A Method for Solving DSGE Models with Dispersed Private Information[1]

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Abstract

We develop a method for applying local approximation techniques to macro models with dispersed private information. It combines and extends existing local approximation methods applied to public information DSGE settings with methods for solving noisy rational expectations models in finance with dispersed private information. We illustrate the method in the context of a two-country dynamic general equilibrium model where agents in both countries have private information about future productivity.

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1 Introduction

Most models in macroeconomics assume that agents have identical information. This stands in sharp contrast to a large literature of asymmetric information models in finance.[1] The usual assumption of identical information in macroeconomics is naturally one of convenience rather than realism. While the same dispersed information that affects asset markets also affects decisions by consumers, firms and workers, solving dynamic stochastic general equilibrium (DSGE) models with that feature is technically challenging. This paper proposes a way to tackle this problem.

We combine and extend methods used to solve noisy rational expectations (NRE) models that are widely used in the finance literature and local approximation methods that are widely used to solve DSGE models in macro economics. DSGE models consider a non-linear general equilibrium setting, including endogenous portfolio choice in recent advances, but abstract from dispersed information. The latter aspect is at the heart of NRE models where agents have private information about the future payoff of an asset. As they trade based on this information, it affects the asset price. There is also an unobserved "noise" demand unrelated to assets' payoffs, for example associated with noise trade, liquidity trade or hedge trade. This noise prevents the price from completely revealing the aggregate of the private information. While rich in terms of information dispersion, NRE models do not connect well to the macro DSGE literature. The models are entirely linear and are partial equilibrium (there is a riskfree asset that is in infinite supply). The closed form solutions that are generally feasible in the NRE literature are clearly not possible in most macro models, which are highly non-linear.

The most commonly used solution method for DSGE macro models, and by far the easiest to understand and implement, is based on local approximations. It involves first solving the deterministic steady state. It then takes a linear Taylor expansion of the model around this steady state to write a set of linear difference equations, which are solved to obtain the first-order solution of the control variables as a function of state variables. One can also proceed to a quadratic Taylor expansion of the equations, which can be used to obtain a second-order solution of the model.

No approximations are required to solve for NRE models thanks to their lin-

[1] See Brunnermeier (2001) for a review.

ear, partial equilibrium, nature. The solution method usually involves three steps. First, an equilibrium asset price is conjectured that depends linearly on the future payoff and the noise. Second, expectations about future asset payoffs are computed by solving a signal extraction problem that uses private information, public information, and information from the asset price. Finally, these expectations (and variance) are substituted in an expression for the optimal portfolio demand, which is then equated to asset supply. The resulting equilibrium asset price is equated to the conjectured one to solve a fixed point problem in the parameters of the conjectured equilibrium price.

Our solution method combines and extends these two methods. We solve a similar signal extraction problem as in the NRE models, which does not involve any approximation. We use the result to first compute the expectations in the model equations, before applying local approximation methods. Various challenges need to be overcome in the process of doing so. In the next section we describe what these challenges are and how we deal with them at a general level. Then in the next two sections we illustrate the method in the context of two specific models.

In Section 3 we consider a simple NRE model. This model can be solved in closed form, so that an approximation is not actually needed. Starting from this simple model has the advantage of elucidating the approximation method and comparing it with the known closed form solution. In Section 4 we apply the method to a more complex two-country DSGE model in which agents make consumption, portfolio and investment decisions.[2] Section 5 concludes.

2 Combining Local Approximation and NRE Solution Methods

In this section we highlight the challenges that arise in combining the local approximation and NRE solution methods and discuss how to handle them.

[2]Papers that apply NRE models to open economy settings include Albuquerque, Bauer and Schneider (2007,2009), Bacchetta and van Wincoop (2006), Brennan and Cao (1997), Gehrig (1993) and Veldkamp and van Nieuwerburgh (2009). But like all models in the NRE literature, these are linear partial equilibrium models, in contrast to the DSGE open economy model with dispersed information that we will consider in Section 4 as an illustration of the method.

2.1 Approximation in Public Information Models

It is useful to start with some basic notation and a broad description of the approximation method in the absence of dispersed information. We begin by defining the "orders" of a variable or equation. Any variable x can be written as:

$$x = x(0) + x(1) + x(2) + ...$$

$x(0)$ is the zero order component that is independent of shocks. In the literature it is often referred to as a deterministic steady state. $x(1)$ is the first order component that is linear in the shocks, or their standard deviation. The second order component, $x(2)$, is linear in the square of shocks, the product of two different shocks, or the variance or covariance of shocks. Third and higher-order components are defined analogously. Formally, denoting the standard deviation of innovations by σ, we write:[3]

$$x(0) = \lim_{\sigma \to 0} x \quad ; \quad x(1) = \sigma \lim_{\sigma \to 0} \frac{x - x(0)}{\sigma} \quad ; \quad x(2) = \sigma^2 \lim_{\sigma \to 0} \frac{x - [x(0) + x(1)]}{\sigma^2} \tag{1}$$

Equations can be similarly decomposed into components of all orders. To illustrate this, consider a set of equations summarized as $E_t \varphi(x_t, x_{t+1}) = 0$, where x_t is a vector of control and state variables. Limiting ourselves for illustrative purposes to a second-order Taylor expansion around the zero-order components, we have (all derivatives are evaluated at $x_t = x_{t+1} = x(0)$):

$$\varphi(x_t, x_{t+1}) = \varphi(x(0), x(0)) + \varphi_1' \hat{x}_t + \varphi_2' \hat{x}_{t+1} + \frac{1}{2} \hat{x}_t' \varphi_{11} \hat{x}_t + \frac{1}{2} \hat{x}_{t+1}' \varphi_{22} \hat{x}_{t+1} + \hat{x}_t' \varphi_{12} \hat{x}_{t+1}$$

where $\hat{x}_t = x_t - x(0)$.

After substituting $\hat{x}_t = x_t(1) + x_t(2) + ...$ in the Taylor expansion we can set $E_t \varphi(x_t, x_{t+1}) = 0$ for zero, first and second-order components:

$$0 = \varphi(x(0), x(0)) \tag{2}$$

$$0 = E_t \left(\varphi_1' x_t(1) + \varphi_2' x_{t+1}(1) \right) \tag{3}$$

$$\begin{aligned} 0 = {} & E_t \left(\varphi_1' x_t(2) + \varphi_2' x_{t+1}(2) \right. \\ & \left. + 0.5 x_t(1)' \varphi_{11} x_t(1) + 0.5 x_{t+1}(1)' \varphi_{22} x_{t+1}(1) + x_t(1)' \varphi_{12} x_{t+1}(1) \right) \end{aligned} \tag{4}$$

[3] If there are multiple shocks we hold the relative standard deviations constant in the approximation method, so that we can index all standard deviations with a single σ.

The standard local approximation method involves first solving the zero-order component of the variables from the zero-order component of the equations (2). We then solve first-order component of the variables from the first-order component of the equations (3), which maps the first-order component of the state variables into the first-order component of the control variables. Finally, we can go a step further and solve the second-order component of the variables from the second-order component of the equations (4).

This basic approximation method can be extended to handle models with endogenous portfolio choice, as recently shown by Devereux and Sutherland (2010) and Tille and van Wincoop (2010).[4] The challenge with portfolio choice is that for example the zero-order solution cannot be completely derived from the zero-order component of the equations as portfolio choice is not well defined in the absence of risk. Specifically the problem arises when there are different investors, such as in an open economy where Home and Foreign agents make different portfolio decisions. With a single representative agent the zero-order component of the portfolio share is simply given by the zero-order component of the relative asset supply, as the agent must hold the available assets in equilibrium.

The extended approximation method distinguishes between the difference in portfolio shares between Home and Foreign agents and all other variables, and the difference between Home and Foreign portfolio Euler equations and all other equations. First the zero-order component of all "other variables" is computed from the zero-order component of all "other equations". This is conditional on the zero-order component of the difference in portfolio shares between Home and Foreign agents. Then the second-order component of the difference in portfolio Euler equations and the first-order component of all other equations is used to jointly solve for the zero-order component of the difference in portfolio shares and the first-order component of all other equations. Finally, this last step can be repeated one order higher to jointly solve for the first-order component of the difference in portfolio shares and the second-order component of all other variables.

In applying these local approximation methods to models with dispersed information, three issues arise, which we now discuss.

[4]This method has been widely applied to open economy DSGE models. Examples are Coeurdacier, Kollmann and Martin (2010), Coeurdacier and Gourinchas (2009), Devereux and Yetman (2010), Ghironi, Lee and Rebucci (2009) and Okawa and van Wincoop (2010).

2.2 Volatility versus Uncertainty

The first issue is to distinguish between volatility and uncertainty. Volatility measures the magnitude of the shocks affecting the economy. The standard deviation of the shocks is by definition first-order. We refer to uncertainty as the precision of agents' private information about these shocks. For example, let u_{t+1} be a future shock with standard deviation σ. Assume that agent i has a private signal $v_t^i = u_{t+1} + \epsilon_{t+1}^i$, where ϵ_{t+1}^i is normal with mean 0 and variance σ_ϵ^2. The errors in the signals are not shocks to the model. We therefore treat them as zero-order. The standard deviation σ_ϵ of the error is therefore a zero-order constant that does not go to zero as σ goes to zero. If instead σ_ϵ were proportional to σ, agents would get very precise signals (but not exactly identical ones) as we let σ become close to zero. As we show in the next section, the dispersion in portfolio shares across investors would then be infinite. Treating σ_ϵ as a zero-order constant ensures a well-defined cross-sectional distribution of portfolio shares as we let $\sigma \to 0$.

2.3 Computing Expectations

The next issue relates to the computation of expectations of variables and equations. When writing the first and second-order components of the equations in (3) and (4), we implicitly assumed that $[E_t(x_{t+1})](o) = E_t[x_{t+1}(o)]$ for order o, i.e. that the first-order component of the expectation of x_{t+1} for instance is equal to the expectation of the first-order component of x_{t+1}. This leads to a set of linear difference equations in the first-order components that can be solved. This implicit assumption is correct when the only information is public information, but not in the presence of private information.

In order to see this, let x_{t+1} be a single variable. In a model with only public information x_{t+1} depends on public information at time t and unknown innovations at time $t+1$. To simplify, let there be just one public information variable s_t and one innovation u_{t+1}, so that $x_{t+1} = \varphi(s_t, u_{t+1})$. Assume that u_{t+1} is drawn from a normal distribution with mean zero and variance σ^2. Using an infinite-order Taylor expansion of φ around $u_{t+1} = 0$ gives

$$x_{t+1} = \varphi(s_t, 0) + \sum_{i=1}^{\infty} \varphi_{u\ldots(i\ times)}(s_t, 0)(u_{t+1})^i \tag{5}$$

where $\varphi_{u\ldots(i\ times)}$ is the $i-th$ order derivative of φ with respect to u_{t+1}.

