Technical Appendix
International Capital Flows

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This Technical Appendix describes all technical details associated with the paper. It is organized in six sections:

1. First-order solution conditional on $k(0)$, which is the zero-order portfolio share invested in domestic assets in each country (so that $k^D(0) = 2k(0) - 1$).
2. Second-order solution conditional on $k^D(1)$
3. First and second order components of Bellman equation
4. Second and third order components of optimal portfolio equations
5. Overall solution method
6. Balance of payments accounting

1 First-order solution conditional on $k(0)$

While the paper describes a general numerical solution, the model is simple enough to allow for an analytical solution of the first-order components of control and state variables conditional on $k(0)$. We first describe the analytical solution and then turn to the numerical solution. The latter has the advantage that it is also applicable to more general structures than the specific model of the paper.

1.1 Analytical solution

The first-order component of the model equations can be computed by log-linearizing around the zero-order (steady state) component of model variables. The latter are $W(0) = 1/\psi$, $R(0) = (1-\psi\theta)/(1-\psi)$, $Q(0) = (1-\psi)/\psi$ and $A(0) = P_F(0) = 1$. The zero-order components of the logs of model variables are simply the logs of these values. Linearizing around these values delivers the following first-order components of model equations (46)-(54) in Appendix A (all equations other than the Bellman equations that will be discussed separately in section 3 below):

$$a_{H,t+1}(1) = \rho a_{H,t}(1) + \epsilon_{H,t+1} \tag{1}$$

$$a_{F,t+1}(1) = \rho a_{F,t}(1) + \epsilon_{F,t+1} \tag{2}$$

$$\begin{aligned} w_{t+1}(1) + p_{t+1}(1) = {} & (1-\psi\theta)\left[k(0) r_{H,t+1}(1) + (1-k(0)) r_{F,t+1}(1)\right] \\ & +\psi\theta a_{H,t+1}(1) + (1-\psi\theta)\left(w_t(1) + p_t(1)\right) \end{aligned} \tag{3}$$

$$\begin{aligned} w^*_{t+1}(1) + p^*_{t+1}(1) = {} & (1-\psi\theta)\left[(1-k(0)) r_{H,t+1}(1) + k(0) r_{F,t+1}(1)\right] \\ & +\psi\theta\left(a_{F,t+1}(1) + p_{F,t+1}(1)\right) + (1-\psi\theta)\left(w^*_t(1) + p^*_t(1)\right) \end{aligned} \tag{4}$$

$$a_{H,t}(1) = \alpha w_t(1) + (1-\alpha) w^*_t(1) + \lambda(\alpha p_t(1) + (1-\alpha) p^*_t(1)) \tag{5}$$

$$\begin{aligned} q_{H,t}(1) = {} & k(0)(w_t(1) + p_t(1)) + (1-k(0))(w^*_t(1) + p^*_t(1)) \\ & +2k^A_t(1) \end{aligned} \tag{6}$$

$$\begin{aligned} q_{F,t}(1) = {} & (1-k(0))(w_t(1) + p_t(1)) + k(0)(w^*_t(1) + p^*_t(1)) \\ & -2k^A_t(1) \end{aligned} \tag{7}$$

$$E_t(r_{H,t+1}(1) - r_{F,t+1}(1)) = 0 \tag{8}$$

The last equation follows from the first-order component of both Home and Foreign portfolio Euler equations. For the asset market clearing conditions (6)-(7) we have used that $k^H_{H,t}(1) = k^A_t(1) + 0.5k^D_t(1)$, $k^F_{H,t}(1) = k^A_t(1) - 0.5k^D_t(1)$, $k^H_{F,t}(1) = 1 - k^H_{H,t}(1)$ and $k^F_{F,t}(1) = 1 - k^F_{H,t}(1)$.

The first order components of consumer price indices and asset returns in equations (57)-(60) of Appendix A are:

$$p_t(1) = (1-\alpha)\, p_{F,t}(1) \tag{9}$$

$$p_t^*(1) = \alpha p_{F,t}(1) \tag{10}$$

$$r_{H,t+1}(1) = \frac{1-\psi}{1-\psi\theta} q_{H,t+1}(1) + \frac{\psi(1-\theta)}{1-\psi\theta} a_{H,t+1}(1) - q_{H,t}(1) \tag{11}$$

$$r_{F,t+1}(1) = \frac{1-\psi}{1-\psi\theta} q_{F,t+1}(1) + \frac{\psi(1-\theta)}{1-\psi\theta}\left(a_{F,t+1}(1) + p_{F,t+1}(1)\right) - q_{F,t}(1) \tag{12}$$

Notice that only the first-order component of the average portfolio share $k_t^A(1)$ enters these equations, not the first-order component of the difference in portfolio shareas, $k_t^D(1)$. In addition, the first-order component of the average portfolio share enters only through the asset market clearing equations.

It is useful to write variables in terms of averages and differences across countries, with the superscript A standing for average and superscript D standing for the difference between countries. We take differences and averages of the sets of equations (1)-(2), (3)-(4), (6)-(7). For (5) substitute $a_{H,t}(1) = a_t^A(1) + 0.5a_t^D(1)$, $a_{F,t}(1) = a_t^A(1) - 0.5a_t^D(1)$, $w_t(1) = w_t^A(1) + 0.5w_t^D(1)$ and $w_t^*(1) = w_t^A(1) - 0.5w_t^D(1)$. Also using (9)-(10), (1)-(8) then becomes

$$a_{t+1}^A(1) = \rho a_t^A(1) + \epsilon_{t+1}^A \tag{13}$$

$$a_{t+1}^D(1) = \rho a_t^D(1) + \epsilon_{t+1}^D \tag{14}$$

$$\begin{aligned} w_{t+1}^A(1) + 0.5p_{F,t+1}(1) &= (1-\psi\theta)\, r_{t+1}^A(1) + \psi\theta(a_{t+1}^A(1) + 0.5p_{F,t+1}(1)) + \\ &\quad \psi\theta\left(w_t^A(1) + 0.5p_{F,t}(1)\right) \end{aligned} \tag{15}$$

$$\begin{aligned} w_{t+1}^D(1) + (1-2\alpha)p_{F,t+1}(1) &= (1-\psi\theta)(2k(0)-1)r_{t+1}^D(1) + \\ &\quad \psi\theta\left(a_{t+1}^D(1) - p_{F,t+1}(1)\right) + (1-\psi\theta)\left(w_t^D(1) + (1-2\alpha)p_{F,t}(1)\right) \end{aligned} \tag{16}$$

$$\begin{aligned} a_t^A(1) + 0.5a_t^D(1) &= w_t^A(1) + 0.5\,(2\alpha-1)\, w_t^D(1) + \\ &\quad \lambda 2\alpha\,(1-\alpha)\, p_{F,t}(1) \end{aligned} \tag{17}$$

$$q_t^A(1) = w_t^A(1) + 0.5p_{F,t}(1) \tag{18}$$

$$q_t^D(1) = (2k(0)-1)w_t^D(1) + (1-2\alpha)(2k(0)-1)p_{F,t}(1) + 4k_t^A(1) \tag{19}$$

$$E_t r_{t+1}^D(1) = 0 \tag{20}$$

Taking the average and difference of the asset returns (11)-(9), we have

$$r^A_{t+1}(1) = \frac{1-\psi}{1-\psi\theta} q^A_{t+1}(1) + \frac{\psi(1-\theta)}{1-\psi\theta}\left[a^A_{t+1}(1) + 0.5 p_{F,t+1}(1)\right] - q^A_t(1) \tag{21}$$

$$r^D_{t+1}(1) = \frac{1-\psi}{1-\psi\theta} q^D_{t+1}(1) + \frac{\psi(1-\theta)}{1-\psi\theta}\left(a^D_{t+1}(1) - p_{F,t+1}(1)\right) - q^D_t(1) \tag{22}$$

Combining (15), (18) and (21) yields

$$w^A_t(1) = a^A_t(1) \tag{23}$$

$$q^A_t(1) = a^A_t(1) + \frac{1}{2} p_{F,t}(1) \tag{24}$$

$$r^A_{t+1}(1) = \left(a^A_{t+1}(1) - a^A_t(1)\right) + \frac{1}{2}\left(p_{F,t+1}(1) - p_{F,t}(1)\right) \tag{25}$$

Using (23), it is immediate from (17) that

$$p_{F,t}(1) = \frac{1}{\lambda 4\alpha(1-\alpha)} a^D_t(1) - \frac{(2\alpha-1)}{\lambda 4\alpha(1-\alpha)} w^D_t(1) \equiv p_a a^D_t(1) + p_w w^D_t(1) \tag{26}$$

For now we make the conjecture that the equity price differential is given by $q^D_t(1) = q_a a^D_t(1) + q_w w^D_t(1)$. This will be verified below, with coefficients q_a and q_w to be determined. The excess return (22) then implies

$$\begin{aligned} r^D_{t+1}(1) &= \frac{1-\psi}{1-\psi\theta}\left[q_a a^D_{t+1}(1) + q_w w^D_{t+1}(1)\right] \\ &\quad + \frac{\psi(1-\theta)}{1-\psi\theta}\left((1-p_a)\, a^D_{t+1}(1) - p^D_w w^D_{t+1}(1)\right) - q_a a^D_t(1) - q_w w^D_t(1) \\ &= m_1 a^D_{t+1}(1) + m_2 a^D_t(1) + m_3 w^D_{t+1}(1) + m_4 w^D_t(1) \end{aligned} \tag{27}$$

where:

$$m_1 = \frac{1-\psi}{1-\psi\theta} q_a + \frac{\psi(1-\theta)}{1-\psi\theta}(1-p_a) \qquad m_2 = -q_a$$

$$m_3 = \frac{1-\psi}{1-\psi\theta} q_w - \frac{\psi(1-\theta)}{1-\psi\theta} p_w \qquad m_4 = -q_w$$

Substituting (27) into (16) we have

$$w^D_{t+1}(1) = \eta_1 a^D_{t+1}(1) + \eta_2 a^D_t(1) + \eta_3 w^D_t(1) \tag{28}$$

where:

$$\begin{aligned}
\eta_1 &= \frac{\left(2\bar{k}-1\right)\left[\left(1-\psi\right)q_a+\psi\left(1-\theta\right)\left(1-p_a\right)\right]+\left(2\alpha-1\right)p_a+\psi\theta\left(1-p_a\right)}{1-\left(2\bar{k}-1\right)\left[\left(1-\psi\right)q_w-\psi\left(1-\theta\right)p_w\right]-\left(2\alpha-1\right)p_w+\psi\theta p_w} \\
\eta_2 &= -\frac{\left(1-\psi\theta\right)\left[\left(2\bar{k}-1\right)q_a+\left(2\alpha-1\right)p_a\right]}{1-\left(2\bar{k}-1\right)\left[\left(1-\psi\right)q_w-\psi\left(1-\theta\right)p_w\right]-\left(2\alpha-1\right)p_w+\psi\theta p_w} \\
\eta_3 &= \frac{\left(1-\psi\theta\right)\left[1-\left(2\bar{k}-1\right)q_w-\left(2\alpha-1\right)p_w\right]}{1-\left(2\bar{k}-1\right)\left[\left(1-\psi\right)q_w-\psi\left(1-\theta\right)p_w\right]-\left(2\alpha-1\right)p_w+\psi\theta p_w}
\end{aligned}$$

Substituting (28) into (27), the zero expected excess return equation (20) leads to two restrictions on the parameters:

$$0 = (\rho m_1 + m_2) + m_3(\rho\eta_1 + \eta_2) \tag{29}$$

$$0 = m_3\eta_3 + m_4 \tag{30}$$

(30) does not depend on q_a and can be used to solve for for q_w:

$$q_w = -\frac{1-\left(2\alpha-1\right)p_w}{1-\left(2\alpha-1\right)p_w+\theta p_w}\left(1-\theta\right)p_w$$

Having solved for q_w, (29) is used to solve for q_a:

$$\begin{aligned}
q_a &= \frac{\left(1-\theta\right)}{1-\left(2\alpha-1\right)p_w+\theta p_w}\frac{\rho\psi}{\left(1-\psi\theta\right)-\rho\left(1-\psi\right)}\left[\left(1-p_a\right)-\left(2\alpha-1\right)p_w\right] \\
&\quad+\frac{\left(1-\theta\right)}{1-\left(2\alpha-1\right)p_w+\theta p_w}p_a\left(2\alpha-1\right)p_w
\end{aligned}$$

q_a and q_w are then used to solve for m_1, m_2, m_3, m_4 and η_1, η_2 and η_3.

The zero-order component of the portfolio share, $k(0)$, does not affect the parameters p_a, p_w, q_a, q_w or m_1, m_2, m_3, m_4. It therefore does not affect the solution of relative prices of goods and assets. $k(0)$ does impact the first-order solution of the model in two ways though. First, it affects the solution of the average portfolio share $k_t^A(1)$, which follows from (19):

$$k_t^A(1) = k_a a_t^D(1) + k_w w_t^D(1) \tag{31}$$

where:

$$\begin{aligned}
k_a &= \frac{1}{4}\left[q_a+\left(2k(0)-1\right)\left(2\alpha-1\right)p_a\right] \\
k_w &= \frac{1}{4}\left[q_w-\left(2k(0)-1\right)\left[1-\left(2\alpha-1\right)p_w\right]\right]
\end{aligned}$$

Second, it affects the accumulations of wealth. Substituting (13) into (28) we have

$$w_{t+1}^{D}(1)=\eta_1\epsilon_{t+1}^{D}+(\rho\eta_1+\eta_2)\,a_t^{D}(1)+\eta_3 w_t^{D}(1) \tag{32}$$

where

$$\begin{aligned}\rho\eta_1+\eta_2 &= \frac{\rho\left[(2\alpha-1)+\theta\psi\left[\lambda 4\alpha\left(1-\alpha\right)-1\right]\right]-(1-\psi\theta)\left(2\alpha-1\right)}{1+(\lambda-1)\,4\alpha\left(1-\alpha\right)-\psi\theta\left(2\alpha-1\right)}\\ \eta_3 &= (1-\psi\theta)\,\frac{1+(\lambda-1)\,4\alpha\left(1-\alpha\right)}{1+(\lambda-1)\,4\alpha\left(1-\alpha\right)-\psi\theta\left(2\alpha-1\right)}\end{aligned}$$

While $\rho\eta_1+\eta_2$ and η_3 do not depend on $k(0)$, η_1 does depend on $k(0)$. The impact of the innovation ϵ_{t+1}^{D} on $w_{t+1}^{D}(1)$ therefore depends on $k(0)$.

Overall we can summarize the first-order solution of all variables other than $k^D(1)$ as follows. The solution for the control variables is

$$p_{F,t}(1)=p_a a_t^{D}(1)+p_w w_t^{D}(1) \tag{33}$$
$$q_t^{D}(1)=q_a a_t^{D}(1)+q_w w_t^{D}(1) \tag{34}$$
$$q_t^{A}(1)=a_t^{A}(1)+0.5p_a a_t^{D}(1)+0.5p_w w_t^{D}(1) \tag{35}$$
$$w_t^{A}(1)=a_t^{A}(1) \tag{36}$$
$$k_t^{A}=k_a a_t^{D}(1)+k_w w_t^{D}(1) \tag{37}$$

The accumulation of the state variables is described by

$$a_{t+1}^{A}(1)=\rho a_t^{A}(1)+\epsilon_{t+1}^{A} \tag{38}$$
$$a_{t+1}^{D}(1)=\rho a_t^{D}(1)+\epsilon_{t+1}^{D} \tag{39}$$
$$w_{t+1}^{D}(1)=\eta_1\epsilon_{t+1}^{D}+(\rho\eta_1+\eta_2)\,a_t^{D}(1)+\eta_3 w_t^{D}(1) \tag{40}$$

1.2 Numerical solution

The solution method described here is the standard first-order solution method that applies more broadly than to the particulars of the model in the paper. The system (1)-(8) consists of 3 state variables and 5 control variables. The vectors of state and control variables are

$$S_t=\left[\begin{array}{ccc} a_t^{D} & w_t^{D} & a_t^{A}\end{array}\right]' \tag{41}$$
$$CV_t=\left[\begin{array}{ccccc} w_t^{A} & p_{F,t} & k_t^{A} & q_{H,t} & q_{F,t}\end{array}\right]' \tag{42}$$

We write the entire vector of model variables as

$$X_t = \begin{bmatrix} (S_t)' \\ (CV_t)' \end{bmatrix} \tag{43}$$

After substituting the expressions for consumer price indices and asset returns, and applying the expectations operator, equations (57)-(60) of Appendix A can be written compactly as

$$E_t g(X_t, X_{t+1}) = 0 \tag{44}$$

The first-order component of model equations follows from a linear expansion around the steady state, which delivers

$$M_1 X_t(1) + M_2 E_t X_{t+1}(1) = 0 \Rightarrow E_t X_{t+1}(1) = M X_t(1)$$

where $M = -(M_2)^{-1} M_1$.

