VARIATIONS ON KURATOWSKI'S 14-SET THEOREM

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1. INTRODUCTION

The following result, from Kuratowski's 1920 dissertation, is known as the *14-set theorem.*

Theorem 1.1. [8] *Let $E \subseteq X$ be a subset of a topological space. The number of distinct sets which can be obtained from E by successively taking closures and complements (in any order) is at most 14. Moreover, there are subsets of the Euclidean line for which 14 is attained.*

In this article we will see what happens when "closure" and "complement" are replaced or supplemented with other basic topological operations.

Question 1.2. Let $\mathcal{I}$ be a subcollection of

$$\{\text{closure, interior, complement, intersection, union}\}.$$

What is the maximum number of distinct sets which can be generated from a single subset of a topological space by successive applications of members of $\mathcal{I}$?

Apparently Question 1.2 will require us to solve $2^5 = 32$ different problems... well, not really. Many of these are redundant, either because different choices of $\mathcal{I}$ allow us to perform the same operations, or because different choices of $\mathcal{I}$ raise algebraically isomorphic questions. (This is explained in Section 4.) Theorem 1.1 answers at least one case, and certainly many others are trivial. Finally, an example of Kuratowski shows that if we allow all five operations, we may obtain infinitely many sets. After a full reckoning, only two questions remain.

Question 1.3.
  (1) What is the maximum number of distinct sets that can be generated from a fixed set in a topological space by successively taking closures, interiors, and intersections (in any order)?
  (2) Same question, but with closures, interiors, intersections, and unions.

Question 1.3(1) was posed by Smith in a 1974 *Monthly* as Problem 5996, with published solution given later by Yu [12]. The numerical answer to Question 1.3(2) was stated by Langford in an abstract in the 1975 *Notices of the AMS* [9], but it seems that no proof had appeared before the first version of the present article was circulated. Recently Gardner and Jackson published a nice article [3] which also solves Question 1.3(2) and gives a thorough discussion of many other aspects of Kuratowski-type problems.

With a little additional work, we will finally solve a generalization which has not appeared elsewhere.

Question 1.4. Same as Question 1.2, with $n \geq 2$ sets initially given.

Our approach to this topic, like Kuratowski's, is almost entirely *algebraic*. The basic language is the "topological calculus" which was developed by the Polish school during the first half of the twentieth century. Prominent figures such as Birkhoff, Stone, Halmos, and Tarski kept this program dynamic, intertwining topology with the related fields of set theory, logic, and lattice theory. (And the incorporation of Hilbert spaces opened up new realms of noncommutative analysis, with von Neumann at the center.) In the present article, the relevant topological calculus is known as a "closure algebra," and the questions above reduce to the calculation of certain algebras generated by a specific partially ordered set. So while our subject is apparently point-set topology, membership of points in sets plays a very minor role!

Theorem 1.1 is actually easy to prove, almost certainly having acquired some cachet from the unusual presence of the number 14. Readers who have never seen it before may enjoy experimenting at Mark Bowron's website `http://www.kuratowski.com`, which allows a visitor to construct an initial set, then counts the number of sets generated by taking closures and complements. (A score of 14 "wins.") Within the substantial literature surrounding this theorem, the idea of considering different collections of topological operations goes back at least to 1927 [13]. Other authors have abstracted the algebraic content, or isolated the specific conditions under which a topological space and subset generate 14 sets. Some of these variations are described in the last section.

This article is intended for the nonspecialist in universal algebra - indeed, it was written by one. Thus we define even basic terms, and do not always give the most general formulations. It is hoped that many readers will find the methods at least as interesting as the answers.

2. Monoids, posets, and the proof of Theorem 1.1

We start with the basics. Let X be an arbitrary set, and let $\mathcal{P}(X)$ be the set of subsets of X. To endow X with a *topology* means to choose a distinguished subset of $\mathcal{P}(X)$, called the *open sets*, which is closed under arbitrary unions and finite intersections, and contains both X and the empty set. The complement of an open set is a *closed set*. The (topological) *closure* of $E \in \mathcal{P}(X)$ is the smallest closed set containing E; the *interior* of E is the largest open set contained in E. Therefore the three functions "closure of," "interior of," and "complement of" can naturally be viewed as operations on $\mathcal{P}(X)$. We will denote them by k, i, and c, respectively, and write them to the left of the set, as is usual for operators (or English sentences). We also denote the collection of maps $\mathcal{P}(X) \to \mathcal{P}(X)$ as $\mathrm{End}(\mathcal{P}(X))$ (for "endomorphisms" – but these maps need not preserve any of the algebraic structure we later attach to $\mathcal{P}(X)$). Thus kiE should be read as "the closure of the interior of E." The reader should be aware that some authors place topological operations to the right of the set, with an opposite rule for composition, and the letters k and c are sometimes switched. With regard to the latter, our choice was made with Kuratowski closure operators in mind.