Taking the expectation of this relation gives

$$E_t x_{t+1} = \varphi(s_t, 0) + \sum_{i=even} \varphi_{u...(i\ times)}(s_t, 0)\sigma^i \frac{(i-1)!!}{i!} \tag{6}$$

where $(i-1)!!$ is the double factorial (product of every other number from 1 to $i-1$). Consider for example the first and second-order components, which are

$$(E_t x_{t+1})(1) = \varphi_s s_t(1) \tag{7}$$

$$(E_t x_{t+1})(2) = \varphi_s s_t(2) + 0.5\varphi_{ss} s_t(1)^2 + 0.5\sigma^2 \varphi_{uu} \tag{8}$$

where φ_s, φ_{ss} and φ_{uu} are the first and second-order derivatives with respect to s_t and u_{t+1}, evaluated at $u_{t+1} = 0$ and $s_t = s(0)$.

Now consider the first- and second-order components of $x_{t+1} = \varphi(s_t, u_{t+1})$:

$$x_{t+1}(1) = \varphi_s s_t(1) + \varphi_u u_{t+1} \tag{9}$$

$$x_{t+1}(2) = \varphi_s s_t(2) + 0.5\varphi_{ss} s_t(1)^2 + \varphi_{su} s_t(1) u_{t+1} + 0.5\varphi_{uu} \left(u_{t+1}\right)^2 \tag{10}$$

It is immediate that the expectation of these terms corresponds to the expressions (7)-(8). It can be shown that the same is the case for higher orders.

The situation is different in the presence of dispersed information. Consider the simple case where $x_{t+1} = u_{t+1}$, and assume for simplicity that there are two pieces of information about u_{t+1}. The first is the unconditional distribution of u_{t+1} which is normal with mean 0 and variance σ^2. The second is a private signal. Agent i observes a signal v_t^i as in Section 2.2. Signal extraction implies

$$E_t u_{t+1} = \frac{v_t^i / \sigma_\epsilon^2}{(1/\sigma_\epsilon^2) + (1/\sigma^2)} \tag{11}$$

The second and third-order components of this are respectively $\sigma^2 \epsilon_{t+1}^i / \sigma_\epsilon^2$ and $\sigma^2 u_{t+1} / \sigma_\epsilon^2$. In contrast, the second and third-order components of u_{t+1} itself are by definition zero, and so are their expectations.

The key implication is that we first need to compute expectations of all equations before distinguishing between the different orders. While it was easy in the example above to compute the expectation of u_{t+1}, things are more difficult when equations depend in a non-linear way on x_{t+1}, while in addition x_{t+1} may depend in a non-linear way on future innovations. We address this non-linearity as follows. First, we take Taylor expansions of the equations. For example, if we need to impose the second-order component of an equation, it is sufficient to use a quadratic

Taylor expansion of the equation. Second, when time $t+1$ control variables enter in equations with an expectation operator, we conjecture a quadratic or cubic solution of the control variable as a function of the state variables.[5] The equations with an expectation operator can then be written as a polynomial (quadratic or cubic) function of variables known at time t and future innovations. We can then compute expectations if we know the distribution of the future innovations.

Even though macro models are in general non-linear, we can still solve a simple linear signal extraction problem to compute the distribution of future innovations. The reason for this is as follows. There are two types of unobserved state variables that result from private information: future fundamentals (through the aggregation of private signals) and noise. For simplicity let there be just one future fundamental and one noise variable. These unobserved state variables affect observed control variables. In our applications in Sections 3 and 4 these observed control variables are only the asset prices, but more generally they can affect other types of control variables as well. We conjecture and verify that the unobserved state variables (future fundamental and noise) affect asset prices in a jointly linear way through a variable h_t. While h_t in general affects asset prices in a non-linear way, it is itself a linear function of the two unknown state variables. Given knowledge of observed state variables that affect asset prices, which we denote by S_t, and the asset prices themselves, agents can extract the value of h_t. As h_t is linear in the unknowns, we then solve a standard linear signal extraction problem.

After computing the expectations we can impose the order components of equations in the standard way as described in Section 2.1 for models with public information. The solution at this stage is however conditional on the weight on the noise relative to the weight on the future fundamental innovation in h_t . We refer to this relative weight as the noise to signal ratio, which we need to solve for.

2.4 Computing Noise to Signal Ratio

In standard NRE models we solve for the noise to signal ratio by imposing asset market equilibrium, equating portfolio demand for assets to asset supply. In our approximation method the same is the case as well. We solve for the noise to signal ratio by equating the first-order components of asset demand and asset supply.

[5] A quadratic conjecture is sufficient if we need to impose the second-order component of an equation.

Doing so requires combining two equations: the first-order component of the asset market clearing, which is imposed anyway in the method described so far, and the third-order component of the average portfolio Euler equation (average across all investors). The first-order component of the asset market clearing equation tells us how large asset demand needs to be, to the first-order, to be equal to asset supply. This needs to be equated to the first-order component of asset demand from the demand or portfolio allocation perspective.

The first-order component of the portfolio share from a demand perspective requires imposing the third-order component of the average portfolio Euler equation. An easy way to think about this is in terms of a simple mean-variance portfolio choice model with two assets. The optimal portfolio share depends on the expected excess return divided by the variance of the excess return. A third-order change in expected return then leads to a first-order portfolio shift as it is divided by the variance, which is second-order. Third-order changes in the expected excess return only show up in the third-order component of portfolio Euler equations. That is why we need to impose the third-order component of the average portfolio Euler equation in order to obtain the first-order component of the average portfolio share from a demand perspective.

3 A Simple Noisy Rational Expectation Model

In this section we consider a simple NRE model. It includes the standard assumptions that, while restrictive, allow for a closed-form solution. We first derive the closed-form solution. We then illustrate the approximation method in this setting. It delivers a solution to the asset price that to the zero, first, second and third-order is identical to the closed-form solution.

3.1 Building blocks

The economy is populated by a unit mass of investors that live for one period. An individual investor i solves an optimal portfolio allocation between a risk-free asset, with an exogenous return r, and a risky asset. Each share of the risky asset yields a payoff f next period, which is normally distributed with mean $\bar{f}$ and variance σ_f^2:

$$f = \bar{f} + \varepsilon^f \quad ; \quad \varepsilon^f \sim N(0, \sigma_f^2) \tag{12}$$

In terms of orders, $\bar{f}$ is zero-order and ε^f is first-order.

Investors have constant absolute risk-aversion preferences. Utility of agent i is $U^i = -Ee^{-c_i}$, where c_i is consumption. Starting with wealth equal to one, consumption next period is:

$$c_i = r + z_i er$$

where z_i is the number of shares of the risky asset purchased by agent i, q is the price of one share of the risky asset and $er = f - rq$ is the excess return on the risky asset. The utility of investor i is then:

$$U^i = -\exp\left[-r - z_i E^i er + \frac{1}{2}(z_i)^2 var^i(f)\right] \tag{13}$$

Utility maximization leads to a standard mean-variance portfolio allocation that reflects the expected excess return on the risky asset scaled by its variance:

$$z_i = \frac{E^i er}{var^i(f)} \tag{14}$$

The expectation and variance have a superscript i because investors have different information. Specifically, investor i receives a private signal v^i about the future payoff innovation:

$$v^i = \varepsilon^f + \epsilon^i \quad ; \quad \epsilon^i \sim N(0, \sigma_v^2) \tag{15}$$

As explained in the previous section, the investor-specific component of the signal, ϵ^i, is zero-order. We follow the standard assumption in the NRE literature that the number of investors is sufficiently large for signal errors to cancel out in aggregate: $\int_0^1 \epsilon^i di = 0$.

The model is closed by imposing market clearing for the risky asset. The asset is in exogenous supply $\bar{b}$. The demand comes from two sources: the utility-maximizing investors, and a random demand b from traders that buy and sell the asset for reasons unrelated to expected payoffs. In the NRE literature these are usually referred to as exogenous noise traders or liquidity traders. We assume $b \sim N(0, \theta\sigma_f^2)$, where θ measures the variance of the noise relative to the unconditional variance of the payoff. The clearing of the asset market is written as:

$$z = \int_0^1 z_i di = \bar{b} - b \tag{16}$$

In terms of orders, $\bar{b}$ is zero-order and b is first-order.

3.2 Solution

As discussed in the introduction, NRE models are solved in three steps. The first step involves conjecturing an equilibrium asset price. We conjecture that the asset price depends on a constant component, $\bar{q}$, and a combination h of payoff shocks and noise shocks:

$$q = \bar{q} + \alpha h = \bar{q} + \alpha\left(\varepsilon^f + \lambda b\right) \tag{17}$$

where $\bar{q}$, α and λ are unknown coefficients. The future payoff innovation ε^f affects the asset price as agents trade based on their private information and the average private signal is ε^f. The presence of noise shocks b prevents the asset price q from completely revealing ε^f. The coefficient λ is the noise to signal ratio that reflects the impact of noise shocks relative to payoff shocks.

The second step of the solution is to compute $E^i\varepsilon^f$ and $var^i(\varepsilon^f)$ by solving a signal extraction problem. There are three sources of information: private signal, public information in the form of the unconditional distribution of f and the asset price. The asset price contains information on the shocks through the combination h, so investors observe h (through the asset price), but not its components. We summarize the three signals about the payoff shock ε^f as follows:

$$Y^i = \begin{pmatrix} 0 \\ v^i \\ h \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\varepsilon^f + \begin{pmatrix} -\varepsilon^f \\ \epsilon^i \\ \lambda b \end{pmatrix} = \iota\varepsilon^f + \epsilon \;\; ; \;\; \Sigma = Var\left(\epsilon\right) = \begin{pmatrix} \sigma_f^2 & 0 & 0 \\ 0 & \sigma_v^2 & 0 \\ 0 & 0 & \lambda^2\theta\sigma_f^2 \end{pmatrix}$$

Using the standard signal extraction technique, the expectation and the variance of the payoff innovation from the perspective of investor i are

$$var\left(\varepsilon^f\right) = \left[\iota'\Sigma^{-1}\iota\right]^{-1} = \frac{1}{\left(\sigma_f^2\right)^{-1} + \left(\sigma_v^2\right)^{-1} + \left(\lambda^2\theta\sigma_f^2\right)^{-1}} \tag{18}$$

$$E^i\left(\varepsilon^f\right) = \left[\iota'\Sigma^{-1}\iota\right]^{-1}\iota'\Sigma^{-1}Y^i = a_v v^i + a_h h \tag{19}$$

where:

$$a_v = \frac{var\left(\varepsilon^f\right)}{\sigma_v^2} \quad ; \quad a_h = \frac{var\left(\varepsilon^f\right)}{\lambda^2\theta\sigma_f^2} \tag{20}$$

The expected payoff innovation depends positively on the private signal v^i and the signal h, with the weight on these signals larger the greater their precision. Both the expectation and the variance of ε^f are functions of the noise to signal ratio λ, which remains to be solved.

