Augmented Lagrangian with Variable Splitting for Faster Non-Cartesian $L_1$-SPIRiT
MR Image Reconstruction: Supplementary Material

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Here, we provide additional results supporting the use of our implementation of non-Cartesian $L_1$-SPIRiT, using a combination of variable splitting, an augmented Lagrangian formulation, and preconditioned conjugate gradients.

I. DATA ACQUISITION

In the paper [1], we described both simulated and real data sets. Figure 1, reprinted from [1], portrays ground truth images for all data sets. To save space, we focused on convergence plots and reconstructed images resulting from our real data experiments only. Additional convergence plots and reconstructions featuring our simulated data sets with eight and 16 channels of $T_1$-weighted spiral data are included in this supplement. The eight-channel simulated data was undersampled by a factor of three, while the 16-channel simulated k-space was undersampled by a factor of four to leverage the increased number of coils.

In the experiments that follow, our reconstructions use the same methods and parameter choices described in the original paper. In particular, we used the same sparsifying transforms, regularization parameters, penalty parameter selections, and conjugate phase image initializations. To evaluate the convergence rates of the different algorithms, our baseline was the same preconditioned Split-Bregman method run for 20,000 iterations.

II. CONVERGENCE RATE COMPARISONS

In the original paper, relative objective function values $f(X) - f^{opt}$ are plotted for both the real $T_1$-weighted spiral 2D data and real $T_2$-weighted radial 2D data. Here, we provide the same plots over a longer time scale, so that long-term convergence behavior of the different algorithms is evident. Since the objective function may vary relatively slowly in a neighborhood around its minimum, the reconstructed image $X$ may actually be farther from its optimum $X^{opt}$ than a small relative objective function value would lead one to believe. Therefore, in this supplement, we also include plots of the normalized root mean squared difference (NRMSD), as introduced in the original text. To minimize the bias any potential non-uniqueness of the optimal reconstruction may have on our conclusions, we used the Split-Bregman baseline image.

The convergence plots for the real $T_1$-weighted spiral 2D data are shown in Fig. 2. Our preconditioned ADMM-based methods both converge more rapidly than preconditioned Split-Bregman iteration and both approaches without preconditioning, in terms of both objective function value and...
NRMSD. This fast convergence occurs even though we use the condition numbers calibrated with the simulated data to tune our method’s penalty parameters. Our proposed method also beats the NLCG method until the NLCG method experiences rapid convergence around 400 seconds into the experiment. The NLCG method flattens out, although the difference in final objective value or NRMSD appears relatively insignificant for this data set.

The convergence rate improvements portrayed in Fig. 3 for the real $T_2$-weighted radially sampled 2D data echo what were observed for the real $T_1$-weighted spiral data. Again, our preconditioned ADMM-based methods converge faster than preconditioned Split-Bregman. The improvement over NLCG is significant in the initial minutes of run time, but the NLCG method undergoes a phase of rapid convergence around 300 seconds, before flattening out.

The convergence plots for the 8-channel simulated spiral 2D data shown in Fig. 4 and the 16-channel simulated spiral 2D data in Fig. 5 both depict far more substantial advantages in both NRMSD and objective function value. The preconditioned forms of the proposed ADMM method converge much more rapidly than either competing method for the simulated data. The non-preconditioned variants have the slowest convergence rates of all. This performance holds true when measuring convergence of either the image or the objective function value. We notice that the difference in convergence rates for using exact $z$-updates versus preconditioned $z$-updates in our proposed algorithm appears to grow noticeably with the larger number of channels.

Reconstructed images from the 8-channel simulated spiral 2D data captured after 50 seconds are displayed in Fig. 6 for NLCG and the preconditioned variants of the Split-Bregman and proposed ADMM-based methods. Figure 7 shows similar reconstructions from the 16-channel simulated spiral 2D data after 100 seconds of reconstruction time. Prominent spiral artifacts are visible in both the magnitude and difference images for the NLCG and Split-Bregman methods, in both sets of reconstructions. The proposed ADMM-based reconstructions contain no such visible artifacts. These images corroborate the substantial difference in reconstruction quality after a limited reconstruction time observed for the two real data sets in the original manuscript. Furthermore, we demonstrate the improvement carries over to larger receive array coils, even using the same condition numbers and shrinkage threshold fraction to tune our algorithm.

III. DISCUSSION AND CONCLUSIONS

The preconditioned variants of the proposed ADMM-based method clearly outperform conventional approaches for non-Cartesian $L_1$-SPIRiT reconstruction, at least for a limited reconstruction time. The uniform quality of the reconstructions of all the data sets using the same condition numbers and sparse threshold fraction demonstrates the generality of both our proposed method and our approach for tuning the penalty parameters, which are known to impact the convergence rate of such algorithms. The importance of preconditioning is also evident here, as the un-preconditioned ADMM-based method converges relatively slowly, as does the un-preconditioned Split-Bregman iteration.
Fig. 3. The (a) NRMSDs (dB) and (b) objective values $f(X) - f^{opt}$ relative to the baseline image objective value $f^{opt}$ are plotted versus time for each of the compared algorithms for the real T2-weighted radial data.

Fig. 4. The (a) NRMSDs (dB) and (b) objective values $f(X) - f^{opt}$ relative to the baseline image objective value $f^{opt}$ are plotted versus time for each of the compared algorithms for the simulated 8-channel spiral data.

REFERENCES

Fig. 5. The (a) NRMSDs (dB) and (b) objective values $f(X) - f^{opt}$ relative to the baseline image objective value $f^{opt}$ are plotted versus time for each of the compared algorithms for the simulated 16-channel spiral data.
Fig. 6. Sum-of-squared combined reconstructed and difference images for the simulated 8-channel data set after 50 seconds of reconstruction time using (a) NLCG, (b) Split-Bregman with diagonal preconditioning, (c) ADMM with preconditioning both $X$ and $z$-updates, and (d) ADMM with preconditioned $X$-updates and exact $z$-updates demonstrate the advantage of rapid convergence on image quality after a fixed amount of time. Zoomed images of the inset region show differences in fine details. The difference images in the right column, relative to the ground truth, are all scaled by 10×.

Fig. 7. Sum-of-squared combined reconstructed and difference images for the simulated 16-channel data set after 100 seconds of reconstruction time using (a) NLCG, (b) Split-Bregman with diagonal preconditioning, (c) ADMM with preconditioning both $X$ and $z$-updates, and (d) ADMM with preconditioned $X$-updates and exact $z$-updates demonstrate the advantage of rapid convergence on image quality after a fixed amount of time. Zoomed images of the inset region show differences in fine details. The difference images in the right column, relative to the ground truth, are all scaled by 5×.