We diagonalize the matrix M':

$$M' = EV \Omega EV^{-1}$$

where EV contains the eigenvectors of M' and Ω is a diagonal matrix with the corresponding eigenvalues. Using the property that $(EV^{-1})' = (EV')^{-1}$ it follows that

$$M = (EV')^{-1} \Omega EV'$$

We define

$$\tilde{X}_t(1) = EV' X_t(1)$$

so that the first-order component of the model becomes

$$E_t \tilde{X}_{t+1}(1) = \Omega \tilde{X}_t(1)$$

The system is well defined when there are as many zero and explosive eigenvalues as there are control variables (that is 5). We set the corresponding elements of $\tilde{X}_t(1)$ to zero. Let $EV'(subs)$ denote the rows of EV' corresponding to the zero or explosive eigenvalues. The first-order component of control variables as a function of state variables is then solved from $EV'(subs)X_t(1) = 0$, which gives

$$CV_t(1) = -(EV'(sub, 4:8))^{-1} EV'(sub, 1:3) S_t(1) \equiv \hat{EV} \cdot S_t(1) \tag{45}$$

In particular, we will use the following notation for the solution of goods and asset prices

$$\begin{aligned} p_{F,t}(1) &= p_s S_t(1) && (46) \\ q_{H,t}(1) &= q_s^H S_t(1) && (47) \\ q_{F,t}(1) &= q_s^F S_t(1) && (48) \end{aligned}$$

The accumulation of the first-order component of the state variables can be described as

$$\begin{aligned} S_{t+1}(1) &= N_1 S_t(1) + N_2 \epsilon_{t+1} && (49) \\ \epsilon_{t+1} &= \left[\begin{array}{cc} \epsilon_{t+1}^H & \epsilon_{t+1}^F \end{array} \right]' \end{aligned}$$

This can be derived as follows. Let B_1 be a 3x8 matrix that extracts the rows of the model equations corresponding to the accumulation of the state variables: the dynamics of the home wealth (3) and the dynamics of both productivity levels, (1)-(2). Without the expectation operator applied to those equations we have

$$B_1 M_1 X_t(1) + B_1 M_2 X_{t+1}(1) = B_3 \epsilon_{t+1} \quad (50)$$

where:

$$B_3 = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{array} \right]$$

Using (45) we write:

$$X_t(1) = \left[\begin{array}{c} S_t(1) \\ CV_t(1) \end{array} \right] = \left[\begin{array}{c} I \\ \hat{EV} \end{array} \right] S_t(1) \equiv B_2 S_t(1) \quad (51)$$

where I is a 3x3 matrix. Proceeding similarly for $X_{t+1}(1)$, we rewrite (50) as:

$$B_1 M_1 B_2 S_t(1) + B_1 M_2 B_2 S_{t+1}(1) = B_3 \epsilon_{t+1}$$

which leads to (49) with

$$N_1 = - \left[B_1 M_2 B_2 \right]^{-1} B_1 M_1 B_2 \qquad N_2 = \left[B_1 M_2 B_2 \right]^{-1} B_3$$

2 Second order solution conditional on $k^D(1)$

2.1 Second-order component of model equations

We now describe the numerical solution of the second-order component of model variables conditional on the first-order component of the portfolio difference, $k_t^D(1)$. Applying equation (6) of the paper, the second order component of the model $E_t g(X_t, X_{t+1}) = 0$ is equal to

$$M_1 X_t(2) + M_2 E_t X_{t+1}(2) + E_t O_2 = 0 \tag{52}$$

where $E_t O_2$ contains the product of first-order components of model variables. These multiply second-order derivatives of the model equations at the steady state. Let $E_t O_2(i)$ be the i'th element of $E_t O_2$, corresponding to equation i of the model. We have

$$E_t O_2(i) = \frac{1}{2} X_t(1)' M_{3,i} X_t(1) + \frac{1}{2} E_t X_{t+1}(1)' M_{4,i} X_{t+1}(1) + X_t(1)' M_{5,i} E_t X_{t+1}(1) \tag{53}$$

where $M_{3,i}$ is the second-order derivative of equation i with respect to X_t and $M_{4,i}$, $M_{5,i}$ are similarly defined.

While $k_t^D(1)$ does not enter the first-order components of model equations, it does enter the second-order component through $E_t O_2$. We solve the second-order component of model variables conditional on a conjectured solution for $k_t^D(1)$, which is $k_t^D(1) = k_s S_t(1)$. The $M_{3,i}$, $M_{4,i}$ and $M_{5,i}$ matrices then depend on k_s. Rather than numerically recomputing these second-order derivatives for each value of k_s we proceed as follows. Portfolio shares enter the second-order component of model equations through

$$k_{H,t}^H(1) = k_t^A(1) + 0.5 k_t^D(1) = k_t^A(1) + 0.5 k_s S_t(1) \tag{54}$$

$$k_{H,t}^F(1) = k_t^A(1) - 0.5 k_t^D(1) = k_t^A(1) - 0.5 k_s S_t(1) \tag{55}$$

and $k_{F,t}^H(1) = 1 - k_{H,t}^H(1)$, $k_{F,t}^F(1) = 1 - k_{H,t}^F(1)$. We first numerically compute the second-order derivatives at $k_s = 0$. We then make an analytical adjustment that adds terms to the second-order derivatives depending on k_s. Specifically, portfolio shares enter through wealth accumulation and asset-market clearing conditions. Using these equations ((48)-(49) and (51)-(52)

in Appendix A of the paper), and focusing on the second-order components that depend on k_s, we have

$$\begin{aligned}
w_{t+1}(2) &= \frac{1-\psi\theta}{2}k_sS_t(1)\begin{pmatrix} r_q\left(q_{H,t+1}(1)-q_{F,t+1}(1)\right)-\left(q_{H,t}(1)-q_{F,t}(1)\right) \\ +\left(1-r_q\right)\left(a^D_{t+1}(1)-p_{F,t+1}(1)\right)\end{pmatrix}+\text{other} \\
w_{t+1}(2)^* &= -\frac{1-\psi\theta}{2}k_sS_t(1)\begin{pmatrix} r_q\left(q_{H,t+1}(1)-q_{F,t+1}(1)\right)-\left(q_{H,t}(1)-q_{F,t}(1)\right) \\ +\left(1-r_q\right)\left(a^D_{t+1}(1)-p_{F,t+1}(1)\right)\end{pmatrix}+\text{other} \\
q_{H,t}(2) &= -\frac{1}{2}k_sS_t(1)\left[\left(2\alpha-1\right)p_{F,t}(1)-w^D_t(1)\right]+\text{other} \\
q_{F,t}(2) &= \frac{1}{2}k_sS_t(1)\left[\left(2\alpha-1\right)p_{F,t}(1)-w^D_t(1)\right]+\text{other}
\end{aligned}$$

where "other" stands for all the other second-order terms that do not depend on k_s and $r_q=(1-\psi)/(1-\psi\theta)$. Starting from the second-order derivatives of the model equations at $k_s=0$, these equations allow us to analytically adjust the second-order derivatives $M_{3,i}$, $M_{4,i}$ and $M_{5,i}$ as a function of k_s.

Substituting the solution of the first-order components of model variables, described by (51) and (49), into the expression for $E_tO_2(i)$, we have

$$\begin{aligned}
E_tO_2(i) &= \frac{1}{2}S_t(1)'B_2'M_{3,i}B_2S_t(1)+\frac{1}{2}S_t(1)'N_1'B_2'M_{4,i}B_2N_1S_t(1) \\
&\quad +\frac{1}{2}E_t\epsilon_{t+1}'N_2'B_2'M_{4,i}B_2N_2\epsilon_{t+1}+S_t(1)'B_2'M_{5,i}B_2N_1S_t(1)
\end{aligned}$$

This is written in a more compact way as:

$$E_tO_2(i)=S_t(1)'K_iS_t(1)+\sigma^2k_i$$

where:

$$\begin{aligned}
K_i &= \frac{1}{2}B_2'M_{3,i}B_2+\frac{1}{2}N_1'B_2'M_{4,i}B_2N_1+B_2'M_{5,i}B_2N_1 \\
k_i &= \text{trace}\left(\frac{1}{2}N_2'B_2'M_{4,i}B_2N_2\right)
\end{aligned}$$

This uses that $var(\epsilon_{t+1})=\sigma_2I$, where I is a 2 by 2 matrix.

It will be useful to write the quadratic terms in $S_t(1)$ as a vector. Writing element i of $S_t(1)$ as $S_{t,i}(1)$, define

$$Y_t\,(2) = \begin{bmatrix} S_{t,1}(1)^2 \\ S_{t,1}(1)S_{t,2}(1) \\ S_{t,1}(1)S_{t,3}(1) \\ S_{t,2}(1)S_{t,1}(1) \\ S_{t,2}(1)^2 \\ S_{t,2}(1)S_{t,3}(1) \\ S_{t,3}(1)S_{t,1}(1) \\ S_{t,3}(1)S_{t,2}(1) \\ S_{t,3}(1)^2 \end{bmatrix}$$

$E_tO_2(i)$ can then be written as a linear function of Y_t:

$$E_tO_2(i) = (K_i^{vec})'Y_t\,(2) + \sigma^2 k_i \tag{56}$$

where:

$$K_i^{vec} = \begin{bmatrix} K_{i,1,1} \\ K_{i,1,2} \\ K_{i,1,3} \\ K_{i,2,1} \\ K_{i,2,2} \\ K_{i,2,3} \\ K_{i,3,1} \\ K_{i,3,2} \\ K_{i,3,3} \end{bmatrix}$$

where $K_{i,x,y}$ is element (x, y) of matrix K_i, and vec denotes the vectorization of a matrix. (52) can then be written in a matrix form as:

$$M_1X_t(2) + M_2E_tX_{t+1}(2) + KY_t\,(2) + k\sigma^2 = 0 \tag{57}$$

where:

$$K = \begin{bmatrix} (K_1^{vec})' \\ \ldots \\ (K_8^{vec})' \end{bmatrix} \qquad k = \begin{bmatrix} k_1 \\ \ldots \\ k_8 \end{bmatrix}$$

To compute the dynamics of $Y_t\,(2)$, start by writing

$$E_tY_{t+1}\,(2) = (E_tS_{t+1}(1)S_{t+1}(1)')^{vec}$$

From (49) we write:

$$E_t S_{t+1}(1) S_{t+1}(1)' = N_1 S_t(1) S_t(1)' N_1' + \sigma^2 N_2 N_2'$$

Write N_1 as:

$$N_1 = \begin{bmatrix} n_1' \\ .. \\ n_3' \end{bmatrix}$$

where n_i' is row i of the matrix N_1. Then element (i, j) of $N_1 S_t(1) S_t(1)' N_1'$ is equal to

$$n_i' S_t(1) S_t(1)' n_j = \left[\left(n_i n_j' \right)^{vec} \right]' Y_t(2) \equiv z_{i,j} Y_t(2)$$

We then have

$$E_t S_{t+1}(1) S_{t+1}(1)' = \begin{bmatrix} z_{1,1} Y_t(2) & .. & z_{1,3} Y_t(2) \\ .. & & .. \\ z_{3,1} Y_t(2) & .. & z_{3,3} Y_t(2) \end{bmatrix} + \sigma^2 N_2 N_2'$$

Also define

$$\tilde{n} = \left(N_2 N_2' \right)^{vec}$$

which implies:

$$\left(E_t S_{t+1}(1) S_{t+1}(1)' \right)^{vec} = Z Y_t(2) + \sigma^2 \tilde{n} \tag{58}$$

where

$$Z = \begin{bmatrix} z_{1,1} \\ .. \\ z_{1,3} \\ z_{2,1} \\ .. \\ z_{2,3} \\ z_{3,1} \\ .. \\ z_{3,3} \end{bmatrix}$$

2.2 Second-order solution for the control variables

The preceding analysis allows us to write the second-order component of model equations as

$$\begin{aligned} 0 &= M_1 X_t(2) + M_2 E_t X_{t+1}(2) + K Y_t(2) + k\sigma^2 \\ E_t Y_{t+1}(2) &= Z Y_t(2) + \sigma^2 \tilde{n} \end{aligned}$$

In order to compute the second-order component of control variables we proceed as we did with the first-order solution. Define $M = -M_2^{-1}M_1$ and diagonalize M': $M' = EV\Omega EV^{-1}$. This implies: $M = (EV')^{-1}\Omega EV'$. Define $\tilde{X}_t(2) = EV'X_t(2)$. Then the system becomes

$$\begin{aligned} E_t\tilde{X}_{t+1}(2) &= \Omega\tilde{X}_t(2) + QY_t + \bar{k}\sigma^2 \\ E_tY_{t+1}(2) &= ZY_t(2) + \sigma^2\tilde{n} \end{aligned}$$

where $\bar{k} = -EV'M_2^{-1}k$ and $Q = -EV'M_2^{-1}K$.

Define the matrix G such that: $GZ - \Omega G = Q$. Specifically, we write

$$G = \begin{bmatrix} g_1' \\ \dots \\ g_8' \end{bmatrix}$$

where g_i' is row i of the matrix G. Let q_i' be row i of matrix Q. Then the row i of $GZ - \Omega G = Q$ becomes

$$g_i'Z - \lambda_i g_i' = q_i'$$

where λ_i is the i'th eigenvalue on the diagonal of the matrix Ω. It follows that

$$Z'g_i - \lambda_i g_i = q_i \Rightarrow g_i = (Z' - \lambda_i I)^{-1} q_i$$

where I is a 9x9 identity matrix.

The two equations of the system are then combined as:

$$E_t(\tilde{X}_{t+1}(2) - GY_{t+1}(2)) = \Omega(\tilde{X}_t(2) - GY_t(2)) + \hat{k}\sigma^2$$

where $\hat{k} = \bar{k} - G\tilde{n}$. We again identify the eigenvalues in Ω that are zero or explosive (as in the first order solution), and set the corresponding rows of $\tilde{X}_t(2) - GY_t - \tilde{k}\sigma^2$ to zero, where $\tilde{k} = (I-\Omega)^{-1}\hat{k}$. This gives $EV'(subs)X_t(2) - G(sub,.)Y_t(2) - \tilde{k}(sub)\sigma^2 = 0$, so that the second order solution of the control variables is

$$\begin{aligned} CV_t(2) &= -(EV'(sub,4:8))^{-1}EV'(sub,1:3)S_t(2) \\ &\quad +(EV'(sub,4:8))^{-1}G(sub,.)Y_t(2) + (EV'(sub,4:8))^{-1}\tilde{k}(sub)\sigma^2 \\ &\equiv \hat{EV}S_t(2) + \hat{G}Y_t(2) + k_c\sigma^2 \end{aligned} \tag{59}$$

We write

$$\hat{G} = \begin{pmatrix} \hat{g}_1 \\ \dots \\ \hat{g}_8 \end{pmatrix}$$

where $\hat{g}_i$ is row i of $\hat{G}$. Therefore for control variable i the part of the second-order solution that depends on the product of first-order component of state variables is $\hat{g}_i Y_t(2)$. We can convert this back to matrix form: $\hat{g}_i Y_t(2) = S_t(1)'\hat{g}_i^m S_t(1)$, where the first three elements of $\hat{g}_i$ make up the first row of $\hat{g}_i^m$, the second three elements make up the second row and the last three elements make up the last row. We continue to use the superscript m below to convert vectors to matrices in this way.

For goods and equity prices we will write this second-order solution as

$$p_{F,t}(2) = p_s S_t(2) + S_t(1)' p_{ss} S_t(1) + k_p \sigma^2 \tag{60}$$
$$q_{H,t}(2) = q_s^H S_t(2) + S_t(1)' q_{ss}^H S_t(1) + k_q^H \sigma^2 \tag{61}$$
$$q_{F,t}(2) = q_s^F S_t(2) + S_t(1)' q_{ss}^F S_t(1) + k_q^F \sigma^2 \tag{62}$$

Note that p_{ss}, q_{ss}^H and q_{ss}^F need not be symmetric. It is ok if the i,j and j,i elements differ, as all that matters is their sum.

2.3 Second-order dynamics of the state variables

We now turn to the dynamic process of the second-order components of the state variables. Let again B_1 be a matrix that extracts the rows corresponding to the state variable accumulation equations. Without the expectation operator applied to accumulation equations for the state variables, we have

$$B_1 M_1 X_t(2) + B_1 M_2 X_{t+1}(2) + B_1 O_2 = 0 \tag{63}$$

Using (59) we write:

$$X_t(2) = \begin{bmatrix} S_t(2) \\ CV_t(2) \end{bmatrix} = \begin{bmatrix} S_t(2) \\ \hat{EV} S_t(2) + \hat{G} Y_t(2) + k_c \sigma^2 \end{bmatrix}$$
$$= B_2 S_t(2) + \bar{G} Y_t(2) + k_x \sigma^2 \tag{64}$$

where:

$$B_2 = \begin{bmatrix} I_{3x3} \\ \hat{EV} \end{bmatrix} \qquad \bar{G} = \begin{bmatrix} 0_{3x9} \\ \hat{G} \end{bmatrix} \qquad k_x = \begin{bmatrix} 0_{3x1} \\ k_c \end{bmatrix}$$

Substituting (64) into (63) we have

$$S_{t+1}(2) = N_1 S_t(2) + B_4 Y_t(2) + B_5 Y_{t+1}(2) + B_6 B_1 O_2 + N_6 \sigma^2 \tag{65}$$

where

$$\begin{aligned}
B_4 &= -\left(B_1 M_2 B_2\right)^{-1} B_1 M_1 \bar{G} \\
B_5 &= -\left(B_1 M_2 B_2\right)^{-1} B_1 M_2 \bar{G} \\
B_6 &= -\left(B_1 M_2 B_2\right)^{-1} \\
N_6 &= -\left(B_1 M_2 B_2\right)^{-1} B_1 (M_1 + M_2) k_x
\end{aligned}$$

First consider the term $B_6 B_1 O_2$ in (65). Element i of O_2

$$\begin{aligned}
O_2(i) &= \frac{1}{2} X_t(1)' M_{3,i} X_t(1) + \frac{1}{2} X_{t+1}(1)' M_{4,i} X_{t+1}(1) + X_t(1)' M_{5,i} X_{t+1}(1) = \\
&= S_t(1)' V_{1,i} S_t(1) + \epsilon'_{t+1} V_{2,i} \epsilon_{t+1} + S_t(1)' V_{3,i} \epsilon_{t+1}
\end{aligned}$$

where

$$\begin{aligned}
V_{1,i} &= \frac{1}{2}\left(B'_2 M_{3,i} B_2 + N'_1 B'_2 M_{4,i} B_2 N_1\right) + B'_2 M_{5,i} B_2 N_1 \\
V_{2,i} &= \frac{1}{2} N'_2 B'_2 M_{4,i} B_2 N_2 \\
V_{3,i} &= B'_2 M_{5,i} B_2 N_2 + N'_1 B'_2 M_{4,i} B_2 N_2
\end{aligned}$$

Here we used $X_t(1) = B_2 S_t(1)$ and $S_{t+1}(1) = N_1 S_t(1) + N_2 \epsilon_{t+1}$.