The last step of the solution imposes market equilibrium. We substitute the moments (18)-(19) into investor i's optimal portfolio (14), aggregate the resulting expression across investors, and then impose the asset market clearing condition (16). This leads to a fixed point problem in the three unknown parameters of the conjectured asset price solution. The solution is:

$$\bar{q} = \frac{\bar{f}}{r} - \frac{\theta\sigma_v^4\sigma_f^2}{1+\theta\sigma_v^2\left(\sigma_v^2+\sigma_f^2\right)}\frac{\bar{b}}{r} \tag{21}$$

$$\alpha = \frac{1+\theta\sigma_v^2\sigma_f^2}{1+\theta\sigma_v^2\left(\sigma_v^2+\sigma_f^2\right)}\frac{1}{r} \tag{22}$$

$$\lambda = \sigma_v^2 \tag{23}$$

3.3 Local Approximation Method

We now apply the local approximation method to the model. As pointed out in section 2, it is important to distinguish between the volatility of the shocks themselves, which is measured by σ_f^2, and uncertainty in the private signals that is measured by σ_v^2. The proper metric of orders is volatility, hence σ_f^2 is second-order while σ_v^2 is a zero-order term. To further illustrate why uncertainty σ_v^2 should not be considered as the order metric, consider the portfolio allocation of investor i, which we write as

$$z_i = \frac{\bar{f} - r\left(\bar{q}+\alpha h\right) + a_h h}{var(\varepsilon^f)} + \frac{1}{\sigma_v^2}v^i \tag{24}$$

The cross-sectional variance of portfolio shares is $var(\epsilon^i)/\sigma_v^4 = 1/\sigma_v^2$. If σ_v^2 is second-order, proportional to σ_f^2, then the cross-sectional distribution of portfolio shares explodes to infinity when $\sigma_f \rightarrow 0$. This is a problem as the local approximation is around the point where $\sigma_f = 0$. If instead σ_v^2 is zero-order, the cross-sectional distribution of portfolio shares remains constant even when $\sigma_f \rightarrow 0$.

3.3.1 Step 1: Expansions of Equations

There are only two equations in this simple model. The first is the asset market clearing equation (16). The second is the portfolio Euler equation, which shows that the expected discounted excess return on the risky asset is zero:

$$E^i e^{-r-z_i er} er = 0 \tag{25}$$

We are only interested in solving for the zero, first and second-order components of the asset price. In the absence of private information, it would be sufficient to impose the zero, first and second-order components of (25). With dispersed information however, this solution remains conditional on the noise to signal ratio λ. As discussed in section 2, computing λ requires imposing the third-order component of the average portfolio Euler equation (25) across investors. We therefore need a cubic expansion of the average portfolio Euler equation.

The zero-order component of (25) immediately gives $q(0) = \bar{f}/r$. Taking a cubic expansion of (25) around $q(0)$ and $z^i(0)$, and averaging across all agents, gives:

$$\bar{E}er + m(0)Eer^3 = zEer^2 \tag{26}$$

where $\bar{E}$ is the average expectation across all agents and $m(0) = 0.5 \int z_i(0)^2$. We have also used that the expectations of er^2 and er^3 are the same for all agents, which is shown below, so that we do not need to index the expectation operator.

3.3.2 Step 2: Computing Expectations

Expectations only show up in the average portfolio Euler equation (26), where the expectations of er, er^2 and er^3 enter. As discussed in Section 2, we need to compute the expectations of these variables before we can impose the order components. The excess payoff er depends on the future innovation ε^f. The distribution of ε^f can be computed from signal extraction.

We conjecture that the asset price q is a (possibly non-linear) function of the combination h of payoff and noise innovations: $h = \varepsilon^f + \lambda b$, so that observing the asset price then reveals the value of h. We also conjecture that the noise to signal ratio λ is a zero-order parameter that needs to be solved. In general the conjecture for the asset price also depends on publicly observed state variables, but those are absent from our simple model here (they are present in the more general model in the next section). We compute the expectation and variance of ε^f by solving the exact same signal extraction problem as in Section 3.2, which gives $\bar{E}\varepsilon^f = a_v\varepsilon^f + a_h h$. It also gives the variance of ε^f, which is the same for all agents.

Splitting the coefficients a_v and a_h in (20) into components of different orders, we have (more details are given in Appendix A):

$$\begin{aligned}
\left[\bar{E}\varepsilon^f\right](0) &= \left[\bar{E}\varepsilon^f\right](2) = [var(\varepsilon^f)](0) = [var(\varepsilon^f)](1) = 0 \\
\left[\bar{E}\varepsilon^f\right](1) &= \frac{1}{1+\lambda^2\theta}h \\
\left[\bar{E}\varepsilon^f\right](3) &= \frac{\lambda^2\theta}{\sigma_v^2\left(1+\lambda^2\theta\right)}\sigma_f^2\left(\varepsilon^f - \frac{h}{1+\lambda^2\theta}\right) \\
[var(\varepsilon^f)](2) &= \frac{\lambda^2\theta}{1+\lambda^2\theta}\sigma_f^2
\end{aligned}$$

We use these results to compute $\bar{E}er$, Eer^2 and Eer^3.

3.3.3 Step 3: Imposing Order Components of Equations

We now proceed to imposing the various order components of the average portfolio Euler equation (26) and the market clearing condition (16).

Starting with the first-order component, (16) gives $z(1) = -b$. The first-order component of (26) implies that the first-order expected excess return is zero: $[\bar{E}er](1) = 0$. Using our results this gives the first-order asset price:

$$\left[\bar{E}\varepsilon^f\right](1) - rq\,(1) = 0 \Rightarrow q\,(1) = \frac{1}{1+\lambda^2\theta}\frac{1}{r}h$$

Next consider second-order components. (16) implies that $z(2) = 0$. The second-order component of (26) is:

$$[\bar{E}er](2) = z(0)[Eer^2](2) = \bar{b}[Eer^2](2) \tag{27}$$

where we used the zero-order component of (16): $z(0) = \bar{b}$. Using that $Eer^2 = var(er) + (E(er))^2$, we have $[Eer^2](2) = [var(er)](2) + 2[Eer](1)[Eer](2)$. As $[Eer](1) = 0$ for all agents[6], $[Eer^2](2)$ is equal to $[var(er)](2) = [var(\varepsilon^f)](2)$. (27) then gives the second-order asset price:

$$[\bar{E}\varepsilon^f](2) - rq(2) = \bar{b}[var(\varepsilon^f)](2) \Rightarrow q\,(2) = -\frac{\lambda^2\theta}{1+\lambda^2\theta}\frac{\bar{b}}{r}\sigma_f^2$$

At this point we have solved for the zero-, first- and second-order components of q and z. This solution however remains conditional on the noise to signal ratio λ, to which we now turn.

[6] $[E^i er](1) = 0$ follows from the first-order component of the portfolio Euler equation (25) for agent i.

We solve for λ by equating $z(1)$ from the supply side to $z(1)$ from the demand side. The supply side reflects the asset market clearing (16), $z(1) = -b$.[7] Intuitively, an increase in demand b by liquidity traders reduces the remaining net supply of the risky asset. $z(1)$ from the demand side follows by imposing the third-order component of the average portfolio Euler equation (26):

$$z(1) = \frac{[\bar{E}er](3) - m(0)[Eer^3](3)}{[var(er)](2)} \tag{28}$$

We have $[Eer^3](3) = 0$. To see that, we use that er has a normal distribution, so that its third moment can be written as $Eer^3 = (E(er))^3 - 3E(er)var(er)$. The third-order component of this is zero because the first-order component of $E(er)$ is zero. We also have $[\bar{E}er](3) = [\bar{E}\varepsilon^f](3) - rq(3)$. Using the expressions for $[\bar{E}\varepsilon^f](3)$ and $[var(\varepsilon^f)](2)$ in Section 3.3.2, substituting the result into (28) and equating it to $z(1) = -b$ from the supply side, we get

$$\sigma_v^2 b + \varepsilon^f - \frac{1}{1+\lambda^2\theta} h - \frac{\sigma_v^2(1+\lambda^2\theta)}{\lambda^2\theta\sigma_f^2} rq(3) = 0 \tag{29}$$

This last equation gives us both λ and $q(3)$. Since we have assumed that q is a function of h, possibly non-linear, $q(3)$ in general depends on h. (29) is then a relationship in b, ε^f and h. For this equation to hold, it must be the case that b and ε^f enter in the same linear combination as in h. This immediately implies that $\lambda = \sigma_v^2$, which is a zero-order constant as conjectured. (29) then implies that

$$q(3) = \frac{\theta^2\sigma_v^6}{r(1+\theta\sigma_v^4)^2} \sigma_f^2 h \tag{30}$$

We have now solved for the zero, first, second and third-order components of the asset price and for the noise to signal ratio λ. It is easily verified that the zero, first, second and third-order components of the exact solution $q = \bar{q} + \alpha h$ are identical to $q(0)$, $q(1)$, $q(2)$, and $q(3)$ from the approximation method.

4 A DSGE Model

We now apply the solution method to a two-country dynamic stochastic general equilibrium model. Agents in each country make savings and portfolio allocation

[7] Note that agents do not know the average portfolio share z, and therefore cannot extract b from this. They only know their own portfolio share z_i.

decisions based on public and dispersed private information. Tille and van Wincoop (2011) use the model to analyze the impact of dispersed information on gross and net international capital flows. While the model is more complex than the simple NRE framework of section 3, we introduce a number of simplifying features that allow us to derive an analytical solution and make it easier to illustrate the method. The description of the model is kept to a minimum, with a full description in Tille and van Wincoop (2011).