Definition 2.1. [8] A **Kuratowski closure operator** on a set X is a map $k \in \mathrm{End}(\mathcal{P}(X))$ which satisfies, for any $E, F \in \mathcal{P}(X)$,

(1) $k\varnothing = \varnothing$;
(2) $kkE = kE$;
(3) $kE \supseteq E$;

(4) $kE \cup kF = k(E \cup F)$.

Any Kuratowski closure operator $k \in \mathrm{End}(\mathcal{P}(X))$ is exactly the topological closure operator for the topology on X whose open sets are $\{ckE \mid E \subseteq X\}$ [8]. So the choice of k is equivalent to the choice of topology on X, while c is independent of topology. We write $I \in \mathrm{End}(\mathcal{P}(X))$ for the identity map and record the following (redundant) consequences of our definitions:

$$(2.1) \qquad k^2 = k, \quad c^2 = I, \quad i = ckc, \quad i^2 = i, \quad ic = ck, \quad kc = ci.$$

Next we recall some definitions from algebra. A *partial order* on a set is a reflexive antisymmetic transitive relation. A standard example is $\leq$ on $\mathbb{R}$, but it is not necessary that any two elements be comparable: $\mathcal{P}(X)$ is a partially ordered set - a *poset* - with partial order given by inclusion. One notices that k and i preserve the ordering, while c reverses it. (This means, for example, that $E \supseteq F \Rightarrow kE \supseteq kF$ - use Definition 2.1(4).) Now the collection of functions from any set into a poset can also be made into a poset, where one function dominates another if and only if this is true pointwise. This induces a partial order on $\mathrm{End}(\mathcal{P}(X))$: for $\varphi, \psi \in \mathrm{End}(\mathcal{P}(X))$,

$$\varphi \geq \psi \quad \Longleftrightarrow \quad \varphi(E) \supseteq \psi(E), \quad \forall E \in \mathcal{P}(X).$$

Then item (3) of Definition 2.1 can be rewritten as $k \geq I$, and evidently $i \leq I$.

The poset $\mathrm{End}(\mathcal{P}(X))$ is also a *monoid*: a set with an associative binary operation (in this case composition) and a unit. (So a monoid is a "group without inverses.") Note that order is preserved by an arbitrary right-composition:

$$\varphi \geq \psi \Rightarrow \varphi\sigma \geq \psi\sigma, \qquad \varphi, \psi, \sigma \in \mathrm{End}(\mathcal{P}(X)).$$

Order is also preserved by left-composition with k or i, but reversed by left-composition with c.

"Ordered monoid" sounds frighteningly abstract, but we will only be concerned here with subsets of $\mathrm{End}(\mathcal{P}(X))$. The advantage in the situation at hand is that we may invoke a familiar friend from group theory (or universal algebra, to those in the know): *presentations*. This just means that we will describe sets of operations in terms of generators and relations, as demonstrated in the following lemma.

Lemma 2.2.

(1) *Let $k, i \in \mathrm{End}(\mathcal{P}(X))$ be the closure and interior operators of a topological space. Then the cardinality of the monoid generated by k and i is at most 7.*

(2) *For a subset of a topological space, the number of distinct sets which can be obtained by successively taking closures and interiors (in any order) is at most 7.*

Proof. Composing $k \geq I$ on the left and right with i gives $iki \geq i$. Composing $i \leq I$ on the left and right with k gives $kik \leq k$. We use both of these to calculate

$$(i)k \leq (iki)k = i(kik) \leq i(k) \Rightarrow ik = ikik;$$
$$k(i) \leq k(iki) = (kik)i \leq (k)i \Rightarrow ki = kiki.$$

Since $k^2 = k$ and $i^2 = i$, the monoid generated by k and i contains exactly

$$\{I, i, ik, iki, k, ki, kik\}$$

(which may not all be distinct). This proves the first part, and the second part is a direct consequence. $\qquad \square$

Using parentheses for "the monoid generated by," the second sentence of Lemma 2.2(1) can be rewritten as $|(k,i)| \leq 7$.

