Lab 9 - Rotational Dynamics

OBJECTIVES

• To study angular motion including angular velocity and angular acceleration.
• To relate rotational inertia to angular motion.
• To verify the principal of Conservation of Mechanical Energy.
• To verify the principal of Conservation of Angular Momentum.

OVERVIEW

We want to study the rotation of a rigid body about a fixed axis. In this motion the distance traveled by a point on the body depends on its distance from the axis of rotation. However, the angle of rotation $\theta$, (also called the angular displacement), the angular velocity $\omega$, and the angular acceleration $\alpha$, are each the same for every point. For this reason, the latter parameters are better suited to describe rotational motion. The unit of angular displacement that is commonly used is the radian. By definition, $\theta$ is given in radians by the relation $\theta = \frac{s}{r}$, where $s$ is the arc length and $r$ is the radius as shown in Figure 1. One radian is the angle, measured at the center of a circle, whose legs subtend on the periphery an arc equal in length to the radius. An angle of 90° thus equals $\pi/2$ radians, a full turn $2\pi$ radians, etc. The angular velocity is the rate of change of the
angular displacement with time. It is equal to the angle through which the body rotates per unit time and is measured in radians per second. The angular acceleration is the rate of change of the angular velocity with time and is measured in radians per second squared. In the limit of very small times, the angular velocity is the derivative of the angular displacement with respect to time and the angular acceleration is the derivative of the angular velocity with respect to time:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

(1)

In linear motion the position, velocity, and acceleration are described by vectors. Rotational quantities can also be described by (axial) vectors. In these experiments, however, you will only have to make use of the magnitudes and signs of these quantities. There will be no explicit reference to their vector character.

Sometimes one needs the parameters of the linear motion of some point on the rotating rigid body. They are related very simply to the corresponding angular quantities. Let \( s \) be the distance a point moves on a circle of radius \( r \) around the axis; let \( v \) be the linear velocity of that point and \( a \) its linear acceleration. Then \( s, v, \) and \( a \) are related to \( \theta, \omega, \) and \( \alpha \) by

$$s = r\theta$$

$$v = r\omega$$

$$a = r\alpha$$

(2)

Let us now imagine a rigid body of mass \( m \) rotating with angular speed \( \omega \) about an axis that is fixed in a particular inertial frame. Each particle of mass \( m_i \) in such a rotating body has a certain amount of kinetic energy \( \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2 \omega^2 \). The total kinetic energy of the body is the sum of the kinetic energies of its particles.

If the body is rigid, as we assume in this section, \( \omega \) is the same for all particles. However, the radius \( r \) may be different for different particles. Hence, the total kinetic energy \( KE \) of the rotating body can be written as

$$KE = \frac{1}{2} \left( \sum m_i r_i^2 \right) \omega^2$$

(3)

The term \( \sum m_i r_i^2 \) is the sum over \( i \) of the products of the masses of the particles by the squares of their respective distances from the axis of rotation. We denote this quantity by the symbol \( I \). \( I \) is called the rotational inertia, or moment of inertia, of the body with respect to the particular axis of rotation. \( I \) has dimensions of \([ML^2]\) and is usually expressed in kg\(\cdot m^2\). For extended bodies, the sum must be replaced by an integral:

$$I = \sum m_i r_i^2 \Rightarrow \int r^2 \, dm = \int \rho r^2 \, dV$$

(4)

If the body has a uniform density (as is the case in this experiment) the integral can be rewritten as \( I = \rho \int r^2 \, dV \). This integral can be easily calculated only for bodies with a simple shape that is symmetrical around the axis of rotation. For example, some simple
shapes are given in Figure 2. Note that the rotational inertia of a body depends on the particular axis about which it is rotating as well as on the shape of the body and the manner in which its mass is distributed.

In terms of rotational inertia, we can now write the kinetic energy of the rotating rigid body as:

$$KE_{rot} = \frac{1}{2} I \omega^2.$$  \hspace{1cm} (5)

This is analogous to the expression for the kinetic energy of translation of a body, $KE_{tran} = \frac{1}{2} m v^2$. We have already seen that the angular speed $\omega$ is analogous to the linear speed $v$. Now we see that the rotational inertia $I$ is analogous to the translational inertial mass $m$. 