Let the three rows of the model corresponding to the state accumulation equations (the rows extracted by B_1) be rows a, b and c:

$$B_1 O_2 = \begin{bmatrix} S_t(1)' V_{1,a} S_t(1) + \epsilon'_{t+1} V_{2,a} \epsilon_{t+1} + S_t(1)' V_{3,a} \epsilon_{t+1} \\ S_t(1)' V_{1,b} S_t(1) + \epsilon'_{t+1} V_{2,b} \epsilon_{t+1} + S_t(1)' V_{3,b} \epsilon_{t+1} \\ S_t(1)' V_{1,c} S_t(1) + \epsilon'_{t+1} V_{2,c} \epsilon_{t+1} + S_t(1)' V_{3,c} \epsilon_{t+1} \end{bmatrix}$$

We can then write

$$B_6 B_1 O_2 = \begin{bmatrix} S_t(1)' \bar{V}_{1,1} S_t(1) + \epsilon'_{t+1} \bar{V}_{1,2} \epsilon_{t+1} + S_t(1)' \bar{V}_{1,3} \epsilon_{t+1} \\ S_t(1)' \bar{V}_{2,1} S_t(1) + \epsilon'_{t+1} \bar{V}_{2,2} \epsilon_{t+1} + S_t(1)' \bar{V}_{2,3} \epsilon_{t+1} \\ S_t(1)' \bar{V}_{3,1} S_t(1) + \epsilon'_{t+1} \bar{V}_{3,2} \epsilon_{t+1} + S_t(1)' \bar{V}_{3,3} \epsilon_{t+1} \end{bmatrix}$$

where

$$\begin{aligned}
\bar{V}_{i,1} &= B_{6,i,1} V_{1,a} + B_{6,i,2} V_{1,b} + B_{6,i,3} V_{1,c} \\
\bar{V}_{i,2} &= B_{6,i,1} V_{2,a} + B_{6,i,2} V_{2,b} + B_{6,i,3} V_{2,c} \\
\bar{V}_{i,3} &= B_{6,i,1} V_{3,a} + B_{6,i,2} V_{3,b} + B_{6,i,3} V_{3,c}
\end{aligned}$$

where $B_{6,x,y}$ is the element (x,y) of matrix B_6.

We now turn to the term $B_5 Y_{t+1}(2)$ in (65):

$$B_5 Y_{t+1}(2) = B_5 \left(S_{t+1}(1) S_{t+1}(1)'\right)^{vec}$$

Using (49) we have:

$$S_{t+1}(1) S_{t+1}(1)' = N_1 S_t(1) S_t(1)' N_1' + N_2 \epsilon_{t+1} \epsilon_{t+1}' N_2' + \\ N_1 S_t(1) \epsilon_{t+1}' N_2' + N_2 \epsilon_{t+1} S_t(1)' N_1'$$

We have already derived that

$$\left(N_1 S_t(1) S_t(1)' N_1'\right)^{vec} = Z Y_t(2)$$

We can similarly derive that

$$\left(N_2 \epsilon_{t+1} \epsilon_{t+1}' N_2'\right)^{vec} = \bar{Z} Y_t^{eps}$$

where

$$Y_t^{eps} = \begin{bmatrix} \epsilon_{H,t}^2 \\ \epsilon_{H,t}\epsilon_{F,t} \\ \epsilon_{F,t}\epsilon_{H,t} \\ \epsilon_{F,t}^2 \end{bmatrix}$$

and

$$\bar{Z} = \begin{bmatrix} \bar{z}_{1,1} \\ .. \\ \bar{z}_{1,3} \\ \bar{z}_{2,1} \\ .. \\ \bar{z}_{2,3} \\ \bar{z}_{3,1} \\ .. \\ \bar{z}_{3,3} \end{bmatrix}$$

with

$$\bar{z}_{i,j} = \left[\left(\bar{n}_i \bar{n}_j'\right)^{vec}\right]'$$

where $\bar{n}_i'$ is row i of N_2.

Next turn to $N_1 S_t(1) \epsilon_{t+1}' N_2'$. Element (i,j) of $N_1 S_t(1) \epsilon_{t+1}' N_2'$ is equal to (n_i' is row i of N_1 and $\bar{n}_j'$ is row j of N_2):

$$n_i' S_t(1) \epsilon_{t+1}' \bar{n}_j = S_t(1)' n_i \bar{n}_j' \epsilon_{t+1}$$

Similarly, element (i,j) of $N_2\epsilon_{t+1}S_t'(1)\,N_1'$ is equal to:

$$\bar{n}_i'\epsilon_{t+1}S_t(1)'n_j = \epsilon_{t+1}'\bar{n}_i n_j' S_t(1) = S_t(1)'n_j\bar{n}_i'\epsilon_{t+1}$$

So the element (i,j) of $N_1S_t(1)\epsilon_{t+1}'N_2' + N_2\epsilon_{t+1}S_t(1)'N_1'$ is:

$$S_t(1)'\left[n_i\bar{n}_j' + n_j\bar{n}_i'\right]\epsilon_{t+1}$$

Therefore row i (out of the three rows) of

$$B_5\left(N_1S_t(1)\epsilon_{t+1}'N_2' + N_2\epsilon_{t+1}S_t(1)'N_1'\right)^{vec}$$

is written as:

$$\begin{aligned}
&B_{5,i,1}S_t(1)'\left[n_1\bar{n}_1' + n_1\bar{n}_1'\right]\epsilon_{t+1} + B_{5,i,2}S_t(1)'\left[n_1\bar{n}_2' + n_2\bar{n}_1'\right]\epsilon_{t+1}\\
&+B_{5,i,3}S_t(1)'\left[n_1\bar{n}_3' + n_3\bar{n}_1'\right]\epsilon_{t+1}\\
&+B_{5,i,4}S_t(1)'\left[n_2\bar{n}_1' + n_1\bar{n}_2'\right]\epsilon_{t+1} + B_{5,i,5}S_t(1)'\left[n_2\bar{n}_2' + n_2\bar{n}_2'\right]\epsilon_{t+1}\\
&+B_{5,i,6}S_t(1)'\left[n_2\bar{n}_3' + n_3\bar{n}_2'\right]\epsilon_{t+1}\\
&+B_{5,i,7}S_t(1)'\left[n_3\bar{n}_1' + n_1\bar{n}_3'\right]\epsilon_{t+1} + B_{5,i,8}S_t(1)'\left[n_3\bar{n}_2' + n_2\bar{n}_3'\right]\epsilon_{t+1}\\
&+B_{5,i,9}S_t(1)'\left[n_3\bar{n}_3' + n_3\bar{n}_3'\right]\epsilon_{t+1}
\end{aligned}$$

where $B_{5,x,y}$ is the element of B_5 on the xth row and the yth column. This is written in a more compact way as:

$$S_t(1)'\bar{N}_{5,i}\epsilon_{t+1}$$

where

$$\bar{N}_{5,i} = \sum_{v=1}^{3}\sum_{w=1}^{3}B_{5,i,v,w}^{m}\left(n_v\bar{n}_w' + n_w\bar{n}_v'\right)$$

where $B_{5,i,v,w}^m$ is element (v,w) of matrix $B_{5,i}^m$, where $B_{5,i}^m$ is:

$$B_{5,i}^m = \begin{pmatrix} B_{5,i,1} & B_{5,i,2} & B_{5,i,3}\\ B_{5,i,4} & B_{5,i,5} & B_{5,i,6}\\ B_{5,i,7} & B_{5,i,8} & B_{5,i,9}\end{pmatrix}$$

Here $B_{5,i}^m$ is the matrix form associated with row i of B_5. Similarly write

$$\begin{aligned}
\bar{N}_{3,i} &= (B_4 + B_5Z)_i^m\\
\bar{N}_{4,i} &= \left(B_5\bar{Z}\right)_i^m
\end{aligned}$$

These are the matrix form of row i of the respective matrices.

Putting all these steps together, (65) becomes

$$S_{t+1}(2) = N_1 S_t(2) + \\ + \begin{pmatrix} S_t(1)' N_{3,1} S_t(1) + \epsilon'_{t+1} N_{4,1} \epsilon_{t+1} + S_t(1)' N_{5,1} \epsilon_{t+1} \\ S_t(1)' N_{3,2} S_t(1) + \epsilon'_{t+1} N_{4,2} \epsilon_{t+1} + S_t(1)' N_{5,2} \epsilon_{t+1} \\ S_t(1)' N_{3,3} S_t(1) + \epsilon'_{t+1} N_{4,3} \epsilon_{t+1} + S_t(1)' N_{5,3} \epsilon_{t+1} \end{pmatrix} + N_6 \sigma^2 \quad (66)$$

where

$$\begin{aligned} N_{3,i} &= \bar{N}_{3,i} + \bar{V}_{i,1} \\ N_{4,i} &= \bar{N}_{4,i} + \bar{V}_{i,2} \\ N_{5,i} &= \bar{N}_{5,i} + \bar{V}_{i,3} \end{aligned}$$

This describes the dynamics of the second-order components of the state variables.

2.4 Expected portfolio return

We finally derive the second-order component of the expected portfolio return, $E_t r^{p,H}_{t+1}(2)$, which is needed when computing the second-order component of the Bellman equation in section 3. We obtain the second-order component of $r^{p,H}_{t+1}$ from the second-order component of equation (61) in Appendix A of the paper. This gives

$$\begin{aligned} r^{p,H}_{t+1}(2) &= r_{H,t+1}(2) + (1-k(0))(r_{F,t+1}(2) - r_{H,t+1}(2) - \tau) + p_t(2) - p_{t+1}(2) \\ &\quad + (r_{H,t+1}(1) - r_{F,t+1}(1)) k^H_{H,t}(1) + \\ &\quad \frac{1}{2} k(0)(1-k(0))(r_{H,t+1}(1) - r_{F,t+1}(1))^2 \end{aligned} \quad (67)$$

Start with the expectation of the last two terms in (67). As the expected excess return is zero to a first order and the portfolio shares are known at time t, we have

$$E_t(r_{H,t+1}(1) - r_{F,t+1}(1)) k^H_{H,t}(1) = k^H_{H,t} E_t(r_{H,t+1}(1) - r_{F,t+1}(1)) = 0$$

From the first-order solution the first-order component of the excess return is proportional to the innovation ϵ^D_{t+1} since the expected excess return is zero to a first order. We write the first-order component of the excess return as $r_{H,t+1}(1) - r_{F,t+1}(1) = r_{DE}\epsilon^D_{t+1}$. Therefore

$$E_t(r_{H,t+1}(1) - r_{F,t+1}(1))^2 = 2r^2_{DE}\sigma^2$$

As shown later (in section 4), the expected excess return is also zero to a second order, so the expectation of the second term in (67) becomes:

$$E_t(r_{F,t+1}(2) - r_{H,t+1}(2) - \tau) = -\tau$$

and (67) is written in expected terms as:

$$E_t r^{p,H}_{t+1}(2) = E_t r_{H,t+1}(2) + p_t(2) - E_t p_{t+1}(2) - (1-k(0))\tau + k(0)(1-k(0))r^2_{DE}\sigma^2 \tag{68}$$

Turning to the consumer prices, the second-order component of the home CPI (equation 57 in Appendix A of the paper) is

$$\begin{aligned} p_t(2) &= (1-\alpha)p_{F,t}(2) - \frac{1}{2}(\lambda - 1)\,\alpha\,(1-\alpha)\,p_{F,t}(1)^2 \\ &\equiv \bar{p}_s S_t(2) + S_t(1)'\bar{p}_{ss}S_t(1) + \bar{k}_p\sigma^2 \end{aligned}$$

where we used (60) and:

$$\begin{aligned} \bar{p}_s &= (1-\alpha)p_s \\ \bar{p}_{ss} &= (1-\alpha)p_{ss} - \frac{1}{2}(\lambda-1)\,\alpha\,(1-\alpha)\,p_s' p_s \\ \bar{k}_p &= (1-\alpha)k_p \end{aligned}$$

Using (66) this implies:

$$E_t p_{t+1}(2) = \bar{p}_s N_1 S_t(2) + S_t(1)'\hat{p}_{ss}S_t(1) + \hat{p}\sigma^2$$

where:

$$\begin{aligned} \hat{p}_{ss} &= N_1'\bar{p}_{ss}N_1 + \sum_{v=1}^{3}\bar{p}_s(v)N_{3,v} \\ \hat{p} &= \bar{p}_s N_6 + \bar{k}_p + \tilde{p} \\ \tilde{p} &= \text{trace}\left[\sum_{v=1}^{3}\bar{p}_s(v)N_{4,v} + N_2'\bar{p}_{ss}N_2\right] \end{aligned}$$

We next turn to $E_t r_{H,t+1}(2)$. Using equation (59) of Appendix A of the paper, the expected second order component of the Home return is

$$\begin{aligned} E_t r_{H,t+1}(2) &= -q_{H,t}(2) + r_q E_t q_{H,t+1}(2) + r_a E_t a_{H,t+1}(2) \\ &+\frac{1}{2} r_{qq} E_t q_{H,t+1}(1)^2 + \frac{1}{2} r_{aa} E_t a_{H,t+1}(1)^2 + r_{qa} E_t q_{H,t+1}(1) a_{H,t+1}(1) \end{aligned} \tag{69}$$

where $r_q = (1-\psi)(1-\psi\theta)^{-1}$, $r_a = 1 - r_q$ and $r_{qq} = r_{aa} = -r_{qa} = r_q(1-r_q)$. Consider the last three terms of (69) first. We can simply substitute the first-order results. Using $q_{H,t+1}(1) = q_s^H S_t(1)$, we have

$$\begin{aligned} E_t q_{H,t+1}(1)^2 &= E_t S_{t+1}(1)'(q_s^H(1))' q_s^H S_{t+1}(1) \\ &= S_t(1)' N_1' (q_s^H)' q_s^H N_1 S_t(1) + e_1 \sigma^2 \end{aligned}$$

where e_1 =trace$\left[N_2'(q_s^H)' q_s^H N_2\right]$. Similarly, writing $a_{H,t+1}(1) = a_s^H S_t(1) + a_E^H \epsilon_{t+1}$, where $a_s^H = (0.5\rho, 0, \rho)$ and $a_E^H = (1, 0)$, we have

$$\begin{aligned} E_t a_{H,t+1}(1)^2 &= S_t(1)' \left(a_s^H\right)' a_s^H S_t(1) + e_3 \sigma^2 \\ &= S_t(1)' \left(a_s^H\right)' a_s^H S_t(1) + a_E^H \left(a_E^H\right)' \sigma^2 \end{aligned}$$

where e_3 =trace$\left[\left(a_E^H\right)' a_E^H\right]$. Finally:

$$\begin{aligned} E_t q_{H,t+1}(1) a_{H,t+1}(1) &= E_t(q_s^H N_1 S_t(1) + q_s^H N_2 \epsilon_{t+1})(a_s^H S_t(1) + a_E^H \epsilon_{t+1}) \\ &= S_t(1)' N_1' \left(q_s^H\right)' a_s^H S_t(1) + e_2 \sigma^2 \end{aligned}$$

where e_2 =trace$\left[N_2' \left(q_s^H\right)' a_E^H\right]$.

Next consider the first three terms of (69), using (61) and (66):

$$\begin{aligned} q_{H,t}(2) &= q_s^H S_t(2) + S_t(1)' q_{ss}^H S_t(1) + k_q^H \sigma^2 \\ E_t q_{H,t+1}(2) &= q_s^H N_1 S_t(2) + S_t(1)' \hat{q}_{ss} S_t(1) + \hat{q} \sigma^2 \end{aligned}$$

where:

$$\begin{aligned} \hat{q}_{ss} &= N_1' q_{ss}^H N_1 + \sum_{v=1}^{3} q_s^H(v) N_{3,v} \\ \hat{q} &= q_s^H N_6 + k_q^H + \tilde{q} \\ \tilde{q} &= \text{trace}\left[\sum_{v=1}^{3} q_s^H(v) N_{4,v} + N_2' q_{ss}^H N_2\right] \end{aligned}$$

The last elements is: $E_t a_{H,t+1}(2) = a_s^H S_t(2)$.

Putting all our results together, (68) becomes:

$$E_t r_{t+1}^{p,H}(2) = r_s S_t(2) + S_t(1)' r_{ss} S_t(1) + \hat{r}\sigma^2 \tag{70}$$

where

$$\begin{aligned} r_s &= -q_s^H + r_q q_s^H N_1 + r_a a_s^H + \bar{p}_s - \bar{p}_s N_1 \\ \hat{r} &= -k_q^H + r_q \hat{q} + \frac{1}{2} r_{qq} e_1 + \frac{1}{2} r_{aa} e_3 + r_{qa} e_2 \\ &\quad + \bar{k}_p - \hat{p} + k(0)(1 - k(0)) r_{DE}^2 - (1 - k(0)) \frac{\tau}{\sigma^2} \\ r_{ss} &= -q_{ss}^H + r_q \hat{q}_{ss} + \frac{1}{2} r_{qq} N_1' (q_s^H)' q_s^H N_1 \\ &\quad + r_{qa} N_1' \left(q_s^H\right)' a_s^H + \frac{1}{2} r_{aa} \left(a_s^H\right)' a_s^H + \bar{p}_{ss} - \hat{p}_{ss} \end{aligned}$$

3 First and second-order components of Bellman equation

3.1 Second order Taylor expansion

The Bellman equation for the Home country is listed in equation (55) of Appendix A and repeated here for convenience:

$$\begin{aligned} & e^{v(0)+v(1)+v(2)+f_H(S_t)} \\ = \; & \beta(1-\psi) E_t e^{v(0)+v(1)+v(2)+f_H(S_{t+1})+(1-\gamma) r_{t+1}^{p,H}} \\ & + \beta\psi E_t e^{(1-\gamma) r_{t+1}^{p,H}} \end{aligned} \tag{71}$$

We only list the zero, first and second-order components of the constant term v since higher order components will not matter for the analysis. We will write the first and second-order derivatives of f_H at $S = 0$ as respectively $H_{1,H}$ and $H_{2,H}$.