Proof of Theorem 1.1. It follows from (2.1) that any word in k, i, c can be reduced to a form in which c appears either as the leftmost element only, or not at all. So by the previous lemma

$$(2.2) \qquad (k,c) = (k,i,c) = \{I, i, ik, iki, k, ki, kik, c, ci, cik, ciki, ck, cki, ckik\}.$$

Thus 14 is an upper bound. To conclude the proof, it suffices to exhibit a so-called *(Kuratowski)* 14-*set*: a subset of a topological space for which all of these operations produce distinct sets. One example is $S = \{0\} \cup (1,2) \cup (2,3) \cup [\mathbb{Q} \cap (4,5)] \subset \mathbb{R}$. $\qquad \square$

We now investigate the order structures of (k,i) and (k,i,c) a little further. Via our basic rules for order we find that

$$i \leq I \leq k; \qquad i \leq iki \leq [\text{either of } ki, ik] \leq kik \leq k.$$

By considering the set S we see that these (and the consequences from transitivity) are the only order relations in (k,i), at least when X contains a copy of $\mathbb{R}$. The order structure of (k,i) is depicted in Figure 1, which is called the *Hasse diagram* of the poset. Here a segment from φ up to ψ means that $\varphi < \psi$ and there is no σ satisfying $\varphi < \sigma < \psi$. (We write $\varphi < \psi$ for $\varphi \leq \psi$ and $\varphi \neq \psi$.)

Again using S, we obtain that there are no necessary relations between the first seven and last seven elements of (2.2). Since left composition with c reverses order, the Hasse diagram of (k,i,c) (Figure 2) consists of two disjoint copies of Figure 1, one of which has been left-composed with c and vertically inverted. It was first drawn by Kuratowski [8]; in essence all of the arguments in this section go back to his dissertation.

3. Boolean lattices and the answer to Question 1.3

We have already mentioned that $\mathcal{P}(X)$ is a poset. Now we want to take advantage of its richer structure as a *Boolean lattice.*
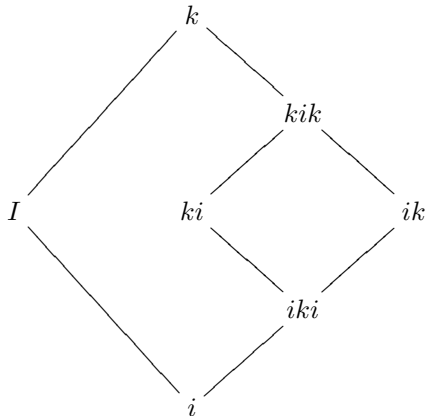

FIGURE 1. The Hasse diagram of (k,i) for topological spaces containing a copy of $\mathbb{R}$.

Recall that a *lattice* is a poset in which any two elements have a least upper bound and a greatest lower bound. We write these binary operations as $\vee$ and $\wedge$, respectively, and refer to them as *join* and *meet*. A lattice is *distributive* if it satisfies the equality

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \qquad \forall\, x, y, z$$

(or equivalently, the same equation with $\vee$ and $\wedge$ everywhere interchanged). Finally, a distributive lattice is *Boolean* if

  (i) it contains a least element 0 and a greatest element 1, and
  (ii) for every element a there is an element b, called a *complement* of a, such that $a \vee b = 1$ and $a \wedge b = 0$.

Complements in Boolean lattices are unique [5, Lemma I.6.1], so that we may view complementation as a third (unary) operation. (Strictly speaking our use of operations means that we have turned our Boolean lattice into a *Boolean algebra*, but we will ignore the distinction.)

The poset $\mathcal{P}(X)$ is a Boolean lattice in which the three operations are nothing but union, intersection, and set complementation. As the collection of all functions from $\mathcal{P}(X)$ into the Boolean lattice $\mathcal{P}(X)$, the poset $\mathrm{End}(\mathcal{P}(X))$ also acquires structure as a Boolean lattice, with pointwise operations. For $\varphi, \psi \in \mathrm{End}(\mathcal{P}(X))$, $E \in \mathcal{P}(X)$, we have

$$(\varphi \vee \psi)E = (\varphi E) \cup (\psi E), \qquad (\varphi \wedge \psi)E = (\varphi E) \cap (\psi E).$$

The complement (in the Boolean sense) of φ is the composition $c\varphi$.