Figure 2
The rotational analog to force is torque (denoted by \( \tau \)). \( \tau \) is related to \( F \) by \( \tau = rF \) (for \( r \) perpendicular to \( F \)). The rotational analog to momentum is angular momentum: \( L = I \omega \). In rotational dynamics, Newton’s second law (\( F = ma = dp/dt \)) becomes:

\[
\tau = I \alpha = dL/dt
\]  

(6)

Recall that in the absence of external forces, linear momentum is conserved. Similarly, in the absence of external torques, angular momentum is conserved. Finally, if no non-conservative forces (such as friction) or torques act, then mechanical energy is conserved.

In Table 1 (below) we compare the translational motion of a rigid body along a straight line with the rotational motion of a rigid body about a fixed axis.

**Table 1. Rectilinear and Rotational Quantities**

<table>
<thead>
<tr>
<th>Rectilinear Motion</th>
<th>Rotation About a Fixed Axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td>Angular displacement</td>
</tr>
<tr>
<td>( x )</td>
<td>( \theta )</td>
</tr>
<tr>
<td>Velocity</td>
<td>Angular velocity</td>
</tr>
<tr>
<td>( v = \frac{dx}{dt} )</td>
<td>( \omega = \frac{d\theta}{dt} )</td>
</tr>
<tr>
<td>Acceleration</td>
<td>Angular acceleration</td>
</tr>
<tr>
<td>( a = \frac{dv}{dt} )</td>
<td>( \alpha = \frac{d\omega}{dt} )</td>
</tr>
<tr>
<td>Mass (translational inertia)</td>
<td>Rotational inertia</td>
</tr>
<tr>
<td>( m = \int \rho dV )</td>
<td>( I = \int r^2 \rho dV )</td>
</tr>
<tr>
<td>Linear momentum</td>
<td>Angular momentum</td>
</tr>
<tr>
<td>( p = mv )</td>
<td>( L = I \omega )</td>
</tr>
<tr>
<td>Kinetic energy</td>
<td>Kinetic energy</td>
</tr>
<tr>
<td>( KE = \frac{1}{2}mv^2 )</td>
<td>( KE = \frac{1}{2}I \omega^2 )</td>
</tr>
<tr>
<td>Force</td>
<td>Torque</td>
</tr>
<tr>
<td>( F = ma )</td>
<td>( \tau = I \alpha )</td>
</tr>
<tr>
<td>Work</td>
<td>Work</td>
</tr>
<tr>
<td>( W = \int F , dx )</td>
<td>( W = \int \tau , d\theta )</td>
</tr>
<tr>
<td>Power</td>
<td>Power</td>
</tr>
<tr>
<td>( P = Fv )</td>
<td>( P = \tau \omega )</td>
</tr>
</tbody>
</table>

**WARNING**: DO NOT MOVE THE ROTATING DISKS UNLESS THE AIR SUPPLY IS ON. THE TA MUST ASSIST WHENEVER DISKS ARE CHANGED.

You will need to have the following apparatus for these experiments:

- Rotational Dynamics kit
- air supply
- meter stick
- 20 g mass
**INVESTIGATION 1: EXPERIMENTAL SET UP**

**Description**

The experimental setup consists of one or two metal disks floating on air cushions but constrained so as to rotate about their common axis. A smaller aluminum disk called the “Torque Pulley” is firmly coupled to the top disk.

Note: We denote the radius of the torque pulley by a lower case \( r \) so as to distinguish it from the dimensions of the main disks.

By wrapping a string around the torque pulley and tugging on it, we can apply a torque (whose magnitude is equal to the product of the string tension and the torque pulley radius, \( \tau = Tr \)) to the system. We will provide this tension by looping the string over a low friction pulley and hanging a small mass on the end. As we will see later, the tension will not simply be \( mg \), the weight of the hanging mass, but will be reduced slightly due to the fact that the mass is accelerating (see equation 8).

![Figure 3](image)

**Activity 1-1: Apparatus Leveling**

1. The clearance between the spindle and disks is only 0.001", and any nicks will damage the low friction support. The Rotational Dynamics apparatus is connected to an air supply with a pressure regulator. The pressure should be preset at about 10 psi (with air flowing) for operation.

2. Arrange the apparatus (see Figure 4 below) so the air-bearing pulley extends over the edge of your lab table. Check with your TA if you are unsure.

3. To ensure that the disk rotates with uniform velocity or acceleration, even with an eccentric load, the apparatus must be leveled accurately.

4. Turn on the air with the top aluminum disk on the spindle. If the disk on the apparatus is not made of aluminum, ask your TA to put it on.

5. Place bubble level on top of the aluminum disk and verify that the apparatus is level. If it is not, adjust the three leveling feet until it is.
Activity 1-2: Disk rotation

1. The two disks can spin independently or together, or the upper disk can spin while the lower disk does not (see Figure 4). These options are controlled using the two valve pins that are provided with the unit. When not in use, these pins can be stored in the valve pin storage holes on the top of the base. Start with the pins in the storage holes.