Taking a second-order Taylor expansion of the left hand side of (71) around $S = 0$ and $v(1) = v(2) = 0$, we get

$$\begin{aligned} & e^{v(0)+v(1)+v(2)+f_H(S_t)} \\ = \; & e^{v(0)} \left[1 + v(1) + v(2) + H_{1,H} S_t\right] \\ & + \frac{1}{2} e^{v(0)} \left[\left[v(1) + v(2) + H_{1,H} S_t\right]^2 + S_t' H_{2,H} S_t\right] \end{aligned}$$

Similarly, the first term on the right-hand side of (71) is expanded around $S=0$, $v(1)=v(2)=0$ and $r^{p,H}_{t+1}=\bar{r}$. Denote $\hat{r}^{p,H}_{t+1}=r^{p,H}_{t+1}-\bar{r}$. A second-order expansion then gives

$$\begin{aligned}
& e^{v(0)+v(1)+v(2)+f_H(S_{t+1})+(1-\gamma)\hat{r}^{p,H}_{t+1}} \\
= \; & e^{v(0)+(1-\gamma)\bar{r}}\left[1+v(1)+v(2)+H_{1,H}S_{t+1}+(1-\gamma)\hat{r}^{p,H}_{t+1}\right] \\
& +\frac{1}{2}e^{v(0)+(1-\gamma)\bar{r}}\left[\begin{array}{c}\left[v(1)+v(2)+H_{1,H}S_{t+1}+(1-\gamma)\hat{r}^{p,H}_{t+1}\right]^2 \\ +S'_{t+1}H_{2,H}S_{t+1}\end{array}\right]
\end{aligned}$$

The last term on the right-hand side of (71) is expanded as

$$\begin{aligned}
e^{(1-\gamma)r^{p,H}_{t+1}} \; = \; & e^{(1-\gamma)\bar{r}}\left[1+(1-\gamma)\hat{r}^{p,H}_{t+1}\right] \\
& +\frac{1}{2}e^{(1-\gamma)\bar{r}}\left[(1-\gamma)\hat{r}^{p,H}_{t+1}\right]^2
\end{aligned}$$

Combining the terms of order zero we get:

$$e^{v(0)}=\frac{\beta\psi e^{(1-\gamma)\bar{r}}}{1-\beta\left(1-\psi\right)e^{(1-\gamma)\bar{r}}}$$

It is convenient to substitute this result into the remaining terms of the second-order expansion of the Bellman equation, which gives

$$\begin{aligned}
& \left[v(1)+v(2)+H_{1,H}S_t\right]+\frac{1}{2}\left[\left[v(1)+v(2)+H_{1,H}S_t\right]^2+S'_tH_{2,H}S_t\right] \\
= \; & \left(1-\psi'\right)E_t\left[\begin{array}{c}\left[v(1)+v(2)+H_{1,H}S_{t+1}+(1-\gamma)\hat{r}^{p,H}_{t+1}\right] \\ +\frac{1}{2}\left[\begin{array}{c}\left[v(1)+v(2)+H_{1,H}S_{t+1}+(1-\gamma)\hat{r}^{p,H}_{t+1}\right]^2 \\ +S'_{t+1}H_{2,H}S_{t+1}\end{array}\right]\end{array}\right] \\
& +\psi'E_t\left[(1-\gamma)\hat{r}^{p,H}_{t+1}+\frac{1}{2}\left[(1-\gamma)\hat{r}^{p,H}_{t+1}\right]^2\right]
\end{aligned} \tag{72}$$

where:

$$\psi'=1-\beta\left(1-\psi\right)\exp\left[(1-\gamma)\bar{r}\right] \tag{73}$$

3.2 First order terms

Focusing on the first-order terms in (72), we have

$$\begin{aligned} v(1) + H_{1,H} S_t(1) &= (1-\psi')\, E_t \left[v(1) + H_{1,H} S_{t+1}(1) + (1-\gamma) r_{t+1}^{p,H}(1) \right] \\ &\quad + \psi' E_t (1-\gamma) r_{t+1}^{p,H}(1) \\ &= (1-\psi')\left(H_{1,H} N_1 S_t(1) + v(1) \right) + (1-\gamma) r_s S_t(1) \end{aligned}$$

where we used (49) and the first order equivalent of (70), namely: $E_t r_{t+1}^{p,H}(1) = r_s S_t(1)$. This clearly implies that:

$$\begin{aligned} v(1) &= 0 \\ H_{1,H} &= (1-\gamma) r_s \left(I - (1-\psi') N_1 \right)^{-1} \end{aligned} \tag{74}$$

where I is a 3x3 identity matrix.

3.3 Second order terms

Now take the second-order terms in (72).

$$\begin{aligned} & H_{1,H} S_t(2) + \frac{1}{2} S_t(1)' \left(H_{2,H} + H_{1,H}' H_{1,H} \right) S_t(1) + v(2) \\ = \; & (1-\psi') E_t \left(H_{1,H} S_{t+1}(2) + v(2) + \frac{1}{2} S_{t+1}(1)' H_{2,H} S_{t+1}(1) \right) + (1-\gamma) E_t r_{t+1}^{p,H}(2) \\ & + \frac{1}{2}(1-\psi') E_t \left(H_{1,H} S_{t+1}(1) + (1-\gamma) r_{t+1}^{p,H}(1) \right)^2 + \frac{1}{2} \psi' E_t \left[(1-\gamma) r_{t+1}^{p,H}(1) \right]^2 \end{aligned}$$

Using (70) and (66) the second order terms become:

$$\begin{aligned} & H_{1,H} S_t(2) + \frac{1}{2} S_t(1)' \left(H_{2,H} + H_{1,H}' H_{1,H} \right) S_t(1) + \psi' v_2 \\ = \; & (1-\psi') H_{1,H} N_1 S_t(2) + (1-\psi') H_{1,H} N_6 \sigma^2 + S_t(1)' F_1 S_t(1) + f_1 \sigma^2 \\ & + \frac{1}{2}(1-\psi') S_t(1)' N_1' H_{2,H} N_1 S_t(1) + f_2 \sigma^2 \\ & + (1-\gamma) r_s S_t(2) + (1-\gamma) S_t(1)' r_{ss} S_t(1) + (1-\gamma) \hat{r} \sigma^2 \\ & + \frac{1}{2}(1-\psi') E_t S_{t+1}(1)' H_{1,H}' H_{1,H} S_{t+1}(1) \\ & + \frac{1}{2}(1-\gamma)^2 E_t \left(r_{t+1}^{p,H}(1) \right)^2 + (1-\psi')(1-\gamma) E_t S_{t+1}(1)' H_{1,H}' r_{t+1}^{p,H}(1) \end{aligned} \tag{75}$$

where

$$\begin{aligned}
F_1 &= (1-\psi')\sum_{v=1}^{3} H_{1,H}(v)N_{3,v} \\
f_1 &= (1-\psi')\text{trace}\left[\sum_{v=1}^{3} H_{1,H}(v)N_{4,v}\right] \\
f_2 &= \frac{1}{2}(1-\psi')\text{trace}\left[N_2' H_{2,H} N_2\right]
\end{aligned}$$

The first-order component of the portfolio return is

$$r_{t+1}^{p,H}(1) = k(0)r_{H,t+1}(1) + (1-k(0))r_{F,t+1}(1) + p_t(1) - p_{t+1}(1) \tag{76}$$

Using

$$\begin{aligned}
r_{H,t+1}(1) &= -q_{H,t}(1) + r_q q_{H,t+1}(1) + r_a a_{H,t+1}(1) \\
r_{F,t+1}(1) &= -q_{F,t}(1) + r_q q_{F,t+1}(1) + r_a a_{F,t+1}(1) + r_a p_{F,t+1}(1)
\end{aligned}$$

the first-order component of the portfolio return can be written as

$$r_{t+1}^{p,H}(1) = r_s S_t(1) + r_E \epsilon_{t+1} \tag{77}$$

where r_s is as in (70) and:

$$r_E = k(0)r_q q_s^H N_2 + (1-k(0))\left(r_q q_s^F N_2 + r_a p_s N_2\right) + \tilde{r} - (1-\alpha)p_s N_2$$

with

$$\tilde{r} = \begin{bmatrix} k(0)r_a & (1-k(0))r_a \end{bmatrix}$$

Using the first order solution for $r_{t+1}^{p,H}(1)$ the last three terms in (75) become:

$$\begin{aligned}
& \frac{1}{2}(1-\psi')E_t S_{t+1}(1)' H_{1,H}' H_{1,H} S_{t+1}(1) \\
& + \frac{1}{2}(1-\gamma)^2 E_t\left(r_{t+1}^{p,H}(1)\right)^2 + (1-\psi')(1-\gamma)E_t S_{t+1}(1)' H_{1,H}' r_{t+1}^{p,H}(1) \\
= \; & \frac{1}{2}(1-\psi')S_t(1)' N_1' H_{1,H}' H_{1,H} N_1 S_t(1) + f_3\sigma^2 + 0.5(1-\gamma)^2 r_E r_E' \sigma^2 \\
& + (1-\psi')(1-\gamma)S_t(1)' N_1' H_{1,H}' r_s S_t(1) + f_4\sigma^2 \\
& + \frac{1}{2}(1-\gamma)^2 S_t(1)' r_s' r_s S_t(1)
\end{aligned}$$

where

$$\begin{aligned} f_3 &= \frac{1}{2}(1-\psi')\text{trace}\left[N_2' H_{1,H}' H_{1,H} N_2\right] \\ f_4 &= (1-\psi')(1-\gamma)\text{trace}\left[N_2' H_{1,H}' r_E\right] \end{aligned}$$

Using these results along with (74), (75) becomes:

$$\begin{aligned} & \frac{1}{2} S_t(1)' \left(H_{2,H} + H_{1,H}' H_{1,H}\right) S_t(1) + \psi' v(2) \\ = & \frac{1}{2} S_t(1)' \left[\begin{array}{c} (1-\psi') N_1' H_{2,H} N_1 + 2F_1 + 2(1-\gamma) r_{ss} + \\ +(1-\psi') N_1' H_{1,H}' H_{1,H} N_1 + 2(1-\psi')(1-\gamma) N_1' H_{1,H}' r_s + \\ +(1-\gamma)^2 r_s' r_s \end{array} \right] S_t(1) \\ & + \left[(1-\psi') H_{1,H} N_6 + f_1 + f_2 + f_3 + f_4 + (1-\gamma)\hat{r} + 0.5(1-\gamma)^2 r_E r_E'\right] \sigma^2 \end{aligned}$$

This implies:

$$\psi' v(2) = \left(\begin{array}{c} (1-\psi') H_{1,H} N_6 + f_1 + f_2 + f_3 + f_4 \\ +(1-\gamma)\hat{r} + 0.5(1-\gamma)^2 r_E r_E' \end{array} \right) \sigma^2$$

and:

$$H_{2,H} = (1-\psi') N_1' H_{2,H} N_1 + H_3 \qquad (78)$$

where

$$\begin{aligned} H_3 &= -H_{1,H}' H_{1,H} + 2F_1 + 2(1-\gamma) r_{ss} + (1-\psi') N_1' H_{1,H}' H_{1,H} N_1 \\ & \quad +2(1-\psi')(1-\gamma) N_1' H_{1,H}' r_s + (1-\gamma)^2 r_s' r_s \end{aligned}$$

To solve for $H_{2,H}$ from (78) we write it in vector notation:

$$H_{2,H}^{vec} = \begin{bmatrix} H_{2,H,1,1} \\ H_{2,H,1,2} \\ H_{2,H,1,3} \\ H_{2,H,2,1} \\ H_{2,H,2,2} \\ H_{2,H,2,3} \\ H_{2,H,3,1} \\ H_{2,H,3,2} \\ H_{2,H,3,3} \end{bmatrix}$$

where $H_{2,H,x,y}$ is element (x, y) of matrix $H_{2,H}$, and vec denotes the vectorization of a matrix.

The element (i,j) of $N_1' H_{2,H} N_1$ is

$$\hat{n}_i' H_{2,H} \hat{n}_j = \left[\left(\hat{n}_i \hat{n}_j' \right)^{vec} \right]' H_{2,H}^{vec} \equiv n_{i,j} H_{2,H}^{vec}$$

where $\hat{n}_i$ is column i of matrix N_1. We write

$$\hat{N} = \begin{bmatrix} n_{1,1} \\ .. \\ n_{1,3} \\ .. \\ n_{3,1} \\ .. \\ n_{3,3} \end{bmatrix}$$

It then follows from (78) that

$$H_{2,H}^{vec} = (1 - \psi') \hat{N} H_{2,H}^{vec} + H_3^{vec}$$

which implies:

$$H_{2,H}^{vec} = (I - (1 - \psi')\hat{N})^{-1} H_3^{vec} \tag{79}$$

where I is a 9x9 identity matrix.

4 Second and third-order components of the optimal portfolio equations

The Euler equations for optimal portfolio choice are used to solve for the difference in portfolio shares. Using (73) and $v(1) = 0$, the Home and Foreign portfolio Euler equations (equations (53) and (54) in Appendix A of the paper) become

$$\begin{aligned} & E_t \left[(1-\psi') e^{v(2)+v(3)+f_H(S_{t+1})} + \psi' \right] e^{-\gamma r_{t+1}^{p,H} + r_{H,t+1}^{H}} \\ = \; & E_t \left[(1-\psi') e^{v(2)+v(3)+f_H(S_{t+1})} + \psi' \right] e^{-\gamma r_{t+1}^{p,H} + r_{F,t+1}^{H} - \tau} \end{aligned} \tag{80}$$

and

$$\begin{aligned} & E_t \left[(1-\psi') e^{v(2)+v(3)+f_F(S_{t+1})} + \psi' \right] e^{-\gamma r_{t+1}^{p,F} + r_{H,t+1}^{H} - \tau} \\ = \; & E_t \left[(1-\psi') e^{v(2)+v(3)+f_F(S_{t+1})} + \psi' \right] e^{-\gamma r_{t+1}^{p,F} + r_{F,t+1}^{F}} \end{aligned}$$

where elements of v higher than third order are omitted as they are not relevant for the analysis of second and third-order terms that follows.

A first order expansion of either relation shows that the expected excess Return is zero to a first order:

$$E_t\left(r_{H,t+1}(1) - r_{F,t+1}(1)\right) = 0$$

4.1 Second order component of optimal portfolio equations

The second-order component of the Home portfolio Euler equation (80) is

$$\begin{aligned} 0 &= E_t(r_{H,t+1}(2) - r_{F,t+1}(2) + \tau) + \frac{1}{2}E_t\left(r^H_{H,t+1}(1)\right)^2 - \frac{1}{2}E_t\left(r^H_{F,t+1}(1)\right)^2 \\ &\quad -\gamma E_t r^{p,H}_{t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) + \\ &\quad (1-\psi')E_t H_{1,H} S_{t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) \end{aligned}$$

Since $r^H_{H,t+1} = r_{H,t+1} + p_t - p_{t+1}$ and $r^H_{F,t+1} = r_{F,t+1} + p_t - p_{t+1}$, it follows that

$$\begin{aligned} &\frac{1}{2}E_t\left(r^H_{H,t+1}(1)\right)^2 - \frac{1}{2}E_t\left(r^H_{F,t+1}(1)\right)^2 \\ &= \frac{1}{2}E_t\left(r_{H,t+1}(1)\right)^2 - \frac{1}{2}E_t\left(r_{F,t+1}(1)\right)^2 + E_t(p_t(1) - p_{t+1}(1))(r_{H,t+1}(1) - r_{F,t+1}(1)) \end{aligned}$$

so that the second-order component of the Home portfolio Euler equation becomes

$$\begin{aligned} 0 &= E_t(r_{H,t+1}(2) - r_{F,t+1}(2) + \tau) + \frac{1}{2}E_t(r_{H,t+1}(1))^2 - \frac{1}{2}E_t(r_{F,t+1}(1))^2 \\ &+E_t\left(p_t(1) - p_{t+1}(1) - \gamma r^{p,H}_{t+1}(1) + (1-\psi')H_{1,H}S_{t+1}(1)\right)(r_{H,t+1}(1) - r_{F,t+1}(1)) \end{aligned} \tag{81}$$

Following similar steps for the Foreign optimal portfolio condition we get

$$\begin{aligned} 0 &= E_t(r_{H,t+1}(2) - r_{F,t+1}(2) - \tau) + \frac{1}{2}E_t(r_{H,t+1}(1))^2 - \frac{1}{2}E_t(r_{F,t+1}(1))^2 \\ &+E_t\left(p^*_t(1) - p^*_{t+1}(1) - \gamma r^{p,F}_{t+1}(1) + (1-\psi')H_{1,F}S_{t+1}(1)\right)(r_{H,t+1}(1) - r_{F,t+1}(1)) \end{aligned} \tag{82}$$

Taking the difference between (81) and (82) we get

$$\begin{aligned} 0 &= 2\tau - E_t\left((p_{t+1}(1) - p_t(1)) - (p_{t+1}(1)^* - p^*_t(1))\right)(r_{H,t+1}(1) - r_{F,t+1}(1)) \\ &\quad -\gamma E_t\left(r^{p,H}_{t+1}(1) - r^{p,F}_{t+1}(1)\right)(r_{H,t+1}(1) - r_{F,t+1}(1)) \\ &\quad +(1-\psi')E_t(H_{1,H} - H_{1,F})S_{t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) \end{aligned}$$

Since the first-order components of Home and Foreign portfolio returns are

$$
\begin{aligned}
r^{p,H}_{t+1}(1) &= k(0)(r_{H,t+1}(1) - r_{F,t+1}(1)) + r_{F,t+1}(1) + p_t(1) - p_{t+1}(1) \\
r^{p,F}_{t+1}(1) &= -k(0)\left(r_{H,t+1}(1) - r_{F,t+1}(1)\right) + r_{H,t+1}(1) + p^*_t(1) - p^*_{t+1}(1)
\end{aligned}
$$

we have

$$
\begin{aligned}
&r^{p,H}_{t+1}(1) - r^{p,F}_{t+1}(1) = (2k(0)-1)(r_{H,t+1}(1) - r_{F,t+1}(1)) + \\
&(p_t(1) - p_{t+1}(1)) - (p^*_t(1) - p^*_{t+1}(1))
\end{aligned}
$$

Use that the first-order solution of the return differential is $r_{H,t+1}(1) - r_{F,t+1}(1) = r_\epsilon \epsilon_{t+1}$, where

$$
r_\epsilon = \left[\begin{array}{cc} r_{DE} & -r_{DE} \end{array} \right]
$$

Also using (74), the second-order component of the difference between the Home and Foreign portfolio Euler equations then becomes

$$
\begin{aligned}
0 \;=\; & 2\tau + (\gamma-1)E_t\left((p_{t+1}(1) - p_t(1)) - (p^*_{t+1}(1) - p^*_t(1))\right)(r_{H,t+1}(1) - r_{F,t+1}(1)) \\
& -\gamma(2k(0)-1)var(r_{H,t+1}(1) - r_{F,t+1}(1)) + (1-\psi')\sigma^2(H_{1,H} - H_{1,F})N_2 r'_\epsilon
\end{aligned}
$$

We can then solve for $k(0)$ as

$$
\begin{aligned}
k(0) \;=\; & \frac{1}{2} + \frac{\tau}{\gamma var(r_{H,t+1}(1) - r_{F,t+1}(1))} + \\
& + \frac{1}{2}\frac{\gamma-1}{\gamma}\frac{E_t\left(p_{t+1}(1) - p^*_{t+1}(1)\right)(r_{H,t+1}(1) - r_{F,t+1}(1))}{var(r_{H,t+1}(1) - r_{F,t+1}(1))} \\
& + \frac{1}{2}\frac{(1-\psi')\sigma^2(H_{1,H} - H_{1,F})N_2 r'_\epsilon}{\gamma var(r_{H,t+1}(1) - r_{F,t+1}(1))}
\end{aligned} \tag{83}
$$

One can also think of this as a solution of the zero-order component of the difference in portfolio shares, which is $2k(0) - 1$.