4.1 Model Description

There are two countries, Home and Foreign, indexed by $i = H,\, F$. Both produce a single consumption good using capital $K_{i,t}$ and labor $N_{i,t}$:

$$Y_{i,t} = A_{i,t} K_{i,t}^{1-\omega} N_{i,t}^{\omega} \tag{31}$$

The labor input is normalized to unity. The real wage $W_{i,t}$ is the marginal product of labor: $W_{i,t} = \omega A_{i,t} \left(K_{i,t}\right)^{1-\omega}$. Capital accumulation reflects investment $I_{i,t}$ and the depreciation rate δ: $K_{i,t+1} = (1-\delta) K_{i,t} + I_{i,t}$. Capital is built by installment firms that convert the consumption good into capital with a quadratic adjustment cost. Investment is then driven by the Tobin's Q relation:

$$I_{i,t}/K_{i,t} = \delta + \left(Q_{i,t} - 1\right)/\xi \tag{32}$$

where $Q_{i,t}$ is the price of one unit of capital in country i, which we refer to as the equity price. The return from purchasing one unit of country i capital at time t consists of the dividend and the capital gain at time $t+1$:

$$R_{i,t+1} = \frac{(1-\omega) A_{i,t+1} \left(K_{i,t+1}\right)^{-\omega} + (1-\delta) Q_{i,t+1}}{Q_{i,t}} \tag{33}$$

Log productivity, $a_{it} = ln(A_{i,t})$, follows an autoregressive process:

$$a_{i,t+1} = \rho a_{i,t} + \varepsilon_{i,t+1} \quad i = H, F \tag{34}$$

where $\varepsilon_{i,t+1}$ has a $N(0, \sigma_a^2)$ distribution and is uncorrelated across countries.

Each agent receives private signals about next period's productivity innovations in both countries. The signals observed by a Home agent j at time t are:

$$v_{j,t}^{H,H} = \varepsilon_{H,t+1} + \epsilon_{j,t}^{H,H} \qquad \epsilon_{j,t}^{H,H} \sim N\left(0, \sigma_{HH}^2\right) \tag{35}$$

$$v_{j,t}^{H,F} = \varepsilon_{F,t+1} + \epsilon_{j,t}^{H,F} \qquad \epsilon_{j,t}^{H,F} \sim N\left(0, \sigma_{HF}^2\right) \tag{36}$$

As before, we assume that the errors of the signals average to zero across investors (e.g. $\int_0^1 \epsilon_{j,t}^{H,H} dj = 0$). Foreign agents receive a similar set of signals. In each country the variance of the signal's error is σ_{HH}^2 for the domestic innovation and σ_{HF}^2 for the foreign innovation. We assume that domestic signals are more precise: $\sigma_{HH}^2 < \sigma_{HF}^2$.

To ensure that equity prices do not fully reveal the future innovations contained in the private signals, we introduce noise through a time-varying cost of investing abroad. A Home agent j investing in Foreign equity receives the return $R_{F,t+1}$ times an iceberg cost $e^{-\tau_{Hj,t}} < 1$. Similarly, a Foreign agent j investing in Home equity receives the return $R_{H,t+1}$ times an iceberg cost $e^{-\tau_{Fj,t}} < 1$. We denote the average of iceberg costs across agents in each country by $\tau_{H,t} = \int_0^1 \tau_{Hj,t} dj$ and $\tau_{F,t} = \int_0^1 \tau_{Fj,t} dj$. The average cost across both countries, $0.5\left(\tau_{H,t} + \tau_{F,t}\right)$, is a second-order constant τ (i.e. proportional to the variance of productivity shocks σ_a^2).

The cross-country difference in the cost follows a stochastic process: $\tau_t^D = \tau_{H,t} - \tau_{F,t} = 2\tau\varepsilon_t^\tau$, where ε_t^τ is the source of noise in the model and has a $N(0, \theta\sigma_a^2)$ distribution. An increase in ε_t^τ leads to a portfolio shift to Home equity that affects the relative asset price. We assume that the individual iceberg cost of a Home agent $\tau_{Hj,t}$ is an infinitely noisy signal of the average cost of Home investors $\tau_{H,t}$ so that she cannot infer the average cost from her own. Alternatively we could assume noise or liquidity traders that exogenously shift between Home and Foreign equity. It generates noise that prevents the relative asset price from revealing relative future productivity that is aggregated through private signals.[8]

We consider an overlapping generations setting where agents live for two periods. This simplifies the portfolio decision by limiting it to one period. Agents supply one unit of labor when young, consume some of their wage income and save the balance in Home and Foreign equity. They consume the proceeds from their investment when old. A young Home agent j at time t maximizes her intertemporal utility of consumption:

$$ln\left(C_{y,t}^{Hj}\right) + \beta E_t^{Hj} ln\left(C_{o,t+1}^{Hj}\right) \tag{37}$$

[8] In this model only the relative asset price will contain information about future productivity. The average asset price is driven by world saving. As we will see below, this only depends on current wages and not on expected future productivity. But this is a special feature that follows from our assumption of log utility and can certainly be generalized (at the cost of complicating the signal extraction problem though).

where $C_{y,t}$ is consumption when young and $C_{o,t+1}$ is consumption when old. The Hj superscript on the expection denotes that expectations can vary across agents as they are computed using private signals. Agent j's income when young consists of the wage $W_{H,t}$. Her consumption when old is given by the return on her savings, $C^{Hj}_{o,t+1} = (W_{H,t} - C^{Hj}_{y,t})R^{p,Hj}_{t+1}$. The portfolio return, $R^{p,Hj}_{t+1}$ reflects the equity returns (33), the iceberg cost on investment abroad, and the portfolio composition with $z_{Hj,t}$ denoting the fraction of wealth invested in Home equity:

$$R^{p,Hj}_{t+1} = z_{Hj,t}R_{H,t+1} + (1 - z_{Hj,t})e^{-\tau_{Hj,t}}R_{F,t+1} \tag{38}$$

The optimization problem of a Foreign agent is similar.

The log utility (37) implies that young agents consume a constant fraction of their wage income: $C^{Hj}_{y,t}/W_{H,t} = C^{Fj}_{y,t}/W_{F,t} = (1+\beta)^{-1}$. The optimal allocation of savings between Home and Foreign equity is given by the portfolio Euler equations, which show that agents pick a portfolio that equalizes the expected discounted return on Home and Foreign equity:

$$E^{Hj}_t \left(R^{p,Hj}_{t+1}\right)^{-1} \left(R_{H,t+1} - R_{F,t+1}e^{-\tau_{Hj,t}}\right) = 0 \tag{39}$$

$$E^{Fj}_t \left(R^{p,Fj}_{t+1}\right)^{-1} \left(R_{H,t+1}e^{-\tau_{Fj,t}} - R_{F,t+1}\right) = 0 \tag{40}$$

The clearing of Home and Foreign equity markets equalizes the value of capital in a country to the value of holdings of the country's equity by investors:

$$Q_{H,t}K_{H,t+1} = \beta(1+\beta)^{-1}\left(W_{H,t}z_{H,t} + W_{F,t}z_{F,t}\right) \tag{41}$$

$$Q_{F,t}K_{F,t+1} = \beta(1+\beta)^{-1}\left(W_{F,t}(1 - z_{H,t}) + W_{F,t}(1 - z_{F,t})\right) \tag{42}$$

where the average portfolio shares invested by Home and Foreign investors in Home equity are denoted $z_{H,t} = \int_0^1 z_{Hj,t}dj$ and $z_{F,t} = \int_0^1 z_{Fj,t}dj$. By Walras' Law we can omit the goods market clearing condition.

4.2 Solution Method

While the model is relatively simple and the solution can be derived analytically, there is still a good deal of algebra involved. We therefore only broadly describe the steps involved, leaving some algebra to Appendix B and full algebraic details to a separate Technical Appendix.

The steps in solving the model are the same as those used for the simple setup of Section 3. While the two-country model is more complex and entails more state variables, the solution method parallels the one in Section 3 with only two main differences. The first is that the model is now dynamic, whereas the one in Section 3 is static. Asset returns then include capital gains, and thus depend on future asset price innovations in addition to the future payoff innovations. We therefore need to make a conjecture on the link between the asset prices and the state variables in order to compute expectations of asset returns (as well as higher moments). We discuss this in more detail in Section 4.5 below.

Second, the average portfolio composition across agents differs between the two countries. As discussed in Section 2.1, the zero-order component of the difference in portfolio shares cannot be computed from the zero-order component of model equations. Instead of sequentially imposing the zero, first and second-order components of the equations, we need to impose the order components of equations based on the method developed by Devereux and Sutherland (2010) and Tille and van Wincoop (2010). We briefly reviewed this method in Section 2.1 and will describe it in more detail in the context of our present model in Section 4.6 below.

Apart from these two modifications, the solution method parallels the one described for the simple model of Section 3. After some preliminaries, we describe the method, following the same three broad steps as in Section 3.

4.3 Some Preliminaries

Before we expand the model equations, it is useful to condense the set of equations somewhat. First, we remove the consumption of young agents. Our assumption of log utility implies that consumption is proportional to wages. Denoting logs with lower case letters, we have $c^i_{y,t} = ln(\omega/(1+\beta)) + a_{it} + (1-\omega)k_{it}$, which solves consumption as a function of state variables without any approximation. Second, we substitute the Tobin's investment equations into the capital accumulation equations to write:

$$e^{k_{i,t+1}-k_{i,t}} = 1 + \left(e^{q_{it}} - 1\right)/\xi \tag{43}$$

The model then consists of 6 equations: the capital accumulation equations (43) of both countries, the asset market clearing conditions (41)-(42) and the portfolio Euler equations (39)-(40). We write these equations as averages of the corresponding Home and Foreign equations, as well as in cross-country differences. We also write the variables as averages and differences. For a variable x we denote the average across countries by $x^A = 0.5(x^H + x^F)$ and the difference across countries by $x^D = x^H - x^F$.

With the exception of portfolio shares, all variables from now on are in logs, denoted with lower case letters. Corresponding to the 6 equations, there are 6 control variables: k^A_{t+1}, k^D_{t+1}, z^A_t, z^D_t, q^A_t and q^D_t. There are 5 state variables: 4 observed state variables that we write as a vector $S_t = (a^D_t, a^A_t, k^D_t, k^A_t)'$ plus a state variable $h_t = \varepsilon^D_{t+1} + \lambda\tau^D_t/\tau$, which combines the unobserved future relative productivity shock ε^D_{t+1} and the current relative iceberg cost τ^D_t. We assume that these unobserved variables affect asset prices only in a joint linear way through h_t. This parallels the assumption we made in Section 3. We verify that this conjecture is correct. As the relative asset price q^D_t only depends on S_t and h_t, agents learn the value of h_t from the relative asset price.[9] A solution of the model involves a mapping from these 5 state variables into the 6 control variables in a way that satisfies the 6 equations.