2. Place one valve pin in the bottom disk valve, located next to the valve pin storage holes. Give the upper disk a spin. Notice that it lies firmly on the lower disk so the two disks spin together. Remove the valve pin and notice how both disks drop onto the base plate.

3. Replace the valve pin in the bottom disk valve and then place the remaining valve pin into the hole in the middle of the upper disk, as in the figure. Now spin the disks in opposite directions. Notice that the two disks now spin independently.

4. Pull the valve pin from the center of the upper disk. The upper disk drops onto the lower disk so that the disks now spin together, as a single rotating body. This is the rotational equivalent of an inelastic collision. You may find that the aluminum disk tends to float a bit, because it is so much lighter than the steel disk. If so, reduce the air pressure slightly.

The theoretical rotational inertia of the rotating disks (annular cylinders) is given by (see Figure 2)

\[
I = \frac{1}{2} M \left( R_1^2 + R_2^2 \right),
\]  

(7)
where $M$ is the mass of the disk, and $R_1$ and $R_2$ are the disk’s outer and inner radii, respectively. The masses are

- Bottom stainless steel: 1.344 kg
- Top stainless steel: 1.357 kg
- Top aluminum: 0.464 kg

The geometric shape of the disks are not identical; however, within the accuracy of your measurement you can use for all three disks:

\[ R_1 = 0.0632 \text{ m and } R_2 = 0.0079 \text{ m}. \]

5. Calculate (if you have not already done so) the rotational inertia of the disks.

<table>
<thead>
<tr>
<th></th>
<th>Rotational Inertia $I$ in kg·m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top stainless steel</td>
<td>___________________________</td>
</tr>
<tr>
<td>Bottom stainless steel</td>
<td>___________________________</td>
</tr>
<tr>
<td>Aluminum</td>
<td>___________________________</td>
</tr>
</tbody>
</table>

**INVESTIGATION 2: ROTATIONAL KINEMATICS AND TORQUE**

**Activity 2-1  How Does Torque Affect An Object's Rotational Motion?**

We want to verify the rotational analogue of the relationship between force and acceleration as given by Equation 6. We will apply a constant torque to an aluminum disk by attaching a mass to a string wrapped around the disk and then hang the mass over a frictionless pulley. The constant force due to gravity acting on the mass ($F_{\text{gravity}} = mg$) is transferred to the disk by the string hanging over the pulley as a constant tension.

**Prediction 2-1:** Explain what will happen to the disk when the mass hanging down over the pulley is released. Does this represent a constant torque?
Question 2-1:  If you have the means to measure the angular velocity of the disk as a function of time, how can you determine if the angular acceleration is constant?

The tension in the string causes a torque on the disk since the string is attached to the disk via a small pulley (with a nominal diameter of 25 mm). When the disk is released, it will start to rotate with constant angular acceleration as the mass falls with constant linear acceleration. We will measure this angular acceleration by plotting the angular velocity of the disk over time. Because $\omega = \alpha t$, if we fit a straight line to the angular velocity data, the slope of the line will give us the angular acceleration of the disk. We will then see if this angular acceleration agrees with the angular acceleration calculated by solving the force equations for the disk and the mass.

Make the measurements using the lighter aluminum top disk. If the disk on the apparatus is not made of aluminum, ask your TA to put it on. Do not replace the disks yourself!

1. Remove the valve pin for the bottom disk so that the bottom disk does not rotate.
2. Cut a piece of thread about 135 cm in length.
3. Measure the diameter of the torque pulley and record the radius below.

   \[ r : \quad \text{mm} \]

4. Referring to above, tie one end of the thread to the hole in the thread holder. Place the thread holder in the recess of the small torque pulley, with the thread passing through the slot in the pulley. Then use the thumbscrew to attach the pulley to the top of the rotating disk, with the flat side of the pulley facing up, so the thread holder is underneath the pulley. Tighten the thumbscrew so the pulley is secure. Make sure that the string is not caught under the pulley! Weigh the “20 g” mass.

   \[ m : \quad \text{g} \]

5. Attach the 20 g mass to the other end of the thread. When the thread is fully extended, the mass should almost touch the floor.

6. Open the experimental file L09.2-1 Angular Velocity. This will set up the computer to graph the angular velocity of the disk in degrees/second.

