$ should be orthogonal to the price, so that $\text{cov}(P_t, \Delta_t) = 0$. Therefore $\text{var}(P_t^*) = \text{var}(P_t - \Delta_t) = \text{var}(P_t) + \text{var}(\Delta_t)$, which implies Corollary 5.

In the early literature on excess volatility of asset prices the emphasis was on expected dividends as a determinant of the stock price. However, time variation in expected return was later recognized as an important second asset pricing determinant\(^{21}\) (these two asset pricing determinants are reflected in $P_t^*$). To this we have now added a third asset pricing determinant, the higher order wedge.

\(^{21}\)Many of the early contributions to this literature, such as Leroy and Porter (1981) and Shiller (1981), showed that $P_t$ is more volatile than $PV_t$ and argued that this is inconsistent with the theory when the discount rate is constant. Since then a consensus has developed that the higher
While the second asset pricing determinant can contribute to an increase in asset price volatility, the third one (the higher order wedge) reduces the volatility of $P_t$.

The finding that the higher order wedge reduces asset price volatility can be explained by a reduced dependence of the asset price on future dividends. This reduced dependence will also typically disconnect the price from future dividends. To see these points we can use (26) and (27) to write:

$$
\Delta_t = N \left( -(I - MH) \epsilon_t^d + M P B \epsilon_t^x \right) \equiv W_D \epsilon_t^d + W_X \epsilon_t^x
$$

The wedge therefore changes the weight in the equilibrium price of future dividend innovations and asset supply innovations. While at this level of generality a formal proof is not available, the higher order wedge is likely to lower the weight of future dividend innovations and increase the weight of unobserved supply shocks. This is because we might expect $P_t^*$ to depend positively on $\epsilon_t^d$ and negatively on $\epsilon_t^x$ and $\Delta_t$ to depend negatively on both types of innovation.

The reasoning is as follows. First, $P_t^*$ is likely to depend positively on $\epsilon_t^d$ because expectations of future dividends should depend positively on private signals, which on average depend positively on future dividend innovations; and $P_t^*$ is likely to depend negatively on $\epsilon_t^x$ because positive supply innovations raise expected risk-premia. Second, to the extent that higher private signals are good news for future dividends, one may expect both $\theta$ and $\Psi$ to have positive elements, leading to positive elements of $\Pi$ as well. The higher order wedge is therefore expected to depend positively on expectational errors about future dividend innovations based on public information. Expectational errors associated with the zero-signals $o^T$ depend negatively on $\epsilon_t^d$, while expectational errors associated with $p_j$ depend negatively on $\epsilon_t^x$ when, plausibly, the elements of $B(L)$ are negative. Thus, $\Delta_t$ depends negatively on $\epsilon_t^d$ and $\epsilon_t^x$.

This educated guess therefore suggests that the wedge lowers the weight of future dividend innovations and increases the weight of unobserved supply shocks. Both effects weaken the relationship between the price and the present value $PV_t$ of future dividends. The higher order wedge can therefore contribute to the weak link

volatility of $P_t$ is due to a time-varying discount rate. There is extensive empirical evidence of time variation of expected returns, which can contribute significantly to asset price volatility. For a nice discussion of the subsequent literature see Campbell, Lo and MacKinlay (1997), p. 275-279.
between $P_t$ and $PV_t$ as first documented by Shiller (1981). The reduced weight of future dividend innovations reduces asset price volatility, while the increased weight of unobserved supply shocks increases asset price volatility. From Corollary 5 it is clear that the former is the strongest of the two effects.

In order to illustrate the general analysis and derive precise results about the impact of the higher order wedge on the equilibrium price, we now turn to a specific example that is contained within the general information structure described so far.

6 An Illustration

In this section we examine a simple case of the model presented in the previous section and solve it numerically. The results illustrate that the higher order wedge disconnects the asset price from the present value of future dividends and that this impact can be large.

Dividends are i.i.d., so that:

$$D_t = \overline{D} + \varepsilon^d_t$$

where $\varepsilon^d_t \sim N(0, \sigma^2_d)$. Moreover, each period investors obtain a single private signal about the dividend $T$ periods later. The adjusted signal is:

$$v^i_t = \varepsilon^d_{t+T} + \varepsilon^{vi}_{t}$$

with $\varepsilon^{vi}_{t} \sim N(0, \sigma^2_v)$. Asset supplies $X_t$ are assumed to be i.i.d. $N(0, \sigma^2_x)$ variables.