4.4 Step 1: Quadratic and Cubic Expansions of Equations

We now turn to the first step towards a solution, which involves Taylor expansions of the equations. The expansions are conducted around the zero-order components of the variables, which are obtained by imposing the zero-order component of the equations. The zero-order components of a^A_t, a^D_t, q^A_t, q^D_t and k^D_t are all zero. Asset market clearing implies that the world portfolio be evenly split across assets: $z^A(0) = 0.5$. Finally, the average capital stock is $k^A(0) = (1/\omega)ln(\beta\omega/(1+\beta))$. As discussed in Section 2.1, the only variable for which we cannot compute the zero-order component this way is the cross-country difference in average portfolio share, $z^D_t = z_{H,t} - z_{F,t}$. For now we take $z^D(0)$ as an unknown parameter that remains to be solved.

We take cubic Taylor expansions of the portfolio Euler equations and quadratic expansions of the 4 other equations. The cubic approximation of the Euler equations is necessary as we need to solve for the third-order components of these equations, for two reasons. First, the method developed by Devereux and Sutherland (2010) and Tille and van Wincoop (2010) implies that the first-order component

[9]While we allow the average asset price q^A_t to depend on h_t as well, in equilibrium q^A_t will only depend on S_t. See the discussion in footnote 8.

of the cross-country difference in the portfolio shares is computed by imposing the third-order component of the *difference* of the portfolio Euler equations. Second, as was the case in Section 3, computing the signal to noise ratio λ in h_t relies on the third-order component of the *average* of the portfolio Euler equations.

From now we denote all variables as deviations from their zero-order components. Quadratic expansions of the average and the difference of the capital accumulation equations (43) are:

$$k_{t+1}^A - k_t^A = \frac{1}{\xi} q_t^A + \frac{1}{2}\frac{\xi - 1}{\xi^2}\left(\left(q_t^A\right)^2 + \frac{1}{4}\left(q_t^D\right)^2\right) \tag{44}$$

$$k_{t+1}^D - k_t^D = \frac{1}{\xi} q_t^D + \frac{\xi - 1}{\xi^2} q_t^A q_t^D \tag{45}$$

Quadratic expansions of the average and the difference of the asset market clearing conditions (41)-(42) are:

$$\begin{aligned} q_t^A + k_{t+1}^A &= a_t^A + (1-\omega) k_t^A - 2\left(z_t^A\right)^2 \\ &\quad + \frac{1 - \left(z^D(0)\right)^2}{8}\left(a_t^D + (1-\omega) k_t^D\right)^2 \\ &\quad - z^D(0) z_t^A \left(a_t^D + (1-\omega) k_t^D\right) \end{aligned} \tag{46}$$

$$\begin{aligned} q_t^D + k_{t+1}^D &= z^D(0)\left(a_t^D + (1-\omega) k_t^D\right) + 4 z_t^A \\ &\quad + z_t^D \left(a_t^D + (1-\omega) k_t^D\right) \end{aligned} \tag{47}$$

For the portfolio-Euler equations (39)-(40) we take cubic expansions and then average across agents within each country. The average and difference of the resulting portfolio Euler equations across the two countries are

$$z_t^A E_t\left(er_{t+1}\right)^2 = \bar{E}_t^A er_{t+1} + \frac{\tau_t^D}{2} + m(0) E_t\left(er_{t+1}\right)^3 \tag{48}$$

$$\left(z_t^D + z^D(0)\right) E_t\left(er_{t+1}\right)^2 = \bar{E}_t^H er_{t+1} - \bar{E}_t^F er_{t+1} + 2\tau \tag{49}$$

where $er_{t+1} = r_{H,t+1} - r_{F,t+1}$ is the log excess return on Home equity. $\bar{E}_t^H er_{t+1}$ and $\bar{E}_t^F er_{t+1}$ are the average expectations of these excess returns across all Home investors and all Foreign investors, respectively, and $\bar{E}_t^A er_{t+1} = 0.5\left(\bar{E}_t^H er_{t+1} + \bar{E}_t^F er_{t+1}\right)$ is the average across all investors worldwide. $m(0)$ is a zero-order constant that is the same for both countries.[10]

[10] In the Technical Appendix there is one additional term on the right hand side of these equations, which is zero to all relevant orders. We therefore omit it here.

A cubic expansion of the excess return that enters these equations is:

$$er_{t+1} = q^D_{t+1} - q^D_t + \delta_1 x^D_{t+1} + \delta_2 x^A_{t+1} x^D_{t+1} + \frac{\delta_3}{24}(x^D_{t+1})^3 + \frac{\delta_3}{2}(x^A_{t+1})^2 x^D_{t+1} \qquad (50)$$

where $x^D_{t+1} = a^D_{t+1} - \omega k^D_{t+1} - q^D_{t+1}$, $x^A_{t+1} = a^A_{t+1} - \omega k^A_{t+1} - q^A_{t+1}$ and δ_1, δ_2 and δ_3 are zero-order constants that depend on ω and β. (50) shows that the excess return depends on the asset prices at time $t+1$.

4.5 Step 2: Computing Expectations

Expectations only enter in the portfolio Euler equations through the expectations of er_{t+1}, er^2_{t+1} and er^3_{t+1}. We first compute these expectations and then split them into their different order components. This is done in three parts. We first make conjectures for the asset prices q^D_t and q^A_t as respectively cubic and quadratic functions of the state variables. Substituting the expressions for the future asset prices q^D_{t+1} and q^A_{t+1} that follow from these conjectures into (50), we can write the excess return as a function of future innovations. Second, we compute the distribution of the future innovations from a signal extraction problem. Finally, we apply the moments of the future innovations from the signal extraction problem to compute the expectations of er_{t+1}, er^2_{t+1} and er^3_{t+1} and then split these into components of different orders.

4.5.1 Conjecture Asset Prices

We make the following cubic conjecture for q^D_t and quadratic conjecture for q^A_t as a function of the state variables S_t and h_t:

$$\begin{aligned} q^D_t &= \alpha_{qD} S_t + \alpha_{5,qD} h_t + S'_t A_{qD} S_t + \beta_{qD} S_t h_t + \mu_{qD} (h_t)^2 + cubic_D(S_t, h_t) \qquad (51) \\ q^A_t &= \alpha_{qA} S_t + \alpha_{5,qA} h_t + S'_t A_{qA} S_t + \beta_{qA} S_t h_t + \mu_{qA} (h_t)^2 \qquad (52) \end{aligned}$$

where $cubic_D(S_t, h_t)$ stands for all 35 cubic terms in the elements of S_t and h_t. Since q^A_{t+1} is multiplied in the excess return expression (50) (through x^A_{t+1}) with terms that are linear and quadratic in other control and state variables, a quadratic approximation of the average asset price is sufficient for a cubic approximation of the excess return.

It is useful to write the state variables S_{t+1} as the sum of variables known at time t and future innovations: $S_{t+1} = (\rho a^D_t, \rho a^A_t, k^D_{t+1}, k^A_{t+1})' + (\varepsilon^D_{t+1}, \varepsilon^A_{t+1}, 0, 0)'$. Using the conjectures (51)-(52) we write q^D_{t+1} and q^A_{t+1} as respectively cubic and quadratic expressions of variables known at time t and future innovations ε^D_{t+1}, ε^A_{t+1} and h_{t+1}. Substituting this into (50), we write the excess return as a cubic expression of variables known at time t and the future innovations. The expectations of the terms involving future innovations are computed from signal extraction, to which we now turn.

4.5.2 Signal Extraction

Agents have no information on the future innovation h_{t+1} beyond its unconditional distribution with mean zero and variance $2(1+2\lambda^2\theta)\sigma_a^2$, and the fact that it is uncorrelated with productivity innovations at time $t+1$.

Home and Foreign investors infer the future productivity innovations, $\varepsilon_{H,t+1}$ and $\varepsilon_{F,t+1}$, using three sources of information: the unconditional distribution of innovations, private signals and the relative asset price q^D_t. Through the relative asset price agents learn $h_t = \varepsilon^D_{t+1} + \lambda\tau^D_t/\tau$.

Consider agent j in the Home country (the problem of a Foreign agent is solved along similar lines). The vector of productivity innovations is $\xi_{t+1} = (\varepsilon_{H,t+1}, \varepsilon_{F,t+1})'$. The Home agent j has a vector of signals $Y^{j,H}_t = \left(h_t, v^{H,H}_{j,t}, v^{H,F}_{j,t}, 0, 0\right)'$ that is linked to ξ_{t+1} through a matrix $X^{j,H}$ and is affected by shocks $\upsilon^{j,H}$:

$$Y^{j,H}_t = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \xi_{t+1} + \begin{pmatrix} \lambda\tau^D_t/\tau \\ \epsilon^{H,H}_{j,t} \\ \epsilon^{H,F}_{j,t} \\ -\varepsilon^H_{t+1} \\ -\varepsilon^F_{t+1} \end{pmatrix} \equiv X^{j,H}\xi_{t+1} + \upsilon^{j,H}_t$$

The expectation and variance of ξ_{t+1} by agent j are given by:

$$E^{j,H}_t\xi_{t+1} = \left[\left(X^{j,H}\right)'\left(R^{j,H}\right)^{-1}\left(X^{j,H}\right)\right]^{-1}\left(X^{j,H}\right)'\left(R^{j,H}\right)^{-1}Y^{j,H}_t \quad (53)$$

$$V^{j,H}_t\left(\xi_{t+1}\right) = \left[\left(X^{j,H}\right)'\left(R^{j,H}\right)^{-1}\left(X^{j,H}\right)\right]^{-1} \quad (54)$$

where $R^{j,H}$ is variance-covariance matrix of $\upsilon^{j,H}_t$, which is diagonal with $diag\left(R^{j,H}\right) = \left(4\lambda^2\theta\sigma_a^2, \sigma^2_{H,H}, \sigma^2_{H,F}, \sigma^2_a, \sigma^2_a\right)'$. A more detailed presentation of the coefficients is given in Appendix B.

We use (53) to write the expectations ε^D_{t+1} and ε^A_{t+1} as functions of h_t and the private signals. Taking the average across agents within each country, these become functions of h_t, ε^D_{t+1} and ε^A_{t+1} as the idiosyncratic components of private signals average to zero. The same is the case for expectations of quadratic and cubic terms in ε^D_{t+1} and ε^A_{t+1}. The coefficients that multiply h_t, ε^D_{t+1} and ε^A_{t+1} in these expectations are functions of σ^2_a and generally have components of different orders. We use these results to split the expectations of linear, quadratic and cubic terms in the future innovations ε^D_{t+1} and ε^A_{t+1} into components of different orders.