7. Rotate the disk until the mass is approximately at the table height. Start the computer and immediately release the disk. Stop the computer when the mass has fallen at least 80 cm.
8. As the mass falls, the disk will rotate with constant angular acceleration and the angular velocity on the graph should increase linearly. Using the mouse, select a region in the middle of the data where you are sure the mass was falling freely. Click on the **Fit** icon on the graph toolbar and select Linear Fit. Record the slope of the fit \((m)\) here:

Slope _________________ deg/s^2.

9. **Print** a copy of the graph with Fit displayed.

As we noted before, the tension in the string is not simply the weight of the mass as the mass is accelerating. We can use Newton’s Second Law to find the tension. The net force on the falling mass is

\[ F_{net} = mg - T = ma \]  

(8)

where \(T\) is the tension in the string, \(m\) is the mass, and \(a\) is the linear acceleration of the mass. Solving for \(T\) gives \(T = mg - ma\).

Because the mass is attached to the edge of the torque pulley, the magnitude of the linear acceleration of the mass is the same as the magnitude of the linear acceleration of any point on the outer edge of the torque pulley. Hence \(a = \alpha r\).

Because the disk is rotating on a nearly frictionless layer of air, the only torque acting on it is from the tension in the string

\[ \tau_{net} = Tr = (mg - ma)r = (mg - m\alpha r)r = I\alpha \]  

(9)

where \(\tau\) is the torque on the disk, \(T\) is the tension in the string, \(r\) is the radius of the torque pulley, and \(\alpha\) is the angular acceleration of the disk. Solving for \(\alpha\) gives:

\[ \alpha = \frac{mgr}{I + mr^2} \]  

(10)

You calculated the rotational inertia previously, so calculate \(\alpha\) using the expression above and compare with the value of the angular acceleration of the disk you obtained from your data.

\[ \alpha = _________________ \text{ rad/s}^2 \text{ (theoretical)} \]

\[ \alpha = _________________ \text{ rad/s}^2 \text{ (experimental)} \]

Difference in \(\alpha\) _______________ %
**Question 2-2:** How did your results compare? If instead of angular velocity data you were given angular position data, what type of fit would you need to perform the find the angular acceleration of the disk?

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**INVESTIGATION 3: CONSERVATION OF ENERGY**

**Activity 3-1 Does Potential Energy Lost Equal Kinetic Energy Gained?**

We want to demonstrate that mechanical energy is conserved in the absence of non-conservative forces. We will apply a torque to a pulley attached to the aluminum disk via a thread attached to a small hanging mass and see if the potential energy lost by the mass as it falls is equal to the gain in the linear kinetic energy of the mass and the rotational kinetic energy of the disk. The weight of the mass supplies a constant torque that accelerates the rotating disk.

Because mechanical energy is conserved as the mass falls, the initial potential energy of the hanging mass $m$ is converted to kinetic energy. After falling a distance $s$, the mass loses an amount of potential energy, $mgs$. The mass has a translational kinetic energy due to its velocity $v$ and rotational kinetic energy due to the upper disk rotating with an angular velocity $\omega$. For energy to be conserved, the following relation must hold:

$$mgs = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$  \hspace{1cm} (11)

where $I$ is found using Equation 7.

**NOTE:** By differentiating both sides of Equation 11 with respect to time we can see that the linear acceleration of the mass is (as stated above) constant. Therefore, the angular acceleration of the disk ($\alpha = a/r$) and the torque applied to the disk ($\tau = I\alpha$) are also constant. Hence, when the disk is released, the mass descends, moving with constant linear acceleration and causing the disks and axle to rotate with constant angular acceleration.

We will now perform an experiment to verify that mechanical energy is conserved by calculating and comparing the left hand and right hand sides of Equation 11.

1. Open the experimental file **L09.3-1 Conservation of Energy**. The Calculator window should be open. Enter your values for $m$, the mass of the hanging weight, $r$,

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1 This distance is the same as the distance that a point on the outer edge of the pulley traces out, $s = r\theta$.

2 $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$
the radius of the torque pulley, and $I$, the rotational inertia of the upper disk. Close the Calculator window.

2. An energy graph will remain visible. The linear kinetic and gravitational potential energies of the hanging mass and the rotational kinetic energy of the disk will be graphed, as will their sum, the total mechanical energy. Note that we have chosen the “zero” for potential energy to be when the mass is at its highest point. The total mechanical energy will therefore initially be zero as well.

3. Turn on the air and rotate the disk until the mass is approximately 90 cm above the floor. **Start the computer**, release the disk, and let the mass drop. **Stop the computer** just before the string runs out on the pulley and the mass “rebounds”.