We approach Question 1.3 in the same way as Theorem 1.1, by enumerating $(k, i, \wedge)$ (respectively $(k, i, \wedge, \vee)$), the algebra of operations in $\mathrm{End}(\mathcal{P}(X))$ which can be expressed in terms of $\{I, k, i, \wedge\}$ (respectively $\{I, k, i, \wedge, \vee\}$). Then we identify a single set which distinguishes all the operations under consideration.

3.1. **Closures, interiors, intersections.** Assuming X contains a copy of $\mathbb{R}$, the order structure of (k, i) is as shown in Figure 1. A first step is to add all irredundant meets to this diagram; we start with $ki \wedge ik$ and then notice that the meet of any
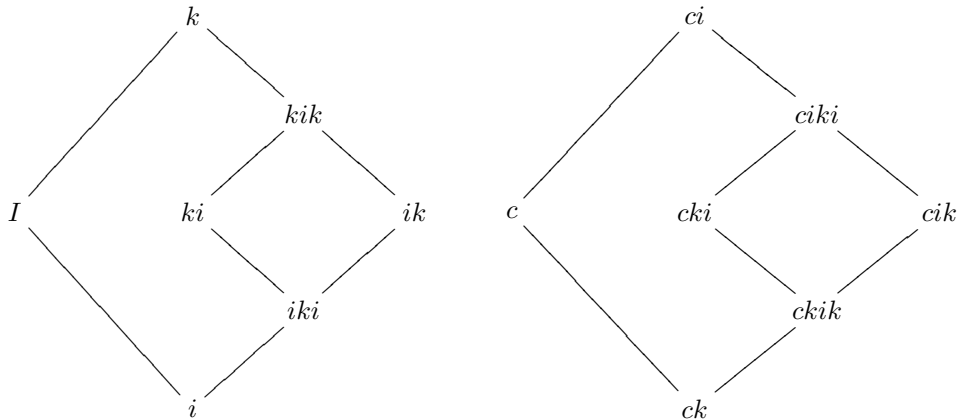

FIGURE 2. The Hasse diagram of (k, i, c) for topological spaces containing a copy of $\mathbb{R}$.
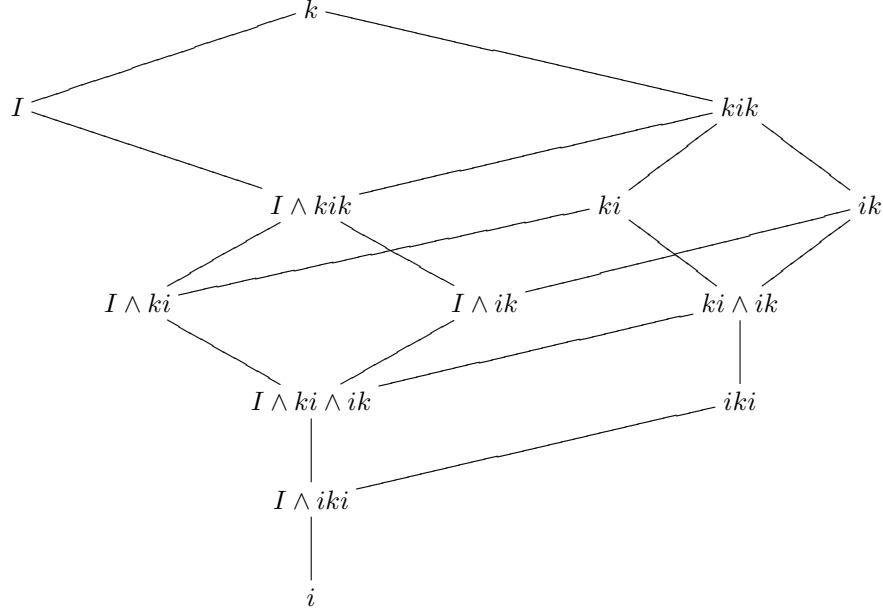
FIGURE 3. The meet semi-lattice generated by the poset (k, i) (also the Hasse diagram of $(k, i, \wedge)$) for topological spaces containing a copy of $\mathbb{R}$.

two elements, both different from I, is already in our poset. It suffices to add the meet of each element with I, and since $k \wedge I = I$ and $i \wedge I = i$, this gives five more elements. The resulting structure is the *meet semi-lattice* generated by the poset (k, i), since it has $\wedge$ but not $\vee$.

