Module Content:
1. One really useful tool in moment of inertia calculations is the parallel axis theorem. Know it and love it.
2. The idea of “principal” moments of inertia is important and useful; we calculate the principal moments of inertia for a (non-symmetric) cross section using inertia transformation equations.

Module Reading, Problems, and Demo:
Reading: Sections 9.3, 10.1-10.7
Problems: Prob. 10-81
Demo: none
Technology: people.virginia.edu/~ejb9z/Weblog
Theory: Inertia Transformations

- first, remember this?

- now the key question is: what angular orientation produces the LARGEST moment of inertia for this cross section?

- we can answer this question using inertia transformation, which basically amount to rotating the cross section in the plane until we find the largest moment of inertia

- The Setup:
  - we have two sets of axes--the original $(x,y)$ and a rotated $(u,v)$ which share an origin
  - we have defined a differential area $dA$ whose coordinates are expressed in both coordinate systems
  - clearly the key is the angle $\theta$
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\[ v = y \cos \theta - x \sin \theta \]
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  v &= y \cos \theta - x \sin \theta
\end{align*}
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\[
\begin{align*}
  dI_u &= v^2 \, dA = (y \cos \theta - x \sin \theta)^2 \, dA \\
  dI_v &= u^2 \, dA = (x \cos \theta + y \sin \theta)^2 \, dA \\
  dI_{uv} &= uv \, dA = (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta) \, dA
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\[
\begin{align*}
    I_u &= I_x \cos^2 \theta + I_y \sin^2 \theta - 2I_{xy} \sin \theta \cos \theta \\
    I_v &= I_x \sin^2 \theta + I_y \cos^2 \theta + 2I_{xy} \sin \theta \cos \theta \\
    I_{uv} &= I_x \sin \theta \cos \theta - I_y \sin \theta \cos \theta + I_{xy}(\cos^2 \theta - \sin^2 \theta)
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I_{uv} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta
\]
Theory: Principal Moment of Inertia

- So now we know the moment of inertia for any coordinate system \((u,v)\), rotated with respect to the original coordinate system \((x,y)\).

- So how do we orient \((u,v)\) such that the moments of inertia are maximum/minimum?

\[
\frac{dI_u}{d\theta} = -2\left(\frac{I_x - I_y}{2}\right) \sin 2\theta - 2I_{xy} \cos 2\theta = 0
\]

- There are two results of this kind of calculation, and they are the angles of orientation for the principal axes of inertia:

\[
\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2}
\]
Theory: Principal Moment of Inertia, II

- the most useful outcome of the remaining algebra and trigonometry is:

where the principal moments of inertia are given directly by the \((x,y)\) moments of inertia

- procedure:
  - determine the moments of inertia using a coordinate system \((x,y)\) which makes the calculation easy
  - use the transformation equation to find the principal (ma/min) moments of inertia for the cross section
Theory: Principal Moment of Inertia, II

- The most useful outcome of the remaining algebra and trigonometry is:

\[
I_{\text{max/min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}
\]

- Where the principal moments of inertia are given directly by the \((x,y)\) moments of inertia.

- Procedure:
  - Determine the moments of inertia using a coordinate system \((x,y)\) which makes the calculation easy.
  - Use the transformation equation to find the principal (ma/min) moments of inertia for the cross section.
Theory: Visualizing Inertia Transformations

- there is a fairly useful way to visualize these inertia transformations, because they actually (as you will soon see) describe a circle:

- square the first and third equations, then add:

- and redefine:
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I_u &= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \\
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- square the first and third equations, then add:

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\left( I_u - \frac{I_x + I_y}{2} \right)^2 + I_{uv}^2 = \left( \frac{I_x - I_y}{2} \right)^2 + I_{xy}^2
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(I_u - a)^2 + I_{uv}^2 = R^2
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\[(I_u - a)^2 + I_{uv}^2 = R^2\]

This is a circle:
- with center at \((a,0)\)
- and a radius of \(R = \sqrt{\ldots}\)
- on a set of coordinate axes \((I, I_{uv})\)
Theory: Describing the Circle

\[ R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \]

\[ I_{\min} = \frac{I_x + I_y}{2} \]

\[ I_{\max} = \frac{I_x - I_y}{2} \]
Theory: Describing the Circle

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maximum $I_{xy}$

$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$
Theory: Describing the Circle

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Theory: Describing the Circle

- in the principal orientations, the product of inertia is zero
- when product of inertia is maximum, moment of inertia is $I_{\text{ave}}$
Example, Mohr’s Circle: Ex. 10.10

- determine the principal moments of inertia using Mohr’s Circle and the centroidal axis.
Problem: z-section

Determine the principal moments of inertia for the beam’s cross-sectional area about the principal axes that have their origin located at the centroid C. Use the equations developed in Section 10.7. For the calculation, assume all corners to be square. Also, draw Mohr’s circle for inertia.
Example: a HW problem

- Determine the moments of inertia and the product of inertia of the beam’s cross section with respect to the u and v axes.
Example: a HW Problem (10-82, changed a little)

- use Mohr’s circle to determine the principal moments of inertia for the cross section