As $T$ increases investors get information further in advance but also have a larger number of relevant private signals each period. Since $V^i_t$ denotes the private information set at time $t$ that is still valuable at time $t + 1$, we have $V^i_t = \{v^i_{t-T+2}, ..., v^i_t\}$ and the average across agents is $\mathbf{V}_t = \{\varepsilon^d_{t+2}, ..., \varepsilon^d_{t+T}\}$ for $T \geq 2$.22 We will write $\mathbf{\Pi}' = \{\pi_1, .., \pi_{T-1}\}$. Proposition 1 then implies

$$\Delta_t = \sum_{s=2}^{T} \pi_{s-1}(E_t\varepsilon^d_{t+s} - \varepsilon^d_{t+s}) = \sum_{s=2}^{T} \pi_{s-1}(E_tD_{t+s} - D_{t+s})$$

22When $T = 1$ private signals today are no longer in the information set tomorrow since tomorrow’s dividend is observed tomorrow. In that case higher order expectations collapse to first order expectations.
Numerical results show that all the elements of $\Pi$ are positive. Therefore, the higher order wedge depends positively on expectational errors about future dividends.

The equilibrium price is given by:

$$P_t = \frac{D}{R - 1} + \sum_{s=1}^{T} a_s\varepsilon_{t+s}^d - \sum_{s=1}^{T} b_s\varepsilon_{t-s+1}^x \quad a_s > 0, \ b_s > 0. \quad (32)$$

Notice that even though supply shocks are not persistent, they have a persistent effect on the asset price since past equilibrium prices (which depend on past supply shocks) are informative about future dividends. The price at time $t$ depends on the last $T$ supply shocks since asset prices over the past $T$ periods contain information about future dividends.\(^{23}\)

Consistent with the educated guess in the previous section, numerical analysis shows that the higher order wedge amplifies the contribution of supply shocks to price volatility, while reducing the contribution of future dividends to price volatility. However, it is not the case that the higher order wedge reduces the weight on all future dividends and amplifies coefficients on all current and past supply shocks.

In order to obtain some insight regarding the role of the higher order wedge in the equilibrium price, consider the case where $T = 2$. In that case

$$\Delta_t = \pi(E_t^{d} \varepsilon_{t+2}^d - \varepsilon_{t+2}^d) \quad (33)$$

We know from Corollary 2 that the expectation error of the time $t + 2$ dividend innovation depends on the errors in the public signals. In this case the adjusted price signals are

$$P_{t}^a = a_1\varepsilon_{t+2}^d + a_2\varepsilon_{t+1}^d + b_1\varepsilon_{t}^x + b_2\varepsilon_{t-1}^x \quad (34)$$

$$P_{t-1}^a = a_1\varepsilon_{t+1}^d + b_1\varepsilon_{t-1}^x \quad (35)$$

These signals provide joint information about $\varepsilon_{t+1}^d$ and $\varepsilon_{t+2}^d$ with errors consisting of $\varepsilon_{t-1}^x$ and $\varepsilon_{t}^x$. The other public signals in the model are the zero-signals, with errors $-\varepsilon_{t+1}^d$ and $-\varepsilon_{t+2}^d$. Thus, from Corollary 2

$$\Delta_t = \delta_1\varepsilon_{t}^x + \delta_2\varepsilon_{t-1}^x + \delta_3\varepsilon_{t+2}^d + \delta_4\varepsilon_{t+1}^d \quad (36)$$

\(^{23}\)Supply shocks at $t - T$ and earlier do not affect the price at time $t$ since they are common knowledge at time $t$ (can be extracted from prices at $t - T$ and earlier) and therefore do not impact the adjusted price signals $p_t$.\(^{22}\)
The precise values of the coefficients, based on implementing (27) for the case where \( T = 2 \), are listed in the Appendix as a function of the parameters of the equilibrium price.