A diagram of this 13-element poset is given in Figure 3. By construction it is closed under $\wedge$, and since i distributes across $\wedge$ it is closed under i. Perhaps surprisingly, it is also closed under k. To prove this, we need to show that for each element φ in Figure 3, $k\varphi$ already appears in Figure 3.

This is clear for the seven elements from Figure 1. For the remaining elements, we start with an easy observation. Since k preserves order, for any $E, F \in \mathcal{P}(X)$ we have

$$k(E \cap F) \subseteq kE, \ k(E \cap F) \subseteq kF \Rightarrow k(E \cap F) \subseteq kE \cap kF.$$

This means that

$$(3.1) \qquad\qquad k(\varphi \wedge \psi) \leq k\varphi \wedge k\psi, \qquad \varphi, \psi \in \mathrm{End}(\mathcal{P}(X)).$$

Now let σ be any of $ki \wedge ik, I \wedge iki, I \wedge ki \wedge ik$, and $I \wedge ki$. Applying (3.1) to σ and reducing gives $k\sigma \leq ki$. But $\sigma \geq i$, so $k\sigma \geq ki$. We conclude that $k\sigma = ki$.

It is left to consider the two elements $k(I \wedge ik)$ and $k(I \wedge kik)$. Applying (3.1) shows that each is $\leq kik$. We claim that $k(I \wedge ik) = kik$, whence the larger element

$k(I \wedge kik)$ is kik as well. We calculate as follows:

$$ik = ik \wedge k(I)$$
$$= ik \wedge k[(I \wedge ik) \vee (I \wedge cik)]$$
$$= ik \wedge [k(I \wedge ik) \vee k(I \wedge cik)]$$
$$= [ik \wedge k(I \wedge ik)] \vee [ik \wedge k(I \wedge cik)].$$

Inspecting the last term,

$$ik \wedge k(I \wedge cik) \leq ik \wedge k(cik) = ik \wedge cik = 0.$$

Here $0 \in \text{End}(\mathcal{P}(X))$ is the map which sends every set to the empty set. We may therefore omit this term from the previous equation, which gives

$$ik = ik \wedge k(I \wedge ik) \Rightarrow ik \leq k(I \wedge ik) \Rightarrow kik \leq k(I \wedge ik).$$

Since the opposite inequality was already established, the claim is proved.

It follows that $(k, i, \wedge)$ has at most thirteen elements, and it can be checked that each of the operations in Figure 3 produces a distinct set when applied to the set

$$(3.2) \qquad T = \left[\left\{\frac{1}{n} : n \in \mathbb{N}\right\}\right] \cup \left[[2,4] - \left\{3 + \frac{1}{n} : n \in \mathbb{N}\right\}\right]$$
$$\cup \left[(5,7] \cap \left(\mathbb{Q} \cup \bigcup_{n=1}^{\infty}\left(6 + \frac{1}{2n\pi}, 6 + \frac{1}{(2n-1)\pi}\right]\right)\right].$$

So the answer to Question 1.3(1) is *thirteen*. (We remind the reader that this result first appeared as the solution to a *Monthly* problem in 1978 [12].)

3.2. **Closures, interiors, intersections, unions.** Still assuming that $\mathbb{R}$ embeds in X to guarantee that $(k, i, \wedge)$ is as large as possible, our first task here is to add all irredundant joins of operations from Figure 3. Since $\text{End}(\mathcal{P}(X))$ is distributive, the resulting set will be closed under joins and meets: it is in fact the distributive lattice generated by Figure 1.

Let us add in the two elements $ki \vee ik$ and $(I \wedge ki) \vee (I \wedge ik)$, and partition our poset into four classes:

(1) i, k;
(2) I;
(3) $iki, ki \wedge ik, ik, ki, ki \vee ik, kik$;
(4) $I \wedge iki, I \wedge ki \wedge ik, I \wedge ik, I \wedge ki, (I \wedge ki) \vee (I \wedge ik), I \wedge kik$.