4.2 Second-order expected excess return

The solution of $k(0)$ is based on the difference between the second-order components of the Home and Foreign portfolio Euler equations. Given the solution for $k(0)$ we now return to the second-order component of the Home portfolio Euler equation (81) in order to solve for the second-order component of the expected excess return. We start by writing

$$
\begin{aligned}
&\frac{1}{2}E_t(r_{H,t+1}(1))^2 - \frac{1}{2}E_t(r_{F,t+1}(1))^2 = \\
&\frac{1}{2}E_t(r_{H,t+1}(1) + r_{F,t+1}(1))(r_{H,t+1}(1) - r_{F,t+1}(1)) = \\
&E_t r^A_{t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) = \\
&E_t\left(a^A_{t+1}(1) + \frac{1}{2}p_{F,t+1}(1)\right)(r_{H,t+1}(1) - r_{F,t+1}(1)) \\
&\quad - \left(a^A_t(1) + \frac{1}{2}p_{F,t}(1)\right)E_t(r_{H,t+1}(1) - r_{F,t+1}(1)) = \\
&\frac{1}{2}E_t p_{F,t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1))
\end{aligned}
$$

where we used (25), the fact that the first-order expected excess return is zero and that ϵ^D_{t+1} is uncorrelated with ϵ^A_{t+1}. In addition we write

$$
\begin{aligned}
&E_t r^{p,H}_{t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) = \\
&\quad k(0)var(r_{H,t+1}(1) - r_{F,t+1}(1)) + E_t r_{F,t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) \\
&\quad +E_t(p_t(1) - p_{t+1}(1))(r_{H,t+1}(1) - r_{F,t+1}(1))
\end{aligned}
$$

Using $r_{F,t+1}(1) = r^A_{t+1}(1) - 0.5r^D_{t+1}(1) = a^A_{t+1}(1) - a^A_{t+1}(1) + 0.5(p_{F,t+1}(1) - p_{F,t}(1)) - 0.5r^D_{t+1}(1)$, we get:

$$
\begin{aligned}
&E_t r_{F,t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) = \\
&\quad -\frac{1}{2}var(r_{H,t+1}(1) - r_{F,t+1}(1)) + \frac{1}{2}E_t p_{F,t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1))
\end{aligned}
$$

The expected product of the portfolio return and excess return is then

$$
\begin{aligned}
E_t r^{p,H}_{t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) \;&=\; \frac{2k(0)-1}{2}var(r_{H,t+1}(1) - r_{F,t+1}(1)) \\
&\quad +\frac{2\alpha-1}{2}E_t p_{F,t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1))
\end{aligned}
$$

Using these results, (81) becomes

$$\begin{aligned} 0 &= E_t(r_{H,t+1}(2) - r_{F,t+1}(2) + \tau) + \\ &\quad (1-\gamma)\frac{2\alpha - 1}{2} E_t p_{F,t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) \\ &\quad -\gamma \frac{2k(0) - 1}{2} var(r_{H,t+1}(1) - r_{F,t+1}(1)) + \\ &\quad (1-\psi')H_{1,H} E_t S_{t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) \end{aligned}$$

Using (83) we get

$$E_t(r_{H,t+1}(2) - r_{F,t+1}(2)) = -\frac{1}{2}(1-\psi')(H_{1,H} + H_{1,F})\sigma^2 N_2 r'_\epsilon \tag{84}$$

(84) shows that the expected excess return is zero to a second order. This can be seen as follows. Because of symmetry, the first two elements of $H_{1,H}$ are equal to minus the first two elements of $H_{1,F}$ as they multiply cross-country differentials. The last element of $H_{1,H}$ is the same as the last element of $H_{1,F}$ as they apply to the worldwide shock. The first two elements of $H_{1,H} + H_{1,F}$ are then zero. Writing w_{DE} as the coefficient multiplying ϵ^D_{t+1} in the first-order solution for w^D_{t+1}, we have

$$\begin{aligned} N_2 r'_\epsilon &= \begin{bmatrix} 1 & -1 \\ w_{DE} & -w_{DE} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} r_{DE} \\ -r_{DE} \end{bmatrix} \\ &= r_{DE} \begin{pmatrix} 2 \\ 2w_{DE} \\ 0 \end{pmatrix} \end{aligned}$$

Since the first two elements of $H_{1,H} + H_{1,F}$ are zero, it follows that $(H_{1,H} + H_{1,F})N_2 r'_\epsilon = 0$ and therefore $E_t(r_{H,t+1}(2) - r_{F,t+1}(2)) = 0$.

4.3 Third order component of Home's optimal portfolio equation

We will denote $\hat{x} = x - x(0)$ for any variable x. A third-order Taylor expansion of the left-hand side of (80), treating the sum of terms in the exponentials as one variable, is equal to (ignoring the multiplication constant $e^{(1-\gamma)\bar{r}}$)

$$\begin{aligned}
&(1-\psi')\left(f_H(S_{t+1}) - \gamma\hat{r}^{p,H}_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1} + v(2) + v(3)\right) \\
&+\frac{1}{2}(1-\psi')\left(f_H(S_{t+1}) - \gamma\hat{r}^{p,H}_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1} + v(2) + v(3)\right)^2 \\
&+\frac{1}{6}(1-\psi')\left(f_H(S_{t+1}) - \gamma\hat{r}^{p,H}_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1} + v(2) + v(3)\right)^3 \\
&+\psi'\left(-\gamma\hat{r}^{p,H}_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1}\right) \\
&+\frac{1}{2}\psi'\left(-\gamma\hat{r}^{p,H}_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1}\right)^2 \\
&+\frac{1}{6}\psi'\left(-\gamma\hat{r}^{p,H}_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1}\right)^3
\end{aligned}$$

The right hand side of (80) is the same except that $\hat{r}_{H,t+1}$ is replaced by $\hat{r}_{F,t+1} - \tau$. Combining both sides of (80) we get

$$\begin{aligned}
0 \;=\; & E_t(\hat{r}_{H,t+1} - \hat{r}_{F,t+1} + \tau) \\
& +\frac{1}{2}E_t\left((\hat{r}_{H,t+1})^2 - (\hat{r}_{F,t+1} - \tau)^2\right) \\
& +(1-\psi')E_t(\hat{r}_{H,t+1} - \hat{r}_{F,t+1} + \tau)\left(f_H(S_{t+1}) + v(2) + v(3)\right) \\
& +E_t(\hat{r}_{H,t+1} - \hat{r}_{F,t+1} + \tau)(-\gamma\hat{r}^{p,H}_{t+1} + \hat{p}_t - \hat{p}_{t+1}) \\
& +O_3
\end{aligned} \tag{85}$$

where

$$\begin{aligned}
O_3 \;=\; & \frac{1}{6}E_t\left((\hat{r}_{H,t+1})^3 - (\hat{r}_{F,t+1})^3\right) + \\
& +\frac{1}{2}E_t\left((\hat{r}_{H,t+1})^2 - (\hat{r}_{F,t+1})^2\right)\left(-\gamma\hat{r}^{p,H}_{t+1} + \hat{p}_t - \hat{p}_{t+1} + (1-\psi')f_H(S_{t+1})\right) \\
& +\frac{1}{2}(1-\psi')E_t(\hat{r}_{H,t+1} - \hat{r}_{F,t+1})\left(f_H(S_{t+1}) - \gamma\hat{r}^{p,H}_{t+1} + \hat{p}_t - \hat{p}_{t+1}\right)^2 \\
& +\frac{1}{2}\psi' E_t(\hat{r}_{H,t+1} - \hat{r}_{F,t+1})\left(-\gamma\hat{r}^{p,H}_{t+1} + \hat{p}_t - \hat{p}_{t+1}\right)^2
\end{aligned} \tag{86}$$

In the expression for O_3 we have omitted τ, $v(2)$ and $v(3)$ since when multiplied with other terms they lead to fourth and higher order terms.

The third-order component of (85) is equal to

$$
\begin{aligned}
0 &= E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) + O_3 && (87)\\
&+ \hat{cov}_t\left(r_{H,t+1} - r_{F,t+1}, r^A_{t+1} + (1-\psi')f_H(S_{t+1}) - \gamma r^{p,H}_{t+1} + p_t - p_{t+1}\right)\\
&+ \tau E_t\left(r^A_{t+1}(1) + (1-\psi')H_{1,H}S_{t+1}(1) - \gamma r^{p,H}_{t+1}(1) + p_t(1) - p_{t+1}(1)\right)
\end{aligned}
$$

where $\hat{cov}_t$ is defined as $\hat{cov}_t(x,y) = E_t x(1)y(2) + E_t x(2)y(1)$ and O_3 is equal to (86) after replacing each variable with its first-order component. $v(2)$ drops out as it only enters the third-order component of (85) when multiplied with the first-order expected excess return, which is zero.

We now simplify the cubic terms in (86) by substituting the first-order solution of all variables. Using that $r_{H,t+1}(1) = r^A_{t+1}(1) + 0.5r^D_{t+1}(1)$ and $r_{F,t+1}(1) = r^A_{t+1}(1) - 0.5r^D_{t+1}(1)$, we have

$$
E_t\left((r_{H,t+1}(1))^3 - (r_{F,t+1}(1))^3\right) = E_t\left(\frac{1}{4}(r^D_{t+1}(1))^3 + 3(r^A_{t+1}(1))^2 r^D_{t+1}(1)\right)
$$

Since $r^D_{t+1}(1) = r_{DE}\epsilon^D_{t+1}$, the first term of the above expression is zero because $E_t(\epsilon^D_{t+1})^3 = 0$. Therefore

$$
E_t\left(r_{H,t+1}(1)^3 - r_{F,t+1}(1)^3\right) = 3E_t(r^A_{t+1}(1))^2 r^D_{t+1}(1) \qquad (88)
$$

Define

$$
l_{t+1}(1) = r^A_{t+1}(1) - \gamma r^{p,H}_{t+1}(1) + p_t(1) - p_{t+1}(1)
$$

Then (86) becomes

$$
\begin{aligned}
O_3 &= \frac{1}{2}(1-\psi')E_t(r_{H,t+1}(1) - r_{F,t+1}(1))\left(l_{t+1}(1) + H_{1,H}S_{t+1}(1)\right)^2\\
&+ \frac{1}{2}\psi' E_t(r_{H,t+1}(1) - r_{F,t+1}(1))(l_{t+1}(1))^2 && (89)
\end{aligned}
$$

We can simplify this further by using the second-order component of Home's optimal portfolio equation, equation (81), which we write as

$$
\begin{aligned}
&E_t(r_{H,t+1}(2) - r_{F,t+1}(2) + \tau) +\\
&E_t(r_{H,t+1}(1) - r_{F,t+1}(1))\left(l_{t+1}(1) + (1-\psi')H_{1,H}S_{t+1}(1)\right) = 0
\end{aligned}
$$

We have shown that $E_t(r_{H,t+1}(2) - r_{F,t+1}(2)) = 0$. After substituting the first order solution for the variables we then must have

$$
r_{DE}E_t\left(l_{t+1}(1) + (1-\psi')H_{1,H}S_{t+1}(1)\right)\epsilon^D_{t+1} = -\tau
$$

The variables in the big parentheses depend on $S_t(1)$, ϵ^A_{t+1} and ϵ^D_{t+1}:

$$l_{t+1}(1) + (1-\psi')H_{1,H}S_{t+1}(1) = A_H S_t(1) + B_H \epsilon^D_{t+1} + C_H \epsilon^A_{t+1} \tag{90}$$

As $E_t \epsilon^D_{t+1} = E_t \epsilon^A_{t+1}\epsilon^D_{t+1} = 0$ we get:

$$2B_H r_{DE}\sigma^2 = -\tau \tag{91}$$

as $E_t\left(\epsilon^D_{t+1}\right)^2 = 2\sigma^2$.

Next use the fact that

$$\begin{aligned} &H_{1,H}S_{t+1}(1) = H_{1,H}N_1 S_t(1) + H_{1,H}N_2\epsilon_{t+1} = \\ &f_{HS}S_t(1) + f_{HD}\epsilon^D_{t+1} + f_{HA}\epsilon^A_{t+1} \end{aligned} \tag{92}$$

where:

$$\begin{aligned} f_{HS} &= H_{1,H}N_1 \\ f_{HD} &= H_{1,H,1} + H_{1,H,2}w_{DE} \\ f_{HA} &= H_{1,H,3} \end{aligned}$$

where we used:

$$\begin{aligned} H_{1,H}N_2\epsilon_{t+1} &= \begin{bmatrix} H_{1,H,1} & H_{1,H,2} & H_{1,H,3} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ w_{DE} & -w_{DE} \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} \epsilon_{Ht+1} \\ \epsilon_{Ft+1} \end{bmatrix} \\ &= \begin{bmatrix} H_{1,H,1} & H_{1,H,2} & H_{1,H,3} \end{bmatrix} \begin{bmatrix} \epsilon^D_{t+1} \\ w_{DE}\epsilon^D_{t+1} \\ \epsilon^A_{t+1} \end{bmatrix} \end{aligned}$$

Substituting (90) and (92) into (89), we get (recall that $E_t(\epsilon^D_{t+1})^3 = E_t(\epsilon^A_{t+1})^3 = E_t(\epsilon^A_{t+1})^2\epsilon^D_{t+1} = E_t\epsilon^A_{t+1}(\epsilon^D_{t+1})^2 = E_t\epsilon^A_{t+1}\epsilon^D_{t+1} = 0$)

$$O_3 = 2\sigma^2 r_{DE}\left(B_H A_H + \psi'(1-\psi')f_{HD}f_{HS}\right)S_t(1) \tag{93}$$

Using (93) the third order expansion of the Home optimal portfolio (87) is written as

$$\begin{aligned} 0 = \; & E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) \\ & + 2\sigma^2 r_{DE}\left(B_H A_H + \psi'(1-\psi')f_{HD}f_{HS}\right)S_t(1) \\ & + \hat{cov}_t\left(r_{H,t+1} - r_{F,t+1}, r^A_{t+1} + (1-\psi')f_H(S_{t+1}) - \gamma r^{p,H}_{t+1} + p_t - p_{t+1}\right) \\ & + \tau E_t\left(r^A_{t+1}(1) + (1-\psi')H_{1,H}S_{t+1}(1) - \gamma r^{p,H}_{t+1}(1) + p_t(1) - p_{t+1}(1)\right) \end{aligned}$$

Focusing for a moment on the last term, using (90), (91) and $E_t\epsilon_{t+1}^D = E_t\epsilon_{t+1}^A = 0$, we write:

$$\tau E_t\left(r_{t+1}^A(1) + (1-\psi')H_{1,H}S_{t+1}(1) - \gamma r_{t+1}^{p,H}(1) + p_t(1) - p_{t+1}(1)\right) =$$
$$\tau A_H S_t(1) = -2B_H r_{DE}\sigma^2 A_H S_t(1)$$

The third-order component of the Home optimal portfolio condition then simplifies to

$$\begin{aligned} 0 &= E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) + 2\sigma^2 r_{DE}\psi'(1-\psi')f_{HD}f_{HS}S_t(1) \\ &\quad + \hat{cov}_t\left(r_{H,t+1} - r_{F,t+1}, r_{t+1}^A + (1-\psi')f_H(S_{t+1}) - \gamma r_{t+1}^{p,H} - p_{t+1}\right) \end{aligned} \tag{94}$$

where p_t is preset and can be ommitted from $\hat{cov}_t$.