The other component of the price is \( P_t^* \), which from (7) can be written as

\[
P_t^* = \sum_{s=1}^{\infty} \frac{1}{R^s} E_t^s D_{t+s} - \gamma \sigma^2 \frac{1}{R} \varepsilon_t^p
\]  

(37)

Not surprisingly, numerical analysis shows that \( P_t^* \) always depends positively on \( \varepsilon_{t+1}^d \) and \( \varepsilon_{t+2}^d \): higher private signals at \( t \) and \( t - 1 \), which themselves depend positively on \( \varepsilon_{t+1}^d \) and \( \varepsilon_{t+2}^d \), raise the expectation of future dividends. Moreover, numerical analysis shows that \( P_t^* \) depends negatively on \( \varepsilon_{t-1}^d \) and \( \varepsilon_t^d \): higher supply at time \( t \) lowers the price by raising the risk premium, while higher supply at \( t - 1 \) lowers the price at \( t - 1 \), which in turn lowers \( P_t^* \) by lowering the expectation of \( \varepsilon_{t+1}^d \) (see (35)).

The Appendix shows that the coefficients \( \delta_1 \) and \( \delta_3 \) in (36) are both negative. The negative sign of \( \delta_3 \) is a result of the zero public signal, since based on the zero-signal alone, the expectational error about \( \varepsilon_{t+2}^d \) is equal to \( -\varepsilon_{t+2}^d \). The negative sign of \( \delta_1 \) is a result of the error in the time \( t \) price signal. As can be expected from (34), an increase in \( P_t^a \) raises the expectation of \( \varepsilon_{t+2}^d \). As \( P_t^a \) depends negatively on \( \varepsilon_t^d \), this shock negatively affects the expectational error of \( \varepsilon_{t+2}^d \) and therefore the higher order wedge. Since \( P_t = P_t^* + \Delta_t \), the negative signs of \( \delta_1 \) and \( \delta_3 \) imply that the higher order wedge leads to a less positive coefficient on \( \varepsilon_{t+2}^d \) in the equilibrium price and a more negative coefficient on \( \varepsilon_t^d \). This amplifying impact of unobserved supply shocks on the price, and the dampening effect of future dividend innovations, are consistent with the intuition developed in the previous section.

However, it is not the case that the impact of each past dividend innovation is amplified and each past asset supply is dampened: the weight of \( \varepsilon_{t-1}^d \) is dampened by the higher order wedge (\( \delta_2 > 0 \)) and that of \( \varepsilon_{t+1}^d \) is amplified (\( \delta_1 > 0 \)). While numerically these effects are far outweighed by the amplification of \( \varepsilon_t^d \) and the dampening of \( \varepsilon_{t+2}^d \), it is nonetheless useful to provide some intuition. We only discuss the dampened impact of \( \varepsilon_{t-1}^d \).\(^{24}\) A related argument can also explain the amplified impact of \( \varepsilon_{t+1}^d \).

\(^{24}\)As shown in the Appendix, \( \delta_2 \) depends positively on \( a_1 b_2 - a_2 b_1 \), which numerically is always positive. The higher order wedge therefore depends positively on \( \varepsilon_{t-1}^d \), therefore dampening its impact on the price.
This can best be understood by considering the price signals (34) and (35). An increase in $P_{t-1}^a$ raises the expectation of $d_{t+1}$, which for a given $P_t^a$ lowers the expectation of $d_{t+2}$. As a result, the expectation of $d_{t+2}$ depends negatively on $P_{t-1}^a$. And since $P_{t-1}^a$ itself depends negatively on $d_{t-1}$, the time $t - 1$ supply shock has a positive impact on the expectational error of $d_{t+1}$ and therefore on the higher order wedge ($\delta_2 > 0$). This result also illustrates that the finding by Allen, Morris and Shin (2006) that HOE give more weight to public signals does not necessarily generalize to all public signals. In this case the higher order wedge depends negatively on the $P_{t-1}^a$ public signal, while the same signal raises $P_t^a$ as a higher $t - 1$ price raises the expectation of the time $t + 1$ dividend.

In order to provide a numerical illustration, Figure 1 shows some results for the parameterization $\sigma_v = \sigma_d = \sigma_x = 0.4$ and $R = 1.02, \gamma = 2$. In panels A and B the parameter $T$ is varied from 2 to 50. Panel A shows the drop in the correlation between the price and the present value of future dividends as a result of the higher order wedge. The difference rises for larger values of $T$. When $T$ is small, the information about most future dividend innovations is public in the form of the zero-signals, so that HOE have little impact. The example shows that for $T = 50$ the impact of HOE is substantial, reducing the correlation between the present discounted value of dividends and the price from 0.82 to 0.29.