It may help to notice that the third class is the right-hand five of Figure 3, plus $ki \vee ik$, while the fourth class is the middle five of Figure 3, plus $(I \wedge ki) \vee (I \wedge ik)$.

Each class above is already a sublattice of $\text{End}(\mathcal{P}(X))$, so an irredundant join $x_1 \vee x_2 \vee \cdots \vee x_n$ can contain at most one x_j from each class. Elements in the first class occur in no irredundant joins. The identity I cannot be involved in an irredundant join except with elements of the third class, which produces six more elements. It is left to consider joins of the third and fourth classes. Using the

distributive law, this turns up 14 more elements:

$$(I \wedge kik) \vee ki \vee ik, (I \wedge kik) \vee ki, (I \wedge kik) \vee ik,$$
$$(I \wedge kik) \vee (ki \wedge ik), (I \wedge kik) \vee iki,$$
$$(I \wedge ki) \vee (I \wedge ik) \vee (ki \wedge ik), (I \wedge ki) \vee (I \wedge ik) \vee iki,$$
$$(I \wedge ki) \vee ik, (I \wedge ki) \vee (ki \wedge ik), (I \wedge ki) \vee iki,$$
$$(I \wedge ik) \vee ki, (I \wedge ik) \vee (ki \wedge ik), (I \wedge ik) \vee iki,$$
$$(I \wedge ki \wedge ik) \vee iki.$$

We conclude that the distributive lattice generated by the poset (k, i) has at most 35 elements, each of which is a join of elements from Figure 3. Since k distributes across joins, this set is closed under left composition with k. The roles of k and i are dual – see the next section for explanation – in the self-dual distributive lattice generated by (k, i), so it is also closed under left composition with i. Finally, fans of drudgery can check that the 35 operations are distinguished by the set T from (3.2). (Skeptics may also consult the more sophisticated example given in [3].) Therefore the answer to Question 1.3(2) is *thirty-five*.

4. Answers to Question 1.2

A complete answer to Question 1.2 is given in Table 1. All of the numbers ≤ 4 in Table 1 are trivial to verify, and some of the repetition is due to the fact that in the presence of c, the inclusion of k or i (respectively $\wedge$ or $\vee$) is equivalent to the inclusion of k and i (respectively $\wedge$ and $\vee$). Other repetition is due to duality, which we now describe.

The dual of a poset is the same underlying set, with the ordering reversed. (So its diagram is turned upside-down.) The Boolean lattice $\mathcal{P}(X)$ is isomorphic with its own dual, via the complementation map c. This means that any operation φ on $\mathcal{P}(X)$ has a dual operation, $\bar{\varphi} = c\varphi c_n$, where by c_n we mean the application of c to each of the n arguments of φ. The action of $\bar{\varphi}$ is vertically opposite to φ; k and i are dual, as are $\wedge$ and $\vee$.

We note that the duality map distributes over composition. For suppose that φ has n arguments, and the jth argument is filled by a function of k_j arguments, with k total arguments in the composition. (It is not necessary that $k = \sum k_j$, because

Operations	$\{I\}$	$\{\wedge\}$	$\{\vee\}$	$\{\wedge, \vee\}$
$\{I\}$	1	1	1	1
$\{i\}$	2	2	2	2
$\{k\}$	2	2	2	2
$\{c\}$	2	4	4	4
$\{i, k\}$	7	13	13	35
$\{i, c\} = \{k, c\} = \{i, k, c\}$	14	∞	∞	∞

TABLE 1. Solution to Question 1.2. Each entry is the maximum cardinality of the algebras generated by the topological operations in its row and column, and also the maximum number of operations in these algebras which can be distinguished by a single subset.

the n functions may have some arguments in common.) Then

$$\overline{\varphi(\psi_1, \psi_2, \ldots, \psi_n)} = c\varphi(\psi_1, \psi_2, \ldots, \psi_n)c_k = c\varphi(\psi_1 c_{j_1}, \psi_2 c_{j_2}, \ldots, \psi_n c_{j_n})$$
$$= c\varphi c_n(c\psi_1 c_{j_1}, c\psi_2 c_{j_2}, \ldots, c\psi_n c_{j_n}) = \bar{\varphi}(\bar{\psi_1}, \bar{\psi_2}, \ldots, \bar{\psi_n}).$$

Thus the restriction of the duality map to operations built out of $\{I, i, k, c, \wedge, \vee\}$ amounts exactly to the interchanges $i \leftrightarrow k$ and $\wedge \leftrightarrow \vee$. This explains, for example, why $|(k, i, \wedge)| = |(k, i, \vee)|$. As for the assertion at the end of the previous section, just observe that for any operation φ in the distributive lattice generated by (k, i), $i\varphi = \overline{k\bar{\varphi}}$, which belongs to the lattice since it is self-dual and closed under left-composition with k.