4.4 Combining the Home and Foreign optimal portfolio equations

Following similar steps for the Foreign country and writing (analogous to (92) for the Home country)

$$H_{1,F}S_{t+1}(1) = f_{FS}S_t(1) + f_{FD}\epsilon_{t+1}^D + f_{FA}\epsilon_{t+1}^A \tag{95}$$

the third-order component of the optimal portfolio equation for the Foreign country is

$$\begin{aligned} 0 &= E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) + 2\sigma^2 r_{DE}\psi'(1-\psi')f_{FD}f_{FS}S_t(1) \\ &\quad + \hat{cov}_t\left(r_{t+1}^D, r_{t+1}^A + (1-\psi')f_F(S_{t+1}) - \gamma r_{t+1}^{p,F} - p_{t+1}^*\right) \end{aligned}$$

where $r_{t+1}^D = r_{H,t+1} - r_{F,t+1}$.

Taking the difference between the third-order component of the Home and Foreign portfolio Euler equations we have

$$\begin{aligned} 0 &= 2\sigma^2 r_{DE}\psi'(1-\psi')(f_{HD}f_{HS} - f_{FD}f_{FS})S_t(1) \\ &\quad + \hat{cov}_t\left(r_{t+1}^D, (1-\psi')(f_H(S_{t+1}) - f_F(S_{t+1})) - \gamma(r_{t+1}^{p,H} - r_{t+1}^{p,F}) - (p_{t+1} - p_{t+1}^*)\right) \end{aligned} \tag{96}$$

The second-order component of Home and Foreign portfolio returns are

$$\begin{aligned} r^{p,H}_{t+1}(2) &= k(0)r_{H,t+1}(2) + (1-k(0))(r_{F,t+1}(2)-\tau) + p_t(2) - p_{t+1}(2) \\ &+\frac{1}{2}k(0)(1-k(0))(r^D_{t+1}(1))^2 + r^D_{t+1}(1)k^H_{H,t}(1) \qquad (97) \\ r^{p,F}_{t+1}(2) &= (1-k(0))\,(r_{H,t+1}(2)-\tau) + k(0)r_{F,t+1}(2) + p^*_t(2) - p^*_{t+1}(2) \\ &+\frac{1}{2}k(0)(1-k(0))(r^D_{t+1}(1))^2 - r^D_{t+1}(1)k^F_{F,t}(1) \qquad (98) \end{aligned}$$

The difference is

$$\begin{aligned} r^{p,H}_{t+1}(2) - r^{p,F}_{t+1}(2) &= (2k(0)-1)r^D_{t+1}(2) + (p_t(2)-p^*_t(2)) - \big(p_{t+1}(2)-p^*_{t+1}(2)\big) \\ &+r^D_{t+1}(1)k^D_t(1) \end{aligned}$$

The first-order component of the portfolio return difference is

$$r^{p,H}_{t+1}(1) - r^{p,F}_{t+1}(1) = (2k(0)-1)r^D_{t+1}(1) + (p_t(1)-p^*_t(1)) - \big(p_{t+1}(1)-p^*_{t+1}(1)\big)$$

Substituting the first and second-order components of portfolio return differences into (96), we have

$$\begin{aligned} 0 &= 2\sigma^2 r_{DE}\psi'(1-\psi')(f_{HD}f_{HS} - f_{FD}f_{FS})S_t(1) \\ &+(1-\psi')c\hat{o}v_t\left(r^D_{t+1}, f_H(S_{t+1}) - f_F(S_{t+1})\right) \\ &-\gamma(2k(0)-1)v\hat{a}r_t(r^D_{t+1}) \qquad (99) \\ &+(\gamma-1)\,c\hat{o}v_t\left(r^D_{t+1}, p_{t+1}-p^*_{t+1}\right) \\ &-\gamma k^D_t(1)var(r^D_{t+1}(1)) \end{aligned}$$

where $v\hat{a}r_t(x) = 2E_t x(1)x(2)$ and $var(r^D_{t+1}(1)) = 2r^2_{DE}\sigma^2$.

We can use this to solve for $k^D_t(1)$:

$$\begin{aligned} k^D_t(1) &= -(2k(0)-1)\frac{v\hat{a}r_t(r^D_{t+1})}{var(r^D_{t+1}(1))} + \frac{\gamma-1}{\gamma}\frac{c\hat{o}v_t(r^D_{t+1}, p_{t+1}-p^*_{t+1})}{var(r^D_{t+1}(1))} \\ &+(1-\psi')\frac{c\hat{o}v_t\left(r^D_{t+1}, f_H(S_{t+1}) - f_F(S_{t+1})\right)}{\gamma var(r^D_{t+1}(1))} \\ &+2\sigma^2 r_{DE}\psi'(1-\psi')\frac{(f_{HD}f_{HS} - f_{FD}f_{FS})S_t(1)}{\gamma var(r^D_{t+1}(1))} \qquad (100) \end{aligned}$$

This corresponds to equation (43) of the paper, where $er_{t+1} = r^D_{t+1}$ and using that from (92) and (95)

$$E_t\left(f_{H,t+1}(1)^2 - f_{F,t+1}(1)^2\right) r^D_{t+1}(1) = 4r_{DE}\sigma^2\left(f_{HS}f_{HD} - f_{FS}f_{FD}\right)$$

where $f_{H,t+1}(1) = H_{1,H}S_{t+1}(1)$ and $f_{F,t+1}(1) = H_{1,F}S_{t+1}(1)$.

4.5 Computing third-order expectations

Let R_1 and R_2 denote, respectively, the first and second-order components of r^D_{t+1}. Similarly, let F_1 and F_2 denote the first and second-order components of $f_H(S_{t+1}) - f_F(S_{t+1})$ and P_1, P_2 the first and second-order components of $p_{t+1} - p^*_{t+1}$. In order to evaluate (100) we then need to compute $E_t R_1 R_2$, $E_t R_1 F_2$, $E_t R_1 P_2$, $E_t R_2 F_1$ and $E_t R_2 P_1$.

The computation of these terms uses the second-order solution to the model. Let's start with the first and second-order terms of r^D_{t+1}. We know that the first order term is $r_{DE}\epsilon^D_{t+1}$. Using the definitions of returns (equations (59)-(60) in Appendix A of the paper), the second-order components are

$$\begin{aligned}
r_{H,t+1}(2) &= -q_{H,t}(2) + r_q q_{H,t+1}(2) + (1-r_q)\, a_{H,t+1}(2) \\
&\quad + \frac{1}{2} r_{qq} \left[(q_{H,t+1}(1))^2 + (a_{H,t+1}(1))^2 - 2q_{H,t+1}(1) a_{H,t+1}(1) \right] \\
r_{F,t+1}(2) &= -q_{F,t}(2) + r_q q_{F,t+1}(2) + (1-r_q)\left(a_{F,t+1}(2) + p_{F,t+1}(2)\right) \\
&\quad + \frac{1}{2} r_{qq} \left[\begin{array}{c} (q_{F,t+1}(1))^2 + (a_{F,t+1}(1))^2 + (p_{F,t+1}(1))^2 + 2a_{F,t+1}(1)p_{F,t+1}(1) \\ -2q_{F,t+1}(1)a_{F,t+1}(1) - 2q_{F,t+1}(1)p_{F,t+1}(1) \end{array} \right]
\end{aligned}$$

where $r_q = (1-\psi)(1-\psi\theta)^{-1}$ and $r_{qq} = r_q(1-r_q)$. The second-order solution for the relative price of Foreign goods and equity prices are (60)-(62).

The second order components in $r_{H,t+1}$ and $r_{F,t+1}$ take four different forms: (i) quadratic in $S_t(1)$, (ii) proportional to σ^2, (iii) quadratic in the innovations ϵ_{t+1} and (iv) product of $S_t(1)$ and innovations. We focus on the difference between $r_{H,t+1}(2)$ and $r_{F,t+1}(2)$. We know that the expected value of this difference is zero. We can then ignore (i) and (ii) as they are known at time t. We can also ignore (iii). The expected value of those terms in R_2 is zero. When multiplying them with first-order terms later on, we can therefore ignore them to the extent that the first-order terms are linear in the state space (the expected product of those terms in R_2 times $S_t(1)$ remains zero.) To the extent that the first order terms are linear in innovations, we can ignore them as well since the expectation of any cubic form of innovations is zero. The only relevant second-order terms are therefore ones that are products of the state space at time t and innovations at $t+1$. We will therefore focus on those terms only.

Using (60)-(62) the second-order component of the excess return is

$$
\begin{aligned}
&R_2 = r_{H,t+1}(2) - r_{F,t+1}(2) = \\
&-\left[\left(q_s^H - q_s^F\right) S_t(2) + S_t(1)' \left(q_{ss}^H - q_{ss}^F\right) S_t(1) + \left(k_q^H - k_q^F\right)\sigma^2\right] \\
&+r_q \left[\left(q_s^H - q_s^F\right) S_{t+1}(2) + S_{t+1}(1)' \left(q_{ss}^H - q_{ss}^F\right) S_{t+1}(1) + \left(k_q^H - k_q^F\right)\sigma^2\right] \\
&+(1-r_q)\left[a_{H,t+1}(2) - a_{F,t+1}(2) - p_s S_{t+1}(2) - S_{t+1}(1)' p_{ss} S_{t+1}(1) - k_p \sigma^2\right] \\
&+\frac{1}{2} r_{qq} \left[\begin{array}{c} (q_{H,t+1}(1))^2 + (a_{H,t+1}(1))^2 - 2q_{H,t+1}(1) a_{H,t+1}(1) \\ -(q_{F,t+1}(1))^2 - (a_{F,t+1}(1))^2 - (p_{F,t+1}(1))^2 - 2a_{F,t+1}(1) p_{F,t+1}(1) \\ +2q_{F,t+1}(1) a_{F,t+1}(1) + 2q_{F,t+1}(1) p_{F,t+1}(1) \end{array}\right]
\end{aligned}
$$

The productivity terms are exactly first-order by assumption, hence $a_{H,t+1}(2) = a_{F,t+1}(2) = 0$. Using (49), (66) and

$$
\begin{aligned}
a_{H,t+1}(1) &= \rho a_{H,t}(1) + \epsilon_{H,t+1} \equiv a_s^H S_t(1) + a_E^H \epsilon_{t+1} \\
a_{F,t+1}(1) &= \rho a_{F,t}(1) + \epsilon_{F,t+1} \equiv a_s^F S_t(1) + a_E^F \epsilon_{t+1}
\end{aligned}
$$

the terms in the expression for $r_{H,t+1}(2) - r_{F,t+1}(2)$ that involve the product of $S_t(1)$ and model innovations are

$$
\begin{aligned}
&\left[r_q \left(q_s^H - q_s^F\right) - (1-r_q) p_s\right] \left(\begin{array}{c} S_t(1)' N_{5,1} \epsilon_{t+1} \\ S_t(1)' N_{5,2} \epsilon_{t+1} \\ S_t(1)' N_{5,3} \epsilon_{t+1} \end{array}\right) \\
&+2S_t(1)' N_1' \left[r_q \left(q_{ss}^H - q_{ss}^F\right) - (1-r_q) p_{ss}\right] N_2 \epsilon_{t+1} \\
&+\frac{1}{2} r_{qq} \left[\begin{array}{c} 2S_t(1)' N_1' \left[\left(q_s^H\right)' q_s^H - \left(q_s^F\right)' q_s^F - (p_s)' p_s + \left(q_s^F\right)' p_s + (p_s)' q_s^F\right] N_2 \epsilon_{t+1} \\ +2S_t(1)' \left[\left(a_s^H\right)' a_E^H - \left(a_s^F\right)' a_E^F\right] \epsilon_{t+1} \\ -2S_t(1)' N_1' \left[\left(q_s^H\right)' a_E^H - \left[\left(q_s^F\right)' - (p_s)'\right] a_E^F\right] \epsilon_{t+1} \\ -2S_t(1)' \left[\left(a_s^H\right)' q_s^H - \left(a_s^F\right)' \left[q_s^F - p_s\right]\right] N_2 \epsilon_{t+1} \end{array}\right]
\end{aligned}
$$

Another way to write these terms is

$$
\begin{aligned}
& S_t(1)' \left[\sum_{v=1}^{3} \left[r_q(q_{s,v}^H - q_{s,v}^F) - (1 - r_q)\, p_{s,v} \right] N_{5,v} \right] \epsilon_{t+1} \\
& + 2S_t(1)' N_1' \left[r_q(q_{ss}^H - q_{ss}^F) - (1 - r_q)\, p_{ss} \right] N_2 \epsilon_{t+1} \\
& + r_{qq} S_t(1)' N_1' \left[(q_s^H)' q_s^H - (q_s^F)' q_s^F - (p_s)'\, p_s + \left(q_s^F\right)' p_s + (p_s)'\, q_s^F \right] N_2 \epsilon_{t+1} \\
& + r_{qq} S_t(1)' \left[(a_s^H)' a_E^H - (a_s^F)' a_E^F \right] \epsilon_{t+1} \\
& - r_{qq} S_t(1)' \left[(a_s^H)' q_s^H - (a_s^F)' q_s^F \right] N_2 \epsilon_{t+1} \\
& - r_{qq} S_t(1)' N_1' \left[(q_s^H)' a_E^H - (q_s^F)' a_E^F \right] \epsilon_{t+1} \\
& - r_{qq} S_t(1)' (a_s^F)' p_s N_2 \epsilon_{t+1} - r_{qq} S_t(1)' N_1' \,(p_s)'\, a_E^F \epsilon_{t+1}
\end{aligned}
$$

where $q_{s,x}^H$ is xth element of the vector q_s^H and similarly for $p_{s,x}$. We can write these terms in a compact way as $S_t(1)'\bar{M}\epsilon_{t+1}$. Since $R_1 = r_{DE}\epsilon_{t+1}^D = r_\epsilon \epsilon_{t+1}$, it follows that

$$
E_t R_1 R_2 = \sigma^2 S_t(1)' \bar{M} r_\epsilon' \tag{101}
$$

We next turn to P_2, the second-order component of $p_{t+1} - p_{t+1}^*$. We have

$$
p_{t+1}(2) = (1-\alpha) p_{F,t+1}(2) - \frac{1}{2}\alpha(1-\alpha)(\lambda - 1)\left(p_{F,t+1}(1)\right)^2 \tag{102}
$$

$$
p_{t+1}^*(2) = \alpha p_{F,t+1}(2) - \frac{1}{2}\alpha(1-\alpha)(\lambda - 1)\left(p_{F,t+1}(1)\right)^2 \tag{103}
$$

Combining this with (60) we have

$$
P_2 = p_{t+1}(2) - p_{t+1}^*(2) = (1 - 2\alpha)\left(p_s S_{t+1}(2) + S_{t+1}(1)' p_{ss} S_{t+1}(1) + k_p \sigma^2\right)
$$

Using (49) and (66) this becomes

$$
\begin{aligned}
P_2 &= (1-2\alpha)p_s N_1 S_t(2) + (1-2\alpha)p_s N_6 \sigma^2 \\
&+(1-2\alpha)S_t(1)'\left(\sum_{v=1}^{3} p_{s,v}N_{3,v}\right)S_t(1) \\
&+(1-2\alpha)\epsilon_{t+1}'\left(\sum_{v=1}^{3} p_{s,v}N_{4,v}\right)\epsilon_{t+1} \\
&+(1-2\alpha)S_t(1)'\left(\sum_{v=1}^{3} p_{s,v}N_{5,v}\right)\epsilon_{t+1} \\
&+(1-2\alpha)S_t(1)'N_1' p_{ss} N_1 S_t\,(1) \\
&+(1-2\alpha)\epsilon_{t+1}' N_2' p_{ss} N_2 \epsilon_{t+1} \\
&+(1-2\alpha)2S_t(1)'N_1' p_{ss} N_2 \epsilon_{t+1} \\
&+(1-2\alpha)k_p \sigma^2
\end{aligned}
$$

Recall that $P_1 = (1-2\alpha)p_s\left[N_1 S_t(1) + N_2\epsilon_{t+1}\right]$. Using these results and focusing on the terms in the cross product of $S_t(1)$ and ϵ_{t+1}we have

$$
\begin{aligned}
E_t R_1 P_2 &= 2(1-2\alpha)\sigma^2 S_t(1)'N_1' p_{ss} N_2 r_\epsilon' \\
&\quad +(1-2\alpha)\sigma^2 S_t(1)'\left(\sum_{v=1}^{3} p_{s,v}N_{5,v}\right) r_\epsilon'
\end{aligned}
\tag{104}
$$

$$
E_t R_2 P_1 = (1-2\alpha)\sigma^2 S_t(1)'\bar{M} N_2' p_s' \tag{105}
$$

Finally consider F_2, the second-order component of $f_H(S_{t+1}) - f_F(S_{t+1})$. We have:

$$
F_2 = (H_{1,H} - H_{1,F})S_{t+1}(2) + \frac{1}{2}S_{t+1}(1)'(H_{2,H} - H_{2,F})S_{t+1}(1)
$$

Using (49) and (66) this becomes

$$
\begin{aligned}
&F_2 = \\
&(H_{1,H} - H_{1,F}) N_1 S_t(2) + (H_{1,H} - H_{1,F}) N_6 \sigma^2 \\
&+S_t(1)' \left(\sum_{v=1}^{3} (H_{1,H,v} - H_{1,F,v}) N_{3,v} \right) S_t(1) \\
&+\epsilon_{t+1}' \left(\sum_{v=1}^{3} (H_{1,H,v} - H_{1,F,v}) N_{4,v} \right) \epsilon_{t+1} \\
&+S_t(1)' \left(\sum_{v=1}^{3} (H_{1,H,v} - H_{1,F,v}) N_{5,v} \right) \epsilon_{t+1} \\
&+\frac{1}{2} S_t(1)' N_1' (H_{2,H} - H_{2,F}) N_1 S_t(1) \\
&+\frac{1}{2} \epsilon_{t+1}' N_2' (H_{2,H} - H_{2,F}) N_2 \epsilon_{t+1} \\
&+\frac{1}{2} S_t(1)' N_1' \left[(H_{2,H} - H_{2,F}) + (H_{2,H} - H_{2,F})' \right] N_2 \epsilon_{t+1}
\end{aligned}
$$