4. **Print** out a copy of the graph for your report.

**Question 3-1:** How well do your results support the contention that mechanical energy is conserved?

**Prediction 3-1:** Discuss what would happen if you applied a frictional torque to the disk. How would the energy graphs be different?

5. Repeat step 3, but this time use your finger to apply a little friction to the spinning disk (but no so much as to keep it from spinning).

6. **Print** out a copy of the graph for your report.

**Question 3-2:** How well do your results agree with your prediction?
INVESTIGATION 4: CONSERVATION OF ANGULAR MOMENTUM

Activity 4-1: Is Angular Momentum Conserved?

Angular momentum, $L = I\omega$, is conserved whenever there are no external torques. In the case of rotating disks that engage each other, all torques are internal and we expect to have conservation of angular momentum.

In this measurement, you will use the optical reader on the Rotational Dynamics apparatus, which counts the black bars on the disks as they pass. Each LED comes on when the corresponding optical reader senses a black bar and goes off when it detects a white bar. Using this data, the computer will display a graph of angular velocity vs. time.

1. Remove the torque pulley and mass. Replace valve stems so both bottom and top disks rotate separately.

2. Ask your TA to remove the top aluminum disk and replace it with the top stainless steel disk.

3. Open experiment file L09.4-1 Angular Momentum. You will see a graph showing the angular velocity of each disk vs. time. Ch 1 should show data from the upper disk and Ch 2 should show data from the lower disk. Verify this by starting the computer and spin one of the disks while holding the other one still. What happens when you reverse the direction of the spin?

4. Perform the following four experiments. Then calculate the initial and final angular momentum and determine whether it is conserved. Enter your data and calculations in Table 4-1. For purposes of this calculation assume that both stainless steel disks have the same rotational inertia. Remember that the angular velocity $\omega$ can be negative. You need to keep track of its sign. Data Studio always indicates a positive number.

Note: Completing Table 4-1 is time consuming. It is crucial for at least one team member to be working on these calculations throughout the experiments.

<table>
<thead>
<tr>
<th>Table 4-1 Conservation of Angular Momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part a</td>
</tr>
<tr>
<td>$\omega$ Initial Top</td>
</tr>
<tr>
<td>$\omega$ Initial Bottom</td>
</tr>
<tr>
<td>$\omega$ Final</td>
</tr>
<tr>
<td>$L$ Initial Total</td>
</tr>
<tr>
<td>$L$ Final</td>
</tr>
<tr>
<td>$L$ Difference</td>
</tr>
</tbody>
</table>

a. **Top disk spinning; bottom disk stationary:** Start the computer, hold the bottom disk stationary, and give the top disk a spin, so that its angular velocity is between 600 and 800 deg/s. Wait for a couple of seconds, let go of the bottom disk and then
pull the valve pin from the top disk so that the top disk falls onto the bottom disk. Wait for at least two full seconds and then stop the computer. Record the angular velocity of each disk just before and after releasing the valve. You will get a better reading by finding a range of data points for each measurement and using the statistics function to find the mean of these values. Enter your data into Table 4-1.

**Question 4-1:** How well do the initial and final angular momentum agree? Is this good enough agreement? Explain.

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**b. Top and bottom disks spinning in the same direction but at different rates:** Perform the same procedure as in part a, but this time spin both disks in the same direction but at different rates, at least 200 deg/s apart. Enter your data into Table 4-1.

**Question 4-2:** How well do the initial and final angular momentum agree? Is this good enough agreement? Explain.

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**c. Top and bottom disks spinning in opposite directions at different rates:** Spin both disks in opposite directions and at different rates. Make sure to record the direction in which each disk is spinning, i.e. clockwise or counter-clockwise. Since the sensors have no way of knowing which direction the disks are spinning, the angular velocity of each disk will be positive on the graph even though they are spinning in opposite directions. Remember that this will make one disk’s angular momentum negative relative to the others. Perform the same procedure as in part a and enter your data into Table 4-1.

**Question 4-3:** How well do the initial and final angular momenta agree? Is this good enough agreement? Explain.
d. **Top and bottom disks spinning in opposite directions at the same rate:** Try to spin the two disks at the same rate but in opposite directions. Follow the same procedure as in part a and enter your data into Table 4-1.

**Question 4-4:** What happened when you removed the pin? How well do the initial and final angular momentum agree? Is this good enough agreement? Explain.

**Question 4-5:** Discuss possible sources of error in this activity.

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Turn off the Rotational Dynamics apparatus, the timers, and the AIRFLOW.
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