Finally we give a simplified version of Kuratowski's example [8] showing that $(k, c, \wedge) = (i, k, c, \wedge, \vee)$ can be infinite. Define a closure operator k on $\mathcal{P}(\mathbb{N})$ by

$$(4.1) \qquad k(A) = \begin{cases} \varnothing, & A = \varnothing; \\ [\min A, \infty) = \{\min A, 1 + \min A, \ldots\}, & \text{otherwise}; \end{cases}$$

for any $A \in \mathcal{P}(\mathbb{N})$. Since k satisfies Definition 2.1, it determines a topology on $\mathbb{N}$, the so-called "left order topology." Let $\varphi = I \wedge [k(k \wedge c)]$, and let $E \subset \mathbb{N}$ be the even natural numbers. The reader can easily check that $\varphi^j(E) = E \cap [2j+2, \infty)$, so the subset E distinguishes infinitely many operations of $(i, k, c, \wedge, \vee)$ for this topological space. Unlike our other examples, E does not distinguish *all* the different operations: for instance $i(E) = ik(E)$ while $i(\{1\}) \neq ik(\{1\})$. The existence of a set which does distinguish all the different operations is addressed in the next section.

5. CLOSURE ALGEBRAS

We pause here for a digression which affords us a more convenient language and sharpens some of the preceding results in surprising ways. These ideas are taken from a beautiful 1944 article of McKinsey and Tarski [10].

A *closure algebra* is a Boolean lattice which is equipped with a closure operator k satisfying the lattice version of Definition 2.1, i.e., with $\varnothing$, $\cup$, and $\supseteq$ replaced with 0, $\vee$, and $\geq$. We naturally define i as the dual operation of k. A closure algebra is *singly-generated* if every element can be obtained from a certain fixed generator by some unary operation built out of $\{I, i, k, c, \wedge, \vee\}$. Of particular interest is the unique (up to isomorphism) *free* singly-generated closure algebra [10, Theorem 5.1], which we refer to here as $\mathcal{F}$. Freeness means that the set of relations is minimal. In other words, if two unary operations agree on the generator of $\mathcal{F}$, they agree on every element of every closure algebra. This has the convenient consequence that inside $\mathcal{F}$, we can identify elements with unary operations; $\mathcal{F}$ *is* the algebra of unary operations expressible in $\{I, i, k, c, \wedge, \vee\}$.

The key observation is that any closure algebra can be identified with a Boolean sublattice of some $\mathcal{P}(X)$, where X is a topological space and k becomes the associated closure operator [10, Theorem 2.4]. Thus Question 1.3 asks about the cardinalities of subalgebras of $\mathcal{F}$ in which only some of $\{I, i, k, c, \wedge, \vee\}$ can be used.

Here are some other striking facts about $\mathcal{F}$ and the theory of closure algebras [10, Theorem 5.10, Theorem 5.17, Appendix IV].

- $\mathcal{F}$ is isomorphic to a sub-closure algebra of the closure algebra of the Euclidean line, so that a certain subset of the line distinguishes all unequal unary closure algebraic operations.

- The problem of deciding whether two expressions define the same operation in all closure algebras is "effectively solvable": there is a decision procedure whose run time is bounded by a function of the complexity of the expressions.
- Whenever two expressions define the same operation in all closure algebras, there is a formal proof – an analogue for closure algebras of Gödel's completeness theorem.

Logicians also know closure algebras (sometimes called "interior algebras") as *meta-mathematical* objects, since they can be used as frameworks for modal and intuitionistic logic [11, Chapter III].

6. Answers to Question 1.4

Now we alter our hypotheses by supposing that $n \geq 2$ sets are initially given. The theorems of McKinsey and Tarski apply to this case as well, so that we may consider the free closure algebra generated by n elements. We will denote it as $\mathcal{F}_n$, with generators $\{F_j^n\}_{1 \leq j \leq n}$. Remarkably, $\mathcal{F}_n$ also embeds in the closure algebra of the Euclidean line.