4.5.3 Excess Return Moments

We use our results to compute the expectations of er_{t+1}, er^2_{t+1} and er^3_{t+1}. As discussed above, we can write the excess return as a cubic expression of variables known at time t and the future innovations ε^D_{t+1}, ε^A_{t+1} and h_{t+1}. Using the results from the signal extraction, we write $\bar{E}^H_t er_{t+1}$ and $\bar{E}^F_t er_{t+1}$ as a cubic function of variables known at time t and the future innovations ε^D_{t+1} and ε^A_{t+1}. The same is the case for expectations of er^2_{t+1} and er^3_{t+1}.

A couple of points are worth making about these expectations. First, the expectations of er^2_{t+1} and er^3_{t+1} are the same for all agents (up to third-order terms) and only depend on variables known at time t. Second, $\bar{E}^H_t er_{t+1}$ and $\bar{E}^F_t er_{t+1}$ depend on the future innovations ε^D_{t+1} and ε^A_{t+1}, but only linearly and only in the third-order component. Finally, this third-order component of the expectation of er_{t+1} is the only term that differs across average Home and Foreign investors. This difference results from the higher precision of signals on domestic innovations ($\sigma^2_{HH} < \sigma^2_{HF}$). Specifically:

$$\left[\bar{E}^H_t er_{t+1} - \bar{E}^F_t er_{t+1}\right](3) = \mu_{er,1}(0)\frac{4\lambda^2\theta}{1+2\lambda^2\theta}\left(\frac{1}{\sigma^2_{HH}} - \frac{1}{\sigma^2_{HF}}\right)\varepsilon^A_{t+1} \qquad (55)$$

where $\mu_{er,1}(0) > 0$ is a function of the known parameters. When future world productivity rises, agents from both countries believe that productivity will rise more in their own country as they have better information about that. Holding ε^D_{t+1} fixed, this then implies that the expectation of the excess return on Home equity rises for Home investors and falls for Foreign investors.

4.6 Step 3: Imposing Equilibrium

We now impose the equations at various orders to compute the solution. This is itself a three-step process. The first two steps follow Devereux and Sutherland

(2010) and Tille and van Wincoop (2010). Step 1 involves imposing the second-order component of the cross-country difference in portfolio Euler equations (49) combined with first-order components of the 5 other equations. This delivers $z^D(0)$ and the first-order components of the so-called "other variables": $q_t^D(1)$, $q_t^A(1)$, $z_t^A(1)$, $k_{t+1}^D(1)$ and $k_{t+1}^A(1)$. These first-order solutions are a function of $S_t(1)$ and h_t. Step 2 involves imposing the third-order component of the cross-country difference in portfolio Euler equations (49) combined with second-order components of the 5 other equations. This delivers $z_t^D(1)$ and the second-order component of the "other variables": $q_t^D(2)$, $q_t^A(2)$, $z_t^A(2)$, $k_{t+1}^D(2)$ and $k_{t+1}^A(2)$. These second-order components are a function of $S_t(1)$, $S_t(2)$ and h_t.

The last step follows from the presence of dispersed information. Analogous to Section 3, we compute the noise to signal ratio λ by equating the first-order component of the average portfolio share z_t^A from the perspective of asset supply (first-order component of the difference in asset market clearing conditions (47)) to that from the perspective of portfolio demand. The latter follows by imposing the third-order component of the average portfolio Euler equation (48).

4.6.1 First-Order of Other Variables and Zero-Order Difference in Portfolio Shares

The first-order solutions of q_t^A and k_{t+1}^A are easily computed from the first-order component of the average capital accumulation equation (44) and the average asset market clearing equation (46):

$$q_t^A(1) = \frac{\xi}{1+\xi}\left[a_t^A(1) - \omega k_t^A(1)\right]$$
$$k_{t+1}^A(1) = \frac{1}{1+\xi}\left[a_t^A(1) + (1+\xi-\omega)\, k_t^A(1)\right]$$

The first-order component of the difference in capital accumulation equations (45) and the difference in asset market clearing conditions (47) gives

$$k_{t+1}^D(1) = k_t^D(1) + \frac{1}{\xi} q_t^D(1) \tag{56}$$

$$4z_t^A(1) = \frac{1+\xi}{\xi} q_t^D(1) + k_t^D(1) - z^D(0)(a_t^D + (1-\omega)k_t^D(1)) \tag{57}$$

This gives the solution for $k_{t+1}^D(1)$ and $z_t^A(1)$ conditional on the first-order component of the relative asset price, $q_t^D(1)$.

From the conjecture (51) we have

$$q_t^D(1) = \alpha_{qD}(0)S_t(1) + \alpha_{5,qD}(0)h_t \tag{58}$$

We can find the coefficients $\alpha_{qD}(0)$ and $\alpha_{5,qD}(0)$ by imposing the first-order component of the average portfolio Euler equation (48), which is

$$\left[\bar{E}_t^A er_{t+1}\right](1) = 0 \tag{59}$$

Computing the expected excess return as described in the previous sub-section, we get $[\bar{E}_t^A er_{t+1}](1) = \alpha_{er}(0)S_t(1) + \alpha_{5,er}(0)h_t$, where $\alpha_{er}(0)$ and $\alpha_{5,er}(0)$ are functions of the parameters $\alpha_{qD}(0)$ and $\alpha_{5,qD}(0)$. Setting the first-order component of the expected excess return equal to zero then implies $\alpha_{er}(0) = \alpha_{5,er}(0) = 0$, from which we compute $\alpha_{qD}(0)$ and $\alpha_{5,qD}(0)$. The latter is a function of the signal to noise ratio λ that remains to be computed. This solution for $q_t^D(1)$ then gives the solution of $k_{t+1}^D(1)$ and $z_t^A(1)$ as a function of $S_t(1)$ and h_t from (56)-(57).

Finally, we impose the second-order component of the cross-country difference in the portfolio Euler equation (49). This gives

$$z^D(0) = \frac{2\tau}{\left[E_t er_{t+1}^2\right](2)} \tag{60}$$

where we used the finding that the second-order component of the expected squared excess returns is the same for all investors worldwide. $\left[E_t er_{t+1}^2\right](2)$ is in turn computed from the first-order solution for the relative asset price (58).[11]

4.6.2 Second-Order of Other Variables and First-Order Difference in Portfolio Shares

We now repeat all of this one order higher. The second-order solutions of q_t^A and k_{t+1}^A are computed from the second-order component of the average capital accumulation equation (44) and the average asset market clearing equation (46):

$$\begin{aligned} q_t^A(2) &= -\frac{\xi\omega}{1+\xi}k_t^A(2) + S_t(1)'A_{qA}(0)S_t(1) + \beta_{qA}(0)S_t(1)h_t + \mu_{qA}(0)h_t^2 \\ k_{t+1}^A(2) &= \frac{1+\xi-\omega}{1+\xi}k_t^A(2) + S_t(1)'A_{kA}(0)S_t(1) + \beta_{kA}(0)S_t(1)h_t + \mu_{kA}(0)h_t^2 \end{aligned}$$

[11] In our particular model $[E_t er_{t+1}^2](2)$ does not depend on $z^D(0)$ as the first-order component of the relative asset price (and therefore the excess return) does not depend on $z^D(0)$. In a more general model though, $[E_t er_{t+1}^2](2)$ does depend on $z^D(0)$ (see Tille and van Wincoop (2010) for an example). In that case $z^D(0)$ needs to be computed as a fixed point problem from (60).

where the 4x4 matrices $A_{qA}(0)$ and $A_{kA}(0)$ are functions of known parameters, and the 1x4 vectors $\beta_{qA}(0)$ and $\beta_{kA}(0)$ and the scalars $\mu_{qA}(0)$ and $\mu_{kA}(0)$ are functions of known parameters and the noise to signal ratio λ.

The second-order component of the difference in capital accumulation equations (45) and the difference in asset market clearing conditions (47) gives

$$\begin{aligned} k_{t+1}^{D}(2) &= k_t^D(2) + \frac{1}{\xi} q_t^D(2) + \frac{\xi - 1}{\xi^2} q_t^A(1) q_t^D(1) && (61) \\ 4z_t^A(2) &= q_t^D(2) + k_{t+1}^D(2) - 2z^D(0)(1-\omega) k_t^D(2) \\ &\quad - z_t^D(1)\left[a_t^D(1) + (1-\omega) k_t^D(1)\right] && (62) \end{aligned}$$

We have already solved for $q_t^A(1)$ and $q_t^D(1)$. This then gives the solution for $k_{t+1}^D(2)$ conditional on $q_t^D(2)$ and the solution of $z_t^A(2)$ conditional on $q_t^D(2)$ and $z_t^D(1)$.

From the conjecture (51) we have

$$\begin{aligned} q_t^D(2) &= \alpha_{qD}(0) S_t(2) + \alpha_{qD}(1) S_t(1) + \alpha_{5,qD}(1) h_t && (63) \\ &\quad + S_t(1)' A_{qD}(0) S_t(1) + \beta_{qD}(0) S_t(1) h_t + \mu_{qD}(0) h_t^2 \end{aligned}$$

We have already solved for $\alpha_{qD}(0)$. We can find the coefficients $\alpha_{qD}(1)$, $\alpha_{5,qD}(1)$, $A_{qD}(0)$, $\beta_{qD}(0)$ and $\mu_{qD}(0)$ by imposing the second-order component of the average portfolio Euler equation (48), which is

$$\left[\bar{E}_t^A er_{t+1}\right](2) = 0 \tag{64}$$

From (50) we write the expected excess return as:

$$\begin{aligned} [\bar{E}_t^A er_{t+1}](2) &= \alpha_{er}(0) S_t(2) + \alpha_{er}(1) S_t(1) + \alpha_{5,er}(1) h_t \\ &\quad + S_t(1)' A_{er}(0) S_t(1) + \beta_{er}(0) S_t(1) h_t + \mu_{er}(0) h_t^2 \end{aligned}$$

where the coefficients are functions of the parameters in the expression (63) for $q_t^D(2)$. Setting the second-order component of the expected excess return equal to zero implies $\alpha_{er}(0) = 0$, $\alpha_{er}(1) = 0$, $\alpha_{5,er}(1) = 0$, $A_{er}(0) = 0$, $\beta_{er}(0) = 0$ and $\mu_{er}(0) = 0$. From this we can compute the parameters in the expression for $q_t^D(2)$. The vector $\alpha_{qD}(0)$ is the same as we had already computed before.