Consistent with Panel A, Panel B shows that the variance of the higher order wedge rises relative to the variance of the price when $T$ increases. The same is the case for $P^a$. When $T = 50$ the variance of the higher order wedge is larger than the variance of $P$, while the variance of $P^a$ is more than twice the variance of $P$. The remaining factor contributing to the variance of the price is a large negative covariance between the higher order wedge and $P^a$. The higher order wedge therefore reduces asset price volatility, as reflected in values of $\text{var}(P^a)/\text{var}(P)$ above 1.

Panel C presents an implication of Corollary 4 for the case where $T = 30$. It shows that impact of the higher order wedge on the correlation between the price and the present value of dividends is maximized for an intermediate level of the quality of private information. The reduction in the correlation is largest (0.51) for $\sigma_v = 0.7$. The impact on the correlation vanishes to zero when either $\sigma_v \to \infty$ or $\sigma_v \to 0$. 

24
7 Conclusion

This paper has analyzed the role of HOE for asset pricing. We have shown that HOE generally differ from first order expectations, and that this difference can have strong implications for the equilibrium asset price. The paper has devoted significant attention towards understanding what determines this new asset pricing determinant, which we called the “higher order wedge,” and how it affects the equilibrium price. A key result is that it depends on expectational errors that weaken the relationship between the price and future dividends.

While our analysis assumes full rationality of investors, the recent literature in behavioral finance implies that expectational errors could be caused by deviations from rationality, such as overconfidence or changing market mood. We conjecture that the insights from our general analysis also apply when expectational errors are caused by factors different from noisy public signals. In particular, the impact of these errors would be amplified by HOE. Combining the dimension of market psychology with our analysis of HOE would bring us close to Keynes’ reasoning on asset prices and closer to understanding asset price movements.

Another natural direction for future research is to quantify the importance of the higher order wedge as an asset pricing determinant. While we have shown that it can be quantitatively very large, its magnitude needs to be evaluated in the context of a somewhat more realistic setup that is calibrated to actual data. In particular, we have maintained the standard assumption in noisy rational expectations models of constant absolute risk aversion. While this simplifies the solution significantly, a more realistic constant rate of relative risk aversion needs to be adopted when confronting the model to the data. More realistic assumptions about the process of dividends and the information structure would need to be considered as well.
Appendix

A Deriving Equation (10)

We will show that (8) can be rewritten as (10). First, applying the same reasoning as in Section 3, for each \( s \) rewrite the difference between the \( s \)-order and first order expectation as

\[
\mathcal{E}_s^s D_{t+s} - \mathcal{E}_t D_{t+s} = (\mathcal{E}_t^2 D_{t+s} - \mathcal{E}_t D_{t+s}) + (\mathcal{E}_t^3 D_{t+s} - \mathcal{E}_t^2 D_{t+s}) + \ldots + (\mathcal{E}_t^s D_{t+s} - \mathcal{E}_t^{s-1} D_{t+s})
\]

Thus we can write

\[
\sum_{s=1}^{\infty} \frac{1}{R^s} \left[ \mathcal{E}_t^s D_{t+s} - \mathcal{E}_t D_{t+s} \right]
\]
as

\[
\frac{1}{R^2} (\mathcal{E}_t^2 D_{t+2} - \mathcal{E}_t D_{t+2}) + \frac{1}{R^3} \left[ (\mathcal{E}_t^2 D_{t+3} - \mathcal{E}_t D_{t+3}) + (\mathcal{E}_t^3 D_{t+3} - \mathcal{E}_t^2 D_{t+3}) \right] + \ldots + \frac{1}{R^s} \left[ (\mathcal{E}_t^2 D_{t+s} - \mathcal{E}_t D_{t+s}) + (\mathcal{E}_t^3 D_{t+s} - \mathcal{E}_t^2 D_{t+s}) + \ldots + (\mathcal{E}_t^s D_{t+s} - \mathcal{E}_t^{s-1} D_{t+s}) \right] + \ldots
\]

We can regroup terms (e.g. the first line corresponds to the sum of the first elements of each line above):

\[
\frac{1}{R^2} \mathcal{E}_t (\mathcal{E}_{t+1} D_{t+2} - D_{t+2}) + \frac{1}{R^3} \mathcal{E}_t (\mathcal{E}_{t+1} D_{t+3} - D_{t+3}) + \ldots + \frac{1}{R^s} \mathcal{E}_t (\mathcal{E}_{t+1} D_{t+s} - D_{t+s}) + \ldots
\]

\[
\frac{1}{R^3} \mathcal{E}_t^2 (\mathcal{E}_{t+2} D_{t+3} - D_{t+3}) + \frac{1}{R^4} \mathcal{E}_t^2 (\mathcal{E}_{t+2} D_{t+4} - D_{t+4}) + \ldots
\]

\[
\ldots + \frac{1}{R^{s+1}} \mathcal{E}_t^{s-1} (\mathcal{E}_{t+s} D_{t+s} - D_{t+s}) + \frac{1}{R^{s+1}} \mathcal{E}_t^{s-1} (\mathcal{E}_{t+s} D_{t+s+1} - D_{t+s+1}) + \ldots
\]