Recall that $F_1 = (H_{1,H} - H_{1,F}) [N_1 S_t(1) + N_2 \epsilon_{t+1}]$. Using these results and focusing on the terms in the cross product of $S_t(1)$ and ϵ_{t+1}, it follows that

$$
\begin{aligned}
E_t R_1 F_2 &= \frac{1}{2} \sigma^2 S_t(1)' N_1' \left[(H_{2,H} - H_{2,F}) + (H_{2,H} - H_{2,F})' \right] N_2 r_\epsilon' \\
&\quad + \sigma^2 S_t(1)' \left(\sum_{v=1}^{3} (H_{1,H,v} - H_{1,F,v}) N_{5,v} \right) r_\epsilon' \qquad (106) \\
E_t R_2 F_1 &= \sigma^2 S_t(1)' \bar{M} N_2' (H_{1,H} - H_{1,F})' \qquad (107)
\end{aligned}
$$

To summarize, we have

$$
\begin{aligned}
E_t R_1 R_2 &= \sigma^2 r_\epsilon \bar{M}' S_t(1) \\
E_t R_1 P_2 &= (1 - 2\alpha) \sigma^2 r_\epsilon \left[2 N_2' (p_{ss})' N_1 + \left(\sum_{v=1}^{3} p_{s,v} N_{5,v} \right)' \right] S_t(1) \\
E_t R_2 P_1 &= (1 - 2\alpha) \sigma^2 p_s N_2 \bar{M}' S_t(1) \\
E_t R_1 F_2 &= \sigma^2 r_\epsilon \left[\begin{array}{c} \frac{1}{2} N_2' \left[(H_{2,H} - H_{2,F}) + (H_{2,H} - H_{2,F})' \right] N_1 \\ + \left(\sum_{v=1}^{3} (H_{1,H,v} - H_{1,F,v}) N_{5,v} \right)' \end{array} \right] S_t(1) \\
E_t R_2 F_1 &= \sigma^2 (H_{1,H} - H_{1,F}) N_2 \bar{M}' S_t(1)
\end{aligned}
$$

Using this (100) becomes

$$k_{t+1}^D(1) = k_s S_t(1) \tag{108}$$

where

$$\begin{aligned}
2\gamma r_{DE}^2 k_s &= (1-\psi')r_\epsilon \frac{1}{2} N_2' \left[(H_{2,H} - H_{2,F}) + (H_{2,H} - H_{2,F})'\right] N_1 \\
&+(1-\psi')r_\epsilon \left(\sum_{v=1}^{3}(H_{1,H,v} - H_{1,F,v})N_{5,v}\right)' \\
&+(1-\psi')(H_{1,H} - H_{1,F})N_2\bar{M}' \\
&-2\gamma(2k(0)-1)r_\epsilon \bar{M}' \\
&+(\gamma-1)(1-2\alpha)2r_\epsilon N_2'(p_{ss})'N_1 + (\gamma-1)(1-2\alpha)r_\epsilon \left(\sum_{v=1}^{3} p_{s,v}N_{5,v}\right)' \\
&+(\gamma-1)(1-2\alpha)p_s N_2 \bar{M}' \\
&+2r_{DE}\psi'(1-\psi')(f_{HD}f_{HS} - f_{FD}f_{FS})
\end{aligned} \tag{109}$$

where we used $var(r_{t+1}^D(1)) = 2\sigma^2 r_{DE}^2$.

4.6 Third-order component of expected excess return

We have already shown that the first and second-order components of the expected excess return are zero. We now turn to computing the third-order component of the expected excess return. We start by taking the sum of the third-order component of the Home portfolio Euler equation in (94) and its foreign equivalent:

$$\begin{aligned}
&2E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) = -2\sigma^2 r_{DE}\psi'(1-\psi')\left(f_{HD}f_{HS} + f_{FD}f_{FS}\right)S_t(1) \\
&-\hat{cov}_t\left(r_{t+1}^D, 2r_{t+1}^A + (1-\psi')\left(f_H(S_{t+1}) + f_F(S_{t+1})\right)\right) \\
&+\hat{cov}_t\left(r_{t+1}^D, \gamma\left(r_{t+1}^{p,H} + r_{t+1}^{p,F}\right) + \left(p_{t+1} + p_{t+1}^*\right)\right)
\end{aligned} \tag{110}$$

Taking the sum of (97)-(98) we write:

$$\begin{aligned}
r_{t+1}^{p,H}(2) + r_{t+1}^{p,F}(2) \; &= r_{H,t+1}(2) + r_{F,t+1}(2) - (1-k(0))2\tau \\
&+p_t(2) + p_t^*(2) - p_{t+1}(2) - p_{t+1}^*(2) \\
&+k(0)(1-k(0))(r_{t+1}^D(1))^2 + r_{t+1}^D(1)\left(k_{H,t}^H(1) - k_{F,t}^F(1)\right)
\end{aligned}$$

Similarly:

$$r_{t+1}^{p,H}(1) + r_{t+1}^{p,F}(1) = r_{H,t+1}(1) + r_{F,t+1}(1) + p_t(1) + p_t^*(1) - p_{t+1}(1) - p_{t+1}^*(1)$$

Using these results we write:

$$\hat{cov}_t\left(r_{t+1}^D, r_{t+1}^{p,H} + r_{t+1}^{p,F}\right) = 2\hat{cov}_t(r_{t+1}^D, r_{t+1}^A)$$
$$-\hat{cov}_t(r_{t+1}^D, p_{t+1} + p_{t+1}^*) + 2k_t^A(1)var(r_{t+1}^D(1))$$

where we used $E_t(r_{t+1}^D(1))^3 = 0$. Substituting this result into (110) yields:

$$\begin{aligned} &E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) = -\sigma^2 r_{DE}\psi'(1-\psi')\left(f_{HD}f_{HS} + f_{FD}f_{FS}\right)S_t(1) \\ &-(1-\gamma)\,\hat{cov}_t\left(r_{t+1}^D, r_{t+1}^A\right) + 0.5\,(1-\gamma)\,\hat{cov}_t\left(r_{t+1}^D, p_{t+1} + p_{t+1}^*\right) \\ &-0.5(1-\psi')\hat{cov}_t\left(r_{t+1}^D, f_H(S_{t+1}) + f_F(S_{t+1})\right) \\ &+\gamma k_t^A(1)var_t\left(r_{t+1}^D(1)\right) \end{aligned} \tag{111}$$

The last term can also be written as $2\sigma^2 r_{DE}^2\gamma k_s^A S_t(1)$ where we have written the first-order solution of the average portfolio share as $k_t^A(1) = k_s^A S_t(1)$.

The term involving the consumer price indexes is also relatively easy to compute from (102)-(103):

$$\hat{cov}_t\left(r_{t+1}^D, p_{t+1} + p_{t+1}^*\right) = \frac{1}{1-2\alpha}(E_t R_1 P_2 + E_t R_2 P_1)$$
$$-\alpha(1-\alpha)(\lambda-1)2\sigma^2 r_\epsilon N_2' p_s' p_s N_1 S_t(1)$$

The two terms in (111) left to compute are then $\hat{cov}_t(r_{t+1}^D, r_{t+1}^A)$ and $\hat{cov}_t(r_{t+1}^D, f_H(S_{t+1}) + f_F(S_{t+1}))$. Let R_1^A and R_2^A denote the first and second-order component of r_{t+1}^A and F_1^A and F_2^A be the first and second-order components of $0.5\,[f_H(S_{t+1}) + f_F(S_{t+1})]$. We therefore need to compute $E_t R_1 R_2^A$, $E_t R_2 R_1^A$, $E_t R_1 F_2^A$ and $E_t R_2 F_1^A$.

In the second-order terms again only products of $S_t(1)$ and ϵ_{t+1} are relevant. These terms in R_2^A are computed analogously to those for R_2 computed above and can be summarized as $S_t(1)'\bar{M}^A\epsilon_{t+1}$, where

$$
\begin{aligned}
&\bar{M}^A = 0.5\left[\sum_{v=1}^{3}\left[r_q q_{s,v}^H + q_{s,v}^F) + (1-r_q)\, p_{s,v}\right] N_{5,v}\right] \\
&+N_1'\left[r_q(q_{ss}^H + q_{ss}^F) + (1-r_q)\, p_{ss}\right] N_2 \\
&+0.5 r_{qq} N_1'\left[(q_s^H)' q_s^H + (q_s^F)' q_s^F + (p_s)'\, p_s - \left(q_s^F\right)'\, p_s - (p_s)'\, q_s^F\right] N_2 \\
&+0.5 r_{qq}\left[(a_s^H)' a_E^H + (a_s^F)' a_E^F\right] \\
&-0.5 r_{qq}\left[(a_s^H)' q_s^H + (a_s^F)' q_s^F\right] N_2 \\
&-0.5 r_{qq} N_1'\left[(q_s^H)' a_E^H + (q_s^F)' a_E^F\right] \\
&+0.5 r_{qq}(a_s^F)' p_s N_2 + 0.5 r_{qq} N_1'\,(p_s)'\, a_E^F
\end{aligned}
$$

Therefore

$$E_t R_1 R_2^A = \sigma^2 S_t(1)' \bar{M}^A r_\epsilon' \tag{112}$$

The innovation component of R_1^A is $r_\epsilon^A \epsilon_{t+1}$, where

$$r_\epsilon^A = 0.5 r_q (q_s^H + q_s^F) N_2 + 0.5(1-r_q) p_s N_2 + 0.5(1-r_q)\left(a_E^H + a_E^F\right)$$

Therefore

$$E_t R_2 R_1^A = \sigma^2 S_t(1)' \bar{M}\left(r_\epsilon^A\right)' \tag{113}$$

Analogous to (106) and (107) we have

$$
\begin{aligned}
E_t R_1 F_2^A &= \frac{1}{4}\sigma^2 S_t(1)' N_1'\left[(H_{2,H} + H_{2,F}) + (H_{2,H} + H_{2,F})'\right] N_2 r_\epsilon' \\
&\quad +0.5\sigma^2 S_t(1)'\left(\sum_{v=1}^{3}\left(H_{1,H,v} + H_{1,F,v}\right) N_{5,v}\right) r_\epsilon' \\
E_t R_2 F_1^A &= 0.5\sigma^2 S_t(1)' \bar{M} N_2' \left(H_{1,H} + H_{1,F}\right)'
\end{aligned}
$$

Substituting these results into (111) we have

$$E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) = \sigma^2 r_3 S_t(1) \tag{114}$$

where

$$
\begin{aligned}
r_3 \;=\; & -r_{DE}\psi'(1-\psi')\left(f_{HD}f_{HS}+f_{FD}f_{FS}\right)-(1-\gamma)\left[r_\epsilon\left(\bar{M}^A\right)'+r_\epsilon^A\bar{M}'\right] \\
& +(1-\gamma)\,r_\epsilon N_2' p_{ss}' N_1 + 0.5\,(1-\gamma)\,r_\epsilon\left(\sum_{v=1}^{3} p_{s,v}N_{5,v}\right)' \\
& +0.5\,(1-\gamma)\,p_s N_2 \bar{M}' - (1-\gamma)\,\alpha(1-\alpha)(\lambda-1)r_\epsilon N_2' p_s' p_s N_1 \\
& -(1-\psi')\frac{1}{4}r_\epsilon N_2'\left[(H_{2,H}+H_{2,F})+(H_{2,H}+H_{2,F})'\right]N_1 \\
& -(1-\psi')0.5 r_\epsilon\left(\sum_{v=1}^{3}\left(H_{1,H,v}+H_{1,F,v}\right)N_{5,v}\right)' \\
& -(1-\psi')0.5\left(H_{1,H}+H_{1,F}\right)N_2\bar{M}' \\
& +2r_{DE}^2\gamma k_s^A
\end{aligned}
$$

5 Solution method

The numerical solution proceeds as follows. Conditional on $k(0)$ we obtain the first-order solution summarized by (45) and (49). The first-order solution also gives us $H_{1,H}$ from (74), which is based on the first-order component of the Bellman equation. $H_{1,F}$ follows by symmetry. We use these results to compute a new value for $k(0)$ from (83). This procedure therefore yields a mapping of $k(0)$ into itself. This mapping is non-linear. The resulting fixed point problem is solved numerically. At this point we have solved for $k(0)$ as well as the first-order component of all variables other than k_t^D.

Next, we conjecture a solution for the first-order component of k_t^D: $k_t^D(1) = k_s S_t(1)$. This affects the second-order component of model equations (see the discussion in section 2.1). Conditional on this first order solution of k_t^D we then obtain the second-order solution summarized by (59) and (66). The second-order solution is then used to compute $H_{2,H}$ from (79), which is based on the second-order component of the Bellman equation. $H_{2,F}$ follows by symmetry. We then use these results to solve for k_s in (109). This then leads to a fixed point problem in k_s, which is solved numerically. At this point we have solved for the first-order components of both k_t^D and k_t^A, which yields the first-order components of all portfolio shares.

6 Balance of payments accounting

6.1 Definitions

At the end of period t home agents have a nominal wealth of $(1-\psi) W_t P_t$ (measured in terms of the Home good) invested in equities, and Foreign agents have a nominal wealth $(1-\psi) W_t^* P_t^*$ invested. The nominal value of the various holdings of equities, as well as the quantity of shares held by each agent, is outlined in the table below:

	Nominal value	Quantity of shares
Home agents' wealth	$(1-\psi) W_t P_t$	
in Home equity	$(1-\psi) k_{H,t}^H W_t P_t$	$G_{H,t}^H = \frac{(1-\psi) k_{H,t}^H W_t P_t}{Q_{H,t}}$
in Foreign equity	$(1-\psi)\left(1-k_{H,t}^H\right) W_t P_t$	$G_{F,t}^H = \frac{(1-\psi)\left(1-k_{H,t}^H\right) W_t P_t}{Q_{F,t}}$
Foreign agents' wealth	$(1-\psi) W_t^* P_t^*$	
in Home equity	$(1-\psi)\left(1-k_{F,t}^F\right) W_t^* P_t^*$	$G_{H,t}^F = \frac{(1-\psi)\left(1-k_{F,t}^F\right) W_t^* P_t^*}{Q_{H,t}}$
in Foreign equity	$(1-\psi) k_{F,t}^F W_t^* P_t^*$	$G_{F,t}^F = \frac{(1-\psi) k_{F,t}^F W_t^* P_t^*}{Q_{F,t}}$

The home net foreign asset position is the difference between gross foreign assets, GA_t, and gross foreign liabilities, GL_t:

$$GA_t = (1-\psi)\left(1-k_{H,t}^H\right) W_t P_t \tag{115}$$

$$GL_t = (1-\psi)\left(1-k_{F,t}^F\right) W_t^* P_t^* \tag{116}$$

$$\begin{aligned} NFA_t &= GA_t - GL_t \qquad (117) \\ &= (1-\psi)\left[\left(1-k_{H,t}^H\right) W_t P_t - \left(1-k_{F,t}^F\right) W_t^* P_t^*\right] \end{aligned}$$

We next turn to trade flows. The value of Home exports during period t (measured in terms of Home good) is simply the consumption of Home goods by Foreign agents:

$$X_t^H = (1-\alpha)\left(P_t^*\right)^\lambda \psi W_t^*$$

Similarly for the value of Foreign exports:

$$X_t^F = (1-\alpha)\left(P_{F,t}\right)^{1-\lambda}\left(P_t\right)^\lambda \psi W_t$$

so the Home trade balance is

$$TB_t = X_t^H - X_t^F = (1-\alpha)\psi\left[\left(P_t^*\right)^\lambda W_t^* - \left(P_{F,t}\right)^{1-\lambda}\left(P_t\right)^\lambda W_t\right]$$

Note that it is also the value of output minus consumption:

$$TB_t = A_{H,t} - \psi P_t W_t = -P_{F,t} A_{F,t} + \psi P_t^* W_t^*$$

which can also be written as:

$$2TB_t = (A_{H,t} - P_{F,t} A_{F,t}) - \psi (P_t W_t - P_t^* W_t^*) \tag{118}$$

We now look at international factor payments in period t. The quantity of foreign shares owned by home residents at the beginning of the period is $G^H_{F,t-1}$. Each share receives $(1-\theta) A_{F,t}$ in terms of Foreign goods, so the payment in terms of Home goods is:

$$GD_t^H = G^H_{F,t-1} (1-\theta) P_{F,t} A_{F,t} \tag{119}$$

Similarly the dividend payments of the Home country are:

$$GD_t^F = G^F_{H,t-1} (1-\theta) A_{H,t} \tag{120}$$

The net dividend income of the home country is then:

$$ND_t = GD_t^H - GD_t^F = (1-\theta) \left[G^H_{F,t-1} P_{F,t} A_{F,t} - G^F_{H,t-1} A_{H,t} \right] \tag{121}$$

The current account is the sum of the trade balance and net dividend income:

$$CA_t = TB_t + ND_t \tag{122}$$

The capital gain on Home gross foreign assets in period t is

$$GK_t^H = G^H_{F,t-1} [Q_{F,t} - Q_{F,t-1}] \tag{123}$$

Similarly the capital gain on Home gross foreign liabilities is

$$GK_t^F = G^F_{H,t-1} [Q_{H,t} - Q_{H,t-1}] \tag{124}$$

and the net capital gain is:

$$NK_t = GK_t^H - GK_t^F = G^H_{F,t-1} [Q_{F,t} - Q_{F,t-1}] - G^F_{H,t-1} [Q_{H,t} - Q_{H,t-1}] \tag{125}$$

Recalling that positions are measured at the end of the periods, the Home gross asset position change between period $t-1$ and t is the sum of gross financial outflows, GF_t^H, and capital gains, GK_t^H:

$$GA_t - GA_{t-1} = GF_t^H + GK_t^H$$

Similarly for the Home gross liability position:

$$GL_t - GL_{t-1} = GF_t^F + GK_t^F$$

In net terms we have:

$$\begin{aligned} NFA_t - NFA_{t-1} &= \left(GF_t^H - GF_t^F\right) + \left(GK_t^H - GK_t^F\right) \\ &= NF_t + NK_t \end{aligned} \tag{126}$$

where NF_t stands for net financial flows (net capital outflows). In addition, the net financial flows have to match the current account:

$$NF_t = CA_t$$

This is simply a consequence of the dynamics of the net foreign assets reflecting the trade balance, net dividend income and net capital gains:

$$NFA_t = NFA_{t-1} + TB_t + ND_t + NK_t \tag{127}$$

We define the passive portfolio share of Home equity as the share when the quantities of assets are held at the zero-order levels, and asset prices take their actual level:

$$\begin{aligned} k_{H,t}^{H,p} &= \frac{Q_{H,t}k(0)}{Q_{H,t}k(0) + Q_{F,t}(1-k(0))} \\ k_{H,t}^{F,p} &= \frac{Q_{H,t}(1-k(0))}{Q_{H,t}(1-k(0)) + Q_{F,t}k(0)} \end{aligned}$$

6.2 First-order components

We focus on the first-order components of all balance of payments variables. Some key zero-order components are:

$$GA(0) = GL(0) = (1-\psi)(1-k(0))W(0) = \frac{1-\psi}{\psi}(1-k(0))$$

$$\begin{aligned} GD^H(0) &= GD^F(0) = (1-k(0))(1-\theta) \quad , \quad GK^H(0) = GK^F(0) = 0 \\ X^H(0) &= X^F(0) = (1-\alpha) \end{aligned}$$

We scale all variables by the zero-order component of GDP, which is 1, so they can all be interpreted as percentage of GDP. The resulting variables are indicated with lower case letters. The asset positions (115)-(117) are:

$$
\begin{aligned}
ga_t(1) &= \frac{1-\psi}{\psi}\left[(1-k(0))(w_t(1)+p_t(1))-k_{H,t}^H(1)\right] \\
gl_t(1) &= \frac{1-\psi}{\psi}\left[(1-k(0))(w_t^*(1)+p_t^*(1))-k_{F,t}^F(1)\right] \\
nfa_t(1) &= \frac{1-\psi}{\psi}\left[\begin{array}{c}(1-k(0))(w_t(1)-w_t^*(1)) \\ +(1-k(0))(p_t(1)-p_t^*(1))-2k_t^A(1)\end{array}\right]
\end{aligned} \tag{128}
$$

where we use $k_t^A(1)=0.5\left(k_{H,t}^H(1)+k_{H,t}^F(1)\right)$.