This solution for $q_t^D(2)$ then gives the solution of $k_{t+1}^D(2)$ from (61). The solution of $z_t^A(2)$ in (62) still depends on $z_t^D(1)$, which we compute by imposing third-order component of the cross-country difference in the portfolio Euler equation (49):

$$z_t^D(1) = \frac{\left[\bar{E}_t^H er_{t+1}\right](3) - \left[\bar{E}_t^F er_{t+1}\right](3)}{\left[E_t\left(er_{t+1}\right)^2\right](2)} - z^D(0) \frac{\left[E_t\left(er_{t+1}\right)^2\right](3)}{\left[E_t\left(er_{t+1}\right)^2\right](2)} \tag{65}$$

As discussed above, the second and third-order component of $E_t er_{t+1}^2$ are the same for all agents worldwide. Computing this expectation as described above, the last term in (65) becomes a linear function of a_t^A and k_t^A, with coefficients that depend on known parameters and the noise to signal ratio λ.

(55) gives the third-order component of the difference in the expected excess return between Home and Foreign investors, which shows up in the numerator of the first term in (65). It is proportional to ε_{t+1}^A. $[E_t er_{t+1}^2](2)$ in the denominator only depends on known parameters and λ. (65) then gives a solution for $z_t^D(1)$ as a linear function of a_t^A, k_t^A and ε_{t+1}^A. Together with the solution for $q_t^D(2)$, this then also gives the solution for $z_t^A(2)$ from (62).

At this point the only parameter that remains to be solved is the noise to signal ratio λ, to which we now turn.

4.6.3 Noise to Signal Ratio

As in Section 3, we compute the noise to signal ratio by equating the first order component of the average portfolio share, $z_t^A(1)$, from the asset supply side to that from the asset demand side. The former has already been solved by imposing the difference in asset market clearing conditions and is given by (57). Using the solution for $q_t^D(1)$, (57) gives $z_t^A(1)$ as a function of $a_t^D(1)$, $k_t^D(1)$ and h_t.

The average portfolio share from the demand side follows by imposing third-order component of the average of portfolio Euler equation (48). This gives

$$z_t^A(1) = \frac{\left[\bar{E}_{t+1}^A er_{t+1}\right](3) + 0.5\tau_t^D}{\left[E_t\left(er_{t+1}\right)^2\right](2)} \tag{66}$$

where we used that $\left[E_t\left(er_{t+1}\right)^3\right](3) = 0$. Using the expressions for the excess returns and our results from the signal extraction, $\left[\bar{E}_{t+1}^A er_{t+1}\right](3)$ is equal to

$$\mu_{er,1}(0)\frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\frac{\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\varepsilon_{t+1}^D + joint_t \tag{67}$$

where $joint_t$ includes terms in S_t and h_t. A full expression is in the Technical Appendix.

We now equate the expression for $z_t^A(1)$ from the supply and demand sides. This gives us a solution for both λ and $q_t^D(3)$, but our main interest is in λ. Note that in the expression for $z_t^A(1)$ from the supply side, ε_{t+1}^D and τ_t^D only enter in a jointly linear way through h_t (which affects $q_t^D(1)$). On the demand side they enter jointly through h_t (through the term $joint_t$ in (67)) and individually. In order for an equilibrium to exist we therefore need to make sure that ε_{t+1}^D and τ_t^D enter the numerator of (66) in the same joint way as in h_t. This leads to an implicit solution for λ that is zero-order as conjectured:

$$\mu_{er,1}(0)\frac{\sigma_{H,H}^2+\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2=\frac{\tau}{\lambda} \tag{68}$$

Equating $z_t^A(1)$ from the supply to the demand side then amounts to setting the coefficients of a cubic expression in S_t and h_t equal to zero. Using (51), this yields $\alpha_{qD}(2)$, $\alpha_{5,qD}(2)$, $A_{qD}(1)$, $\beta_{qD}(1)$, $\mu_{qD}(1)$ and zero-order components of parameters associated with the cubic terms in the conjecture for q_t^D. This gives $q_t^D(3)$. For most purposes however, first and second-order components are sufficient.

5 Conclusion

We develop a method for applying local approximation techniques to macro models with dispersed private information. This combines and extends existing local approximation methods that are widely used to solve macro models with public information with methods for solving NRE models in finance that have dispersed private information. When the method is applied to a simple linear NRE model for which a closed form solution can easily be derived, the zero, first, second and third-order components of the asset price solution are the same as in the closed form solution. We then showed how to apply the method to a more complex two-country DSGE model.

While we have purposefully kept the DSGE model relatively simple in order to better illustrate the local approximation method, there are no fundamental obstacles to applying the same method to larger scale models with more state and control variables and more sources of private information. In the model here the private information only affects portfolio choice. But more generally it also affects saving, leisure and investment choices. Private information is aggregated only through the relative asset price in the model, which then itself becomes a source of information. In more general settings it may be aggregated in a variety of ways through other macro variables as well. This will lead to larger signal extraction problems, but does not fundamentally alter the method itself.

Appendix A

This Appendix computes order components of $\bar{E}\varepsilon^f = a_v\varepsilon^f + a_h h$ and $var(\varepsilon^f)$ in the simple model of Section 3. Signal extraction gives the general experessions for $var\left(\varepsilon^f\right)$, a_v and a_h. These are functions of σ_f^2 and thus have different orders. We compute the orders by expanding the coefficients with respect to σ_f around $\sigma_f = 0$. For instance:

$$a_h = a_h|_{\sigma_f=0} + \left.\frac{\partial a_h}{\partial \sigma_f}\right|_{\sigma_f=0} \sigma_f + \frac{1}{2} \left.\frac{\partial^2 a_h}{(\partial \sigma_f)^2}\right|_{\sigma_f=0} \sigma_f^2 + \frac{1}{3} \left.\frac{\partial^3 a_h}{(\partial \sigma_f)^3}\right|_{\sigma_f=0} \sigma_f^3 + ...$$

The first term on the right-hand side is the zero-order component, the second term the first-order component, and so on. These expansions show that a_v has a second-order component, as does $var\left(\varepsilon^f\right)$, and a_h has zero- and second-order components:

$$\begin{aligned} a_v(2) &= \frac{\lambda^2\theta}{\sigma_v^2\left(1+\lambda^2\theta\right)}\sigma_f^2 \quad ; \quad var\left(\varepsilon^f\right)(2) = \frac{\lambda^2\theta}{1+\lambda^2\theta}\sigma_f^2 \\ a_h(0) &= \frac{1}{1+\lambda^2\theta} \quad ; \quad a_h(2) = -\frac{\lambda^2\theta}{\sigma_v^2\left(1+\lambda^2\theta\right)^2}\sigma_f^2 \end{aligned}$$

All other components are zero: $a_v(0) = a_v(1) = a_v(3) = 0$, $var\left(\varepsilon^f\right)(0) = var\left(\varepsilon^f\right)(1) = var\left(\varepsilon^f\right)(3) = 0$ and $a_h(1) = a_h(3) = 0$.

Using these results and that $h(1) = h$, we have

$$\begin{aligned} \left[\bar{E}\varepsilon^f\right](0) &= 0 \\ \left[\bar{E}\varepsilon^f\right](1) &= a_v(0)\varepsilon^f + a_h(0)h = \frac{1}{1+\lambda^2\theta}h \\ \left[\bar{E}\varepsilon^f\right](2) &= a_v(1)\varepsilon^f + a_h(1)h = 0 \\ \left[\bar{E}\varepsilon^f\right](3) &= a_v(2)\varepsilon^f + a_2(2)h = \frac{\lambda^2\theta\sigma_f^2}{\sigma_v^2\left(1+\lambda^2\theta\right)}\left[\varepsilon^f - \frac{h}{1+\lambda^2\theta}\right] \end{aligned}$$

Appendix B

The coefficients in the inferences of a Home investor j (53)-(54) are given by:

$$\begin{pmatrix} E_t^{j,H}\varepsilon_{H,t+1} \\ E_t^{j,H}\varepsilon_{F,t+1} \end{pmatrix} = \begin{pmatrix} \alpha_{\varepsilon H,h}^{j,H} & \alpha_{\varepsilon H,vH}^{j,H} & \alpha_{\varepsilon H,vF}^{j,H} \\ \alpha_{\varepsilon F,h}^{j,H} & \alpha_{\varepsilon F,vH}^{j,H} & \alpha_{\varepsilon F,vF}^{j,H} \end{pmatrix} \begin{pmatrix} h_t \\ v_{j,t}^{H,H} \\ v_{j,t}^{H,F} \end{pmatrix}$$

$$V_t^{j,H}\left(\xi_{t+1}\right) = \frac{\sigma_a^2}{V} \begin{pmatrix} \frac{1}{4\lambda^2\theta} + \frac{\sigma_a^2}{\sigma_{H,F}^2} + 1 & \frac{1}{4\lambda^2\theta} \\ \frac{1}{4\lambda^2\theta} & \frac{1}{4\lambda^2\theta} + \frac{\sigma_a^2}{\sigma_{H,H}^2} + 1 \end{pmatrix}$$

where:

$$V = \left(\frac{\sigma_a^2}{\sigma_{H,H}^2} + 1\right)\left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + 1\right) + \frac{1}{4\lambda^2\theta}\left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + \frac{\sigma_a^2}{\sigma_{H,H}^2} + 2\right)$$

$$\alpha_{\varepsilon H,h}^{j,H} = \frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + 1\right)\frac{1}{4\lambda^2\theta}$$

$$\alpha_{\varepsilon H,vH}^{j,H} = \frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + 1 + \frac{1}{4\lambda^2\theta}\right)\frac{\sigma_a^2}{\sigma_{H,H}^2}$$

$$\alpha_{\varepsilon H,vF}^{j,H} = \frac{1}{V}\frac{1}{4\lambda^2\theta}\frac{\sigma_a^2}{\sigma_{H,F}^2}$$

$$\alpha_{\varepsilon F,h}^{j,H} = -\frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,H}^2} + 1\right)\frac{1}{4\lambda^2\theta}$$

$$\alpha_{\varepsilon F,vH}^{j,H} = \frac{1}{V}\frac{1}{4\lambda^2\theta}\frac{\sigma_a^2}{\sigma_{H,H}^2}$$

$$\alpha_{\varepsilon F,vF}^{j,H} = \frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,H}^2} + 1 + \frac{1}{4\lambda^2\theta}\right)\frac{\sigma_a^2}{\sigma_{H,F}^2}$$