Using the definition \( PV_t = \sum_{s=1}^{\infty} \frac{1}{R^s} D_{t+s} \), this can be written as:

\[
\frac{1}{R} \mathcal{E}_t (\mathcal{E}_{t+1} PV_{t+1} - PV_{t+1}) + \frac{1}{R^2} \mathcal{E}_t^2 (\mathcal{E}_{t+2} PV_{t+2} - PV_{t+2}) +
\]

26
\[
\frac{1}{R^{s-1}}E_i^t (E_{t+s-1}PV_{t+s-1} - PV_{t+s-1}) + ... \\
\]

which gives (10).

**B Deriving Equations (22) and (23)**

From the definition of \( PV_{t+1} \) it follows that

\[
E_t^i PV_{t+1} = \sum_{s=1}^{\infty} \frac{\hat{D}_{t+s+1}}{R^s} + d^t E_t^i e^d_{t+1} \quad (39)
\]

where \( \hat{D}_{t+s+1} = c_{s+1}e^d_{t+1} + c_{s+2}e^d_{t} + ... \) contains the component of \( D_{t+s+1} \) known at time \( t + 1 \), and \( d \) is a vector of length \( T \) with element \( i \) equal to

\[
\frac{1}{R^{t-1}} \sum_{s=1}^{\infty} c_s 
\]

This sum is well-defined because \( c_s \) is assumed to approach a finite number as \( s \to \infty \). Substituting (21) at \( t + 1 \) into (39) gives (22), where \( \theta' = d'M_3, \beta' = d'M_2, \gamma' = d'M_1 \). Using that \( E_t^i \nabla_{t+1} = E_t^i G e^d_{t+1} \) yields (23) when using (21) at \( t + 1 \), with \( \Psi' = GM_3, \mu' = GM_2, \lambda' = GM_1 \).

**C Proof of Proposition 1**

Given (14), the investor’s expectation of (25) is:

\[
E_i^t (E_{t+1}PV_{t+1} - E_{t+1}PV_{t+1}) = \theta'(E_t^i \nabla_t - \nabla_t^i) \quad (40)
\]

Taking the integral on both sides with respect to the measure \( \mu(i) \), we have

\[
E_t (E_{t+1}PV_{t+1} - PV_{t+1}) = \theta'(E_t^i \nabla_t - \nabla_t^i) \quad (41)
\]

The other terms in (10) involve HOE of future expectational errors. Consider the deviation at \( t + s \):

\[
E_t^s (E_t^s PV_{t+s} - PV_{t+s}) \quad \text{it can be rewritten as} \quad E_t^{s-1}E_{t+s-1}(E_{t+s}PV_{t+s} - PV_{t+s}). \quad \text{Using (41) at time} \ t + s - 1 \text{ we can write:}
\]

\[
E_t^s (E_t^s PV_{t+s} - PV_{t+s}) = \theta'E_t^{s-1} (E_{t+s-1}^t \nabla_{t+s-1} - \nabla_{t+s-1}) \quad (42)
\]
This implies that investors at time $t$ need to compute HOE of information available to investors at time $t + s - 1$.

Using (23) and following the same reasoning as to get (41), it then follows that

$$E_t(E_{t+1} \nabla_{t+1} - \nabla_t) = \Psi'(E_t \nabla_t - \nabla_t)$$

Similarly $E_{t+s-2}(E_{t+s-1} \nabla_{t+s-1} - \nabla_{t+s-1}) = \Psi'(E_{t+s-2} \nabla_{t+s-2} - \nabla_{t+s-2})$. This can be substituted into (42) and we can work backwards using (23) to get

$$E_t(E_{t+s} PV_{t+s} - PV_{t+s}) = \theta'(\Psi')^{s-1}(E_t \nabla_t - \nabla_t) \quad (43)$$

Substituting this into (10), the summation gives the expression for $\Delta_t$ in Proposition 1.