The trade flows and trade balance are:

$$
\begin{aligned}
x_t^H(1) &= (1-\alpha)\left[w_t^*(1)+\lambda p_t^*(1)\right] \\
x_t^F(1) &= (1-\alpha)\left[w_t(1)+\lambda p_t(1)+(1-\lambda)p_{F,t}(1)\right] \\
tb_t(1) &= (1-\alpha)\left[\begin{array}{c}-(w_t(1)-w_t^*(1)) \\ -\lambda(p_t(1)-p_t^*(1))-(1-\lambda)p_{F,t}(1)\end{array}\right]
\end{aligned}
$$

From (118) we also have:

$$
\begin{aligned}
tb_t(1) &= \frac{1}{2}\left(a_{H,t}(1)-p_{F,t}(1)-a_{F,t}(1)\right) \\
&\quad -\frac{1}{2}\left(p_t(1)+w_t(1)-p_t^*(1)-w_t^*(1)\right)
\end{aligned}
$$

The dividend flows are:

$$
\begin{aligned}
& gd_t^H(1)= \\
& \quad (1-\theta)\left[(1-k(0))\left[p_{F,t}(1)+a_{F,t}(1)+w_{t-1}(1)+p_{t-1}(1)-q_{F,t-1}(1)\right]-k_{H,t-1}^H(1)\right] \\
& gd_t^F(1)= \\
& \quad (1-\theta)\left[(1-k(0))\left[a_{H,t}(1)+w_{t-1}^*(1)+p_{t-1}^*(1)-q_{H,t-1}(1)\right]-k_{F,t-1}^F(1)\right] \\
& nd_t(1)= \\
& \quad (1-\theta)(1-k(0))\left[\begin{array}{c}p_{F,t}(1)-(a_{H,t}(1)-a_{F,t}(1)) \\ +\left(w_{t-1}(1)-w_{t-1}^*(1)\right)+\left(p_{t-1}(1)-p_{t-1}^*(1)\right) \\ +(q_{H,t-1}(1)-q_{F,t-1}(1))\end{array}\right] \\
& \quad -(1-\theta)2k_{t-1}^A(1)
\end{aligned} \tag{129}
$$

The current account is

$$ca_t(1) = tb_t(1) + nd_t(1)$$

The capital gains are

$$\begin{aligned} gk_t^H(1) &= \frac{1-\psi}{\psi}(1-k(0))\,\Delta q_{F,t}(1) \\ gk_t^F(1) &= \frac{1-\psi}{\psi}(1-k(0))\,\Delta q_{H,t}(1) \\ nk_t(1) &= \frac{1-\psi}{\psi}(1-k(0))\,[\Delta q_{F,t}(1) - \Delta q_{H,t}(1)] \end{aligned} \tag{130}$$

where $\Delta x_t = x_t - x_{t-1}$. The gross financial flows (gross outflows and gross inflows) are

$$\begin{aligned} & gf_t^H(1) = \Delta ga_t(1) - gk_t^H(1) = \\ & \quad \frac{1-\psi}{\psi}\left[(1-k(0))\,[\Delta w_t(1) + \Delta p_t(1) - \Delta q_{F,t}(1)] - \Delta k_{H,t}^H(1)\right] \\ & gf_t^F(1) = \Delta gl_t(1) - gk_t^F(1) = \\ & \quad \frac{1-\psi}{\psi}\left[(1-k(0))\,[\Delta w_t^*(1) + \Delta p_t^*(1) - \Delta q_{H,t}(1)] - \Delta k_{F,t}^F(1)\right] \end{aligned}$$

We can also check that

$$ca_t(1) = gf_t^H(1) - gf_t^F(1)$$

6.3 Financial flows and valuation effects

The changes in positions can be decomposed between financial flows and capital gains:

$$\begin{aligned} \Delta ga_t(1) &= gf_t^H(1) + gk_t^H(1) \\ \Delta gl_t(1) &= gf_t^F(1) + gk_t^F(1) \\ \Delta nfa_t(1) &= ca_t(1) + nk_t(1) \end{aligned}$$

The net valuation effect can be split between real exchange rate movements (that is movements in the relative price of Foreign goods) and equity price changes. For this purpose we write the Foreign equity price in terms of Foreign goods, denoted $q^*_{F,t}$. This gives

$$nk_t(1) = \frac{1-\psi}{\psi}(1-k(0))\left[\underbrace{-\Delta q_{H,t}(1)}_{\text{equity price on liabilities}} + \underbrace{\Delta q^*_{F,t}(1)}_{\text{equity price on assets}} + \underbrace{\Delta p_{F,t}(1)}_{\text{real exchange rate}}\right]$$

The passive portfolio shares are:

$$k^{H,p}_{H,t}(1) = k^{F,p}_{H,t}(1) = k^p_t(1) = k(0)(1-k(0))[q_{H,t}(1) - q_{F,t}(1)] \qquad (131)$$

6.4 The drivers of capital flows

The changes in Home gross foreign assets and liabilities are written as:

$$\begin{aligned}\Delta GA_t &= (1-\psi)\left[\left(1-k^H_{H,t-1}\right)\Delta(W_tP_t) - W_tP_t\Delta k^H_{H,t}\right]\\ \Delta GL_t &= (1-\psi)\left[\left(1-k^F_{F,t-1}\right)\Delta(W^*_tP^*_t) - W^*_tP^*_t\Delta k^F_{F,t}\right]\end{aligned}$$

The changes of invested wealth stem from savings (labor income plus dividend income minus consumption) and capital gains on Home and Foreign equity. Using the dynamics of home wealth we get the relation for the Home country:

$$(1-\psi)\Delta(W_tP_t) = S_t + (1-\psi)W_{t-1}P_{t-1}\left[k^H_{H,t-1}\frac{\Delta Q_{H,t}}{Q_{H,t-1}} + \left(1-k^H_{H,t-1}\right)\frac{\Delta Q_{F,t}}{Q_{F,t-1}}\right]$$

where home savings are:

$$\begin{aligned}S_t &= \theta A_{H,t} - \psi W_tP_t \\ &\quad + (1-\theta)(1-\psi)W_{t-1}P_{t-1}\left[k^H_{H,t-1}\frac{A_{H,t}}{Q_{H,t-1}} + \left(1-k^H_{H,t-1}\right)\frac{P_{F,t}A_{F,t}}{Q_{F,t-1}}\right]\end{aligned} \qquad (132)$$

Using the dynamics of Foreign wealth we get the relation for the Foreign country:

$$(1-\psi)\Delta(W^*_tP^*_t) = S^*_t + (1-\psi)W^*_{t-1}P^*_{t-1}\left[\left(1-k^F_{F,t-1}\right)\frac{\Delta Q_{H,t}}{Q_{H,t-1}} + k^F_{F,t-1}\frac{\Delta Q_{F,t}}{Q_{F,t-1}}\right]$$

where foreign savings are:

$$\begin{aligned}S^*_t &= \theta P_{F,t}A_{F,t} - \psi W^*_tP^*_t \\ &\quad + (1-\theta)(1-\psi)W^*_{t-1}P^*_{t-1}\left[\left(1-k^F_{F,t-1}\right)\frac{A_{H,t}}{Q_{H,t-1}} + k^F_{F,t-1}\frac{P_{F,t}A_{F,t}}{Q_{F,t-1}}\right]\end{aligned} \qquad (133)$$

We can show that $S_t + S_t^* = 0$.

Gross financial outflows are the change in gross assets minus capital gains on Home agents' holding of Foreign equity:

$$\begin{aligned} GF_t^H &= \Delta GA_t - GK_t^H \\ &= \left(1 - k_{H,t-1}^H\right) S_t - (1-\psi) W_t P_t \Delta k_{H,t}^H \\ &\quad + k_{H,t-1}^H \left(1 - k_{H,t-1}^H\right)(1-\psi) W_{t-1} P_{t-1} \left[\frac{\Delta Q_{H,t}}{Q_{H,t-1}} - \frac{\Delta Q_{F,t}}{Q_{F,t-1}}\right] \end{aligned} \tag{134}$$

Similarly for gross financial inflows:

$$\begin{aligned} GF_t^F &= \Delta GL_t - GK_t^F \\ &= \left(1 - k_{F,t-1}^F\right) S_t^* - (1-\psi) W_t^* P_t^* \Delta k_{F,t}^F \\ &\quad - k_{F,t-1}^F \left(1 - k_{F,t-1}^F\right)(1-\psi) W_{t-1}^* P_{t-1}^* \left[\frac{\Delta Q_{H,t}}{Q_{H,t-1}} - \frac{\Delta Q_{F,t}}{Q_{F,t-1}}\right] \end{aligned} \tag{135}$$

The first-order components of (132)-(133) are

$$\begin{aligned} s_t(1) &= \theta a_{H,t}(1) - (w_t(1) + p_t(1)) \\ &\quad + (1-\theta) \left[\begin{array}{c} (w_{t-1}(1) + p_{t-1}(1)) + k(0)(a_{H,t}(1) - q_{H,t-1}(1)) \\ + (1 - k(0))(a_{F,t}(1) + p_{F,t}(1) - q_{F,t-1}(1)) \end{array}\right] \\ s_t^*(1) &= \theta (a_{F,t}(1) + p_{F,t}(1)) - (w_t^*(1) + p_t^*(1)) \\ &\quad + (1-\theta) \left[\begin{array}{c} \left(w_{t-1}^*(1) + p_{t-1}^*(1)\right) + (1 - k(0))(a_{H,t}(1) - q_{H,t-1}(1)) \\ + k(0)(a_{F,t}(1) + p_{F,t}(1) - q_{F,t-1}(1)) \end{array}\right] \end{aligned}$$

Note that $s_t^*(1) = -s_t(1)$, hence:

$$\begin{aligned} s_t(1) &= \frac{1}{2}(s_t(1) - s_t^*(1)) \\ &= \frac{1}{2}\theta [a_{H,t}(1) - a_{F,t}(1) - p_{F,t}(1)] \\ &\quad - \frac{1}{2}(w_t(1) - w_t^*(1) + p_t(1) - p_t^*(1)) \\ &\quad + \frac{1-\theta}{2} \left[\begin{array}{c} w_{t-1}(1) - w_{t-1}^*(1) + p_{t-1}(1) - p_{t-1}^*(1) \\ + (2k(0) - 1)[a_{H,t}(1) - a_{F,t}(1) - p_{F,t}(1)] \\ - (2k(0) - 1)(q_{H,t-1}(1) - q_{F,t-1}(1)) \end{array}\right] \end{aligned} \tag{136}$$

Using (131), the first-order components of (134)-(135) are written as:

$$gf_t^H(1) = (1-k(0))\, s_t(1) - \frac{1-\psi}{\psi}\left(\Delta k_{H,t}^H(1) - \Delta k_t^p(1)\right) \quad (137)$$

$$gf_t^F(1) = -(1-k(0))\, s_t(1) + \frac{1-\psi}{\psi}\left(\Delta k_{H,t}^F(1) - \Delta k_t^p(1)\right) \quad (138)$$

where we used $\Delta k_{H,t}^F(1) + \Delta k_{F,t}^F(1) = 0$. The net portfolio flows are

$$nf_t(1) = 2(1-k(0))\, s_t(1) - 2\frac{1-\psi}{\psi}\left(\Delta k_t^A(1) - \Delta k_t^p(1)\right)$$

we can check that $nf_t(1) = ca_t(1)$.

6.5 Components of external adjustment

The rates of return on Home and Foreign equity reflect a capital gain and a dividend return:

$$R_{H,t+1} = 1 + \frac{Q_{H,t+1} - Q_{H,t}}{Q_{H,t}} + D_{H,t+1}$$

$$R_{F,t+1} = 1 + \frac{Q_{F,t+1} - Q_{F,t}}{Q_{F,t}} + D_{F,t+1}$$

where:

$$D_{H,t+1} = (1-\theta)\frac{A_{H,t+1}}{Q_{H,t}} \qquad D_{F,t+1} = (1-\theta)\frac{P_{F,t+1}A_{F,t+1}}{Q_{F,t}}$$

First-order components are

$$d_{H,t+1}(1) = a_{H,t+1}(1) - q_{H,t}(1) \qquad d_{F,t+1}(1) = a_{F,t+1}(1) + p_{F,t+1}(1) - q_{F,t}(1)$$

and the return differentials are

$$d_{H,t+1}(1) - d_{F,t+1}(1) = a_{H,t+1}(1) - a_{F,t+1}(1) - p_{F,t+1}(1) - (q_{H,t}(1) - q_{F,t}(1))$$

$$r_{H,t+1}(1) - r_{F,t+1}(1) = \frac{1-\psi}{1-\psi\theta}\left[(q_{H,t+1}(1) - q_{F,t+1}(1)) - (q_{H,t}(1) - q_{F,t}(1))\right] + \frac{\psi(1-\theta)}{1-\psi\theta}(d_{H,t+1}(1) - d_{F,t+1}(1))$$

Using (128), (129) and (130) the dynamics of the Home country's net foreign assets are

$$\begin{aligned} nfa_{t+1}(1) &= nfa_t(1) + tb_{t+1}(1) + nd_{t+1}(1) + nk_{t+1}(1) \\ &= nfa_t(1) + tb_{t+1}(1) + \frac{\psi(1-\theta)}{1-\psi} nfa_t(1) \\ &\quad - (1-k(0)) \frac{1-\psi\theta}{\psi} (r_{H,t+1}(1) - r_{F,t+1}(1)) \end{aligned}$$

where we used the fact that

$$nd_{t+1}(1) = \frac{\psi(1-\theta)}{1-\psi} nfa_t(1) - (1-\theta)(1-k(0))(d_{H,t+1}(1) - d_{F,t+1}(1))$$

We rewrite the dynamics of the net foreign asset position as

$$\begin{aligned} nfa_{t+1}(1) &= tb_{t+1}(1) + R(0) \cdot nfa_t(1) \\ &\quad - GA(0) \cdot R(0)(r_{H,t+1}(1) - r_{F,t+1}(1)) \end{aligned}$$

where $GA(0) = \frac{1-\psi}{\psi}(1-k(0))$ is the zero-order component of the gross asset position. Iterating forward we get

$$nfa_t(1) = -\sum_{s=1}^{\infty} \left(\frac{1}{R(0)}\right)^s tb_{t+s}(1) + GA(0) \sum_{s=1}^{\infty} \left(\frac{1}{R(0)}\right)^{s-1} (r_{H,t+s}(1) - r_{F,t+s}(1)) \tag{139}$$

(139) shows that a net debt ($nfa_t(1) < 0$) has to be offset by future trade surpluses ($tb_{t+s}(1) > 0$) or a higher return on Foreign equity than on Home equity, that is a higher return on home assets than liabilities ($r_{H,t+s}(1) - r_{F,t+s}(1) < 0$).

Future returns can in turn be split between capital gains and dividend yields:

$$nfa_t(1) = -\sum_{s=1}^{\infty} \left(\frac{1}{R(0)}\right)^s tb_{t+s}(1) - \sum_{s=1}^{\infty} \left(\frac{1}{R(0)}\right)^s [ndy_{t+s}(1) + nk_{t+s}(1)]$$

where nk_{t+s} is given by (130) and:

$$ndy_{t+1}(1) = -(1-\theta)(1-k(0))(d_{H,t+1}(1) - d_{F,t+1}(1))$$