The inferences for a Foreign investor are computed along similar lines. The inferred variance is the same across all investors: $V_t^{j,F}\left(\xi_{t+1}\right) = V_t^{j,H}\left(\xi_{t+1}\right) = V_t\left(\xi_{t+1}\right)$. The coefficients in the expectations are:

$$\begin{aligned} \alpha_{\varepsilon H,h}^{j,F} &= -\alpha_{\varepsilon F,h}^{j,H} \quad ; \quad \alpha_{\varepsilon F,h}^{j,F} = -\alpha_{\varepsilon H,h}^{j,H} \\ \alpha_{\varepsilon H,vH}^{j,F} &= \alpha_{\varepsilon F,vF}^{j,H} \quad ; \quad \alpha_{\varepsilon F,vH}^{j,F} = \alpha_{\varepsilon H,vF}^{j,H} \\ \alpha_{\varepsilon H,vF}^{j,F} &= \alpha_{\varepsilon F,vH}^{j,H} \quad ; \quad \alpha_{\varepsilon F,vF}^{j,F} = \alpha_{\varepsilon H,vH}^{j,H} \end{aligned}$$

We now turn to the zero-, first- and second- order of the various coefficients. The inferred variance only has second-order terms:

$$\left[V_t\left(\xi_{t+1}\right)\right](2) = \frac{1}{2\left(1 + 2\lambda^2\theta\right)} \begin{pmatrix} 1 + 4\lambda^2\theta & 1 \\ 1 & 1 + 4\lambda^2\theta \end{pmatrix} \sigma_a^2$$

The coefficients on the asset price h_t have zero- and second-order terms:

$$\begin{aligned}
\left[\alpha^{j,H}_{\varepsilon H,h}\right](0) &= -\left[\alpha^{j,H}_{\varepsilon F,h}\right](0) = \frac{1}{2\left(1+2\lambda^2\theta\right)} \\
\left[\alpha^{j,H}_{\varepsilon H,h}\right](2) &= \frac{\sigma^2_{H,H} - \left(1+4\lambda^2\theta\right)\sigma^2_{H,F}}{4\left(1+2\lambda^2\theta\right)^2\sigma^2_{H,H}\sigma^2_{H,F}}\sigma^2_a \\
\left[\alpha^{j,H}_{\varepsilon F,h}\right](2) &= \frac{\left(1+4\lambda^2\theta\right)\left(\sigma^2_{H,H}\right) - \sigma^2_{H,F}}{4\left(1+2\lambda^2\theta\right)^2\sigma^2_{H,H}\sigma^2_{H,F}}\sigma^2_a
\end{aligned}$$

The coefficients on the private signals only have second-order terms:

$$\begin{aligned}
\left[\alpha^{j,H}_{\varepsilon H,vH}\right](2) &= \frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma^2_a}{\sigma^2_{H,H}} \quad ; \quad \left[\alpha^{j,H}_{\varepsilon H,vF}\right](2) = \frac{1}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma^2_a}{\sigma^2_{H,F}} \\
\left[\alpha^{j,H}_{\varepsilon F,vH}\right](2) &= \frac{1}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma^2_a}{\sigma^2_{H,H}} \quad ; \quad \left[\alpha^{j,H}_{\varepsilon F,vF}\right](2) = \frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma^2_a}{\sigma^2_{H,F}}
\end{aligned}$$

Using the various orders of the coefficients, we compute the various order of the expected innovations. While the signal from the asset price h_t only has a first-order component, the private signals include both a zero-order component through the idiosyncratic signal, and a first-order component through the common term reflecting the true innovation.

The zero-order component of expected innovations are zero for all agents. All investors also agree on the first-order component of expected innovations, which reflects the asset price:

$$\left[E^{j,H}_t\varepsilon_{H,t+1}\right](1) = \left[E^{j,F}_t\varepsilon_{H,t+1}\right](1) = -\left[E^{j,H}_t\varepsilon_{F,t+1}\right](1) = -\left[E^{j,F}_t\varepsilon_{F,t+1}\right](1) = \frac{1}{2\left(1+2\lambda^2\theta\right)}h_t$$

The second-order component of expected innovations reflects the idisosyncratic components of private signals, and thus averages to zero in each country:

$$\begin{aligned}
\left[E^{j,H}_t\varepsilon_{H,t+1}\right](2) &= \left[\left(1+4\lambda^2\theta\right)\frac{\epsilon^{H,H}_{j,t}}{\sigma^2_{H,H}} + \frac{\epsilon^{H,F}_{j,t}}{\sigma^2_{H,F}}\right]\frac{\sigma^2_a}{2\left(1+2\lambda^2\theta\right)} \\
\left[E^{j,H}_t\varepsilon_{F,t+1}\right](2) &= \left[\frac{\epsilon^{H,H}_{j,t}}{\sigma^2_{H,H}} + \left(1+4\lambda^2\theta\right)\frac{\epsilon^{H,F}_{j,t}}{\sigma^2_{H,F}}\right]\frac{\sigma^2_a}{2\left(1+2\lambda^2\theta\right)} \\
\left[E^{j,F}_t\varepsilon_{H,t+1}\right](2) &= \left[\left(1+4\lambda^2\theta\right)\frac{\epsilon^{F,H}_{j,t}}{\sigma^2_{H,F}} + \frac{\epsilon^{F,F}_{j,t}}{\sigma^2_{H,H}}\right]\frac{\sigma^2_a}{2\left(1+2\lambda^2\theta\right)} \\
\left[E^{j,F}_t\varepsilon_{F,t+1}\right](2) &= \left[\frac{\epsilon^{F,H}_{j,t}}{\sigma^2_{H,F}} + \left(1+4\lambda^2\theta\right)\frac{\epsilon^{F,F}_{j,t}}{\sigma^2_{H,H}}\right]\frac{\sigma^2_a}{2\left(1+2\lambda^2\theta\right)}
\end{aligned}$$

The third-order component of expected innovations reflects the asset price and the common component of private signals. It differs between Home and Foreign agents as each put more weight on the signals on domestic innovations that are more precise:

$$\begin{aligned}
\left[E_t^{j,H}\varepsilon_{H,t+1}\right](3) &= \left[\frac{\sigma_{H,H}^2-\left(1+4\lambda^2\theta\right)\sigma_{H,F}^2}{2\left(1+2\lambda^2\theta\right)\sigma_{H,H}^2\sigma_{H,F}^2}h_t+\left(1+4\lambda^2\theta\right)\frac{\varepsilon_{H,t+1}}{\sigma_{H,H}^2}+\frac{\varepsilon_{F,t+1}}{\sigma_{H,F}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)} \\
\left[E_t^{j,H}\varepsilon_{F,t+1}\right](3) &= \left[\frac{\left(1+4\lambda^2\theta\right)\sigma_{H,H}^2-\sigma_{H,F}^2}{2\left(1+2\lambda^2\theta\right)\sigma_{H,H}^2\sigma_{H,F}^2}h_t+\frac{\varepsilon_{H,t+1}}{\sigma_{H,H}^2}+\left(1+4\lambda^2\theta\right)\frac{\varepsilon_{F,t+1}}{\sigma_{H,F}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)} \\
\left[E_t^{j,F}\varepsilon_{H,t+1}\right](3) &= \left[-\frac{\left(1+4\lambda^2\theta\right)\sigma_{H,H}^2-\sigma_{H,F}^2}{2\left(1+2\lambda^2\theta\right)\sigma_{H,H}^2\sigma_{H,F}^2}h_t+\left(1+4\lambda^2\theta\right)\frac{\varepsilon_{H,t+1}}{\sigma_{H,F}^2}+\frac{\varepsilon_{F,t+1}}{\sigma_{H,H}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)} \\
\left[E_t^{j,F}\varepsilon_{F,t+1}\right](3) &= \left[-\frac{\sigma_{H,H}^2-\left(1+4\lambda^2\theta\right)\sigma_{H,F}^2}{2\left(1+2\lambda^2\theta\right)\sigma_{H,H}^2\sigma_{H,F}^2}h_t+\frac{\varepsilon_{H,t+1}}{\sigma_{H,F}^2}+\left(1+4\lambda^2\theta\right)\frac{\varepsilon_{F,t+1}}{\sigma_{H,H}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}
\end{aligned}$$

We compute the expectations of cross products of innovations using the definition of the covariance, namely: $[Exy](2) = [cov(x,y)](2) + [Ex](1)[Ey](1)$. All investors have the same second-order component of expected cross-products:

$$\begin{aligned}
\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^2\right](2) &= \left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^2\right](2) = \frac{1}{4\left(1+2\lambda^2\theta\right)^2}\left(h_t\right)^2+\frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \\
\left[E_t^{j,H}\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right](2) &= -\frac{1}{4\left(1+2\lambda^2\theta\right)^2}\left(h_t\right)^2+\frac{1}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2
\end{aligned}$$

The third-order component of expected cross-products of innovations reflects the idisosyncratic components of private signals, and thus averages to zero in each country:

$$
\begin{aligned}
\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^2\right](3) &= \left[\left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}+\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)^2}h_t \\
\left[E_t^{j,H}\left(\varepsilon_{F,t+1}\right)^2\right](3) &= -\left[\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}+\left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)^2}h_t \\
\left[E_t^{j,H}\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right](3) &= \left[\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}-\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}\right]\frac{\lambda^2\theta\sigma_a^2}{\left(1+2\lambda^2\theta\right)^2}h_t \\
\left[E_t^{j,F}\left(\varepsilon_{H,t+1}\right)^2\right](3) &= \left[\left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2}+\frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)^2}h_t \\
\left[E_t^{j,F}\left(\varepsilon_{F,t+1}\right)^2\right](3) &= -\left[\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2}+\left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)^2}h_t \\
\left[E_t^{j,F}\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right](3) &= \left[\frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}-\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2}\right]\frac{\lambda^2\theta\sigma_a^2}{\left(1+2\lambda^2\theta\right)^2}h_t
\end{aligned}
$$

We finally compute the expectations of cubic products using $[E(x^3)](3) = ([Ex](1))^3 + 3([Ex](1))[var(x)](2)$ and $[E(yx^2)](3) = 2[cov(x,y)](2)[Ex](1) + [Ey](1)[E(x^2)](2)$. All agents share the same expectations for the third-order components of cubic products:

$$
\begin{aligned}
\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^3\right](3) &= -\left[E_t^{j,H}\left(\varepsilon_{F,t+1}\right)^3\right](3) \\
&= \left[\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\right]^3+3\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \\
\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^2\varepsilon_{F,t+1}\right](3) &= -\left[E_t^{j,H}\varepsilon_{H,t+1}\left(\varepsilon_{F,t+1}\right)^2\right](3) \\
&= -\left[\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\right]^3+\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\frac{1-4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2
\end{aligned}
$$

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