## D Equilibrium Price

First conjecture that the price takes the form (16). It is useful to express $P_{t+1}$ in function of observable and unobservable variables:

$$P_{t+1} = \frac{\bar{D}}{R - 1} + a_1 \varepsilon_{t+T+1}^d + b_1 \varepsilon_{t+1}^x + a^* \epsilon_t^d + b^* \epsilon_t^x + A^*(L) \varepsilon_t^d + B^*(L) \varepsilon_{t-T}^x \quad (44)$$

where $a^* = (a_{T+1}, a_T, ..., a_2)$, $b^* = (b_{T+1}, ..., b_2)$ and $A^*(L) = a_{T+2} + a_{T+3}L + ...$ (with a similar definition for $B^*(L)$). From (18), we can derive $E_t(\epsilon_t^d)$ in function of $p_t$ and $E_t(\epsilon_t^d)$. Thus, we have:

$$E_t^i(P_{t+1}) = \frac{\bar{D}}{R - 1} + \psi E_t^i(\epsilon_t^d) + \psi^P p_t + A^*(L) \varepsilon_t^d + B^*(L) \varepsilon_{t-T}^x \quad (45)$$

This can be substituted into (4) to give:

$$P_t = \frac{\bar{D}}{R - 1} + \bar{\psi} E_t(\epsilon_t^d) + \bar{A}(L) \varepsilon_{t+T}^d + \bar{B}(L) \varepsilon_t^x \quad (46)$$

Averaging (21) over investors and substituting into (46) gives an expression for $P_t$ that has the same form as (16). We then need to solve a fixed point problem.

For the variance we have:

$$\sigma_t^2 = \text{var}_i(P_{t+1} + D_{t+1}) = (c_1 + a_1)^2 \sigma_d^2 + b_1^2 \sigma_x^2 + \psi' \text{var}_i(\epsilon_t^d) \psi \quad (47)$$

The standard signal extraction formulas give:

$$\text{var}_i(\epsilon_t^d) = \sigma_d^2 (I - MH) \quad (48)$$

which can be substituted into (47).
E  The Higher Order Wedge for $T = 2$

Without going over the extensive algebra, implementation of (27) for the case where $T = 2$ delivers (36) where

$$\delta_1 = \pi \frac{\sigma_d^2 \sigma_v^2 g_1}{G} \left( \sigma_v^2 \sigma_d^2 + (\sigma_v^2 + \sigma_d^2) g_1^2 \sigma_x^2 \right)$$  \hspace{1cm} (49)$$

$$\delta_2 = \pi \frac{\sigma_d^4 \sigma_v^4 g_2}{G}$$  \hspace{1cm} (50)$$

$$\delta_3 = -\pi \frac{1}{G} \left( g_1^4 \sigma_v^2 \sigma_x^4 (\sigma_v^2 + \sigma_d^2) + \sigma_d^2 \sigma_v^4 \sigma_x^4 (g_1^2 + g_2^2) \right)$$  \hspace{1cm} (51)$$

$$\delta_4 = -\pi \frac{\sigma_d^2 \sigma_v^4 g_1 g_2 \sigma_x^2}{G}$$  \hspace{1cm} (52)$$

and where

$$G = \sigma_d^4 \sigma_v^4 + (2 g_1^2 + g_2^2)(\sigma_v^2 + \sigma_d^2) \sigma_d^2 \sigma_v^2 \sigma_x^2 + (\sigma_d^2 + \sigma_v^2)^2 g_1^4 \sigma_x^4 > 0$$  \hspace{1cm} (53)$$

$$g_1 = \frac{b_1}{a_1} < 0$$  \hspace{1cm} (54)$$

$$g_2 = \frac{a_1 b_2 - a_2 b_1}{a_1^2}$$  \hspace{1cm} (55)$$

Numerically $g_2$ is always positive since $b_2$ is substantially smaller in absolute size than $b_1$. Therefore $\delta_1 < 0$, $\delta_2 > 0$, $\delta_3 < 0$ and $\delta_4 > 0$. 
References


Panel A: Correlation between Price and PV

Panel B: Variance Decomposition of the Price

Panel C: Private Signal Noise and Higher Order Wedge

* Benchmark parameters: $T=30$, $\sigma_d=\sigma_v=\sigma_x=0.4$; $R=1.02$; $\gamma=2$. 