At first glance Question 1.4 may seem intractable or at least extremely tedious. In the presence of $\wedge$ or $\vee$ the cardinalities grow at least exponentially: for example, the two sets $T \times \mathbb{R}, \mathbb{R} \times T \subset \mathbb{R}^2$ obviously generate at least 13^2 distinct subsets under k, i, and $\wedge$. It turns out, however, that all of the hard work is done, and the situation stabilizes nicely for $n \geq 2$. A complete solution to Question 1.4 is given in Table 2; below we explain the key points.

Using a subscript to denote the ambient algebra, we first claim that $|(k, \wedge)_{\mathcal{F}_2}| = \infty$, proved in almost the same way as $|\mathcal{F}| = \infty$. Let k be the closure operator of (4.1), and let $E = E_0$ and O be the even and odd natural numbers, respectively. For $j \geq 1$, define inductively elements of the closure algebra generated by E and O by

$$E_j = E_{j-1} \cap k(kE_{j-1} \cap O).$$

Then the $E_j = E \cap [2j + 2, \infty)$ are all distinct, establishing the claim. By duality and inclusions, this justifies every occurrence of ∞ in Table 2.

The first column of Table 2 consists of algebras with unary operations only, so the results of Table 1 can be applied to one generator at a time. For the last three entries in the fourth row of Table 2, the algebra under consideration is the free

Operations	$\{I\}$	$\{\wedge\}$	$\{\vee\}$	$\{\wedge, \vee\}$
$\{I\}$	n	$2^n - 1$	$2^n - 1$	D_n
$\{i\}$	$2n$	$3^n - 1$	∞	∞
$\{k\}$	$2n$	∞	$3^n - 1$	∞
$\{c\}$	$2n$	2^{2^n}	2^{2^n}	2^{2^n}
$\{i, k\}$	$7n$	∞	∞	∞
$\{i, c\} = \{k, c\} = \{i, k, c\}$	$14n$	∞	∞	∞

Table 2. Solution to Question 1.4. Each number is the cardinality of the subalgebra of $\mathcal{F}_n$ generated by the operations in its row and column.

Boolean algebra with n generators, which is well-known to have 2^{2^n} elements [5, Theorem II.2.2(iii)].

The elements of $(\wedge)_{\mathcal{F}_n}$ have the form $F_{j_1}^n \wedge F_{j_2}^n \wedge \cdots \wedge F_{j_k}^n$, where $1 \le k \le n$ and the $j_k \in \{1, 2, \ldots n\}$ are distinct. Apparently they are in one-to-one correspondence with the $2^n - 1$ nonempty subsets of the n generators. The situation for $(i, \wedge)_{\mathcal{F}_n}$ is similar; now any element is a meet in which for each F_j^n, one of three possibilities holds: F_j^n is present, iF_j^n is present, or both are absent. (Recall that i distributes across $\wedge$.) Since we do not admit the empty meet, this allows $3^n - 1$ possibilities. The other occurrences of $2^n - 1$ and $3^n - 1$ in Table 2 follow from duality.

Finally, $(\wedge, \vee)_{\mathcal{F}_n}$ is the free distributive lattice on n generators; determining its cardinality D_n is sometimes called *Dedekind's problem*. No explicit formula for D_n is known, but asymptotically $\log_2 D_n \sim C(n, \lfloor \frac{n}{2} \rfloor)$, where $C(n, k)$ denotes the binomial coefficient and $\lfloor x \rfloor$ the greatest integer $\le x$ [7].

Each of the finite formulas of Table 2 extends to the case $n = 1$.

7. Other variations

It has not been our goal to survey the wealth of literature concerning extensions of the 14-set theorem. The interested reader may consult the article [3], which takes a sophisticated algebraic approach and contains many references. Here we simply indicate some of the other directions in which the theorem has been generalized.

A natural idea is to study subalgebras of $\mathcal{F}$ in which the generating operations include other topological operations built out of $\{k, \wedge, c\}$. This need not be too abstract – for example, the operation "boundary of" (considered in Kuratowski-type theorems by many authors) is $k \wedge kc$. And of course many basic topological constructions are not closure algebraic. Kuratowski himself [8] noted that the operation "accumulation points of," which sends $E \in \mathcal{P}(X)$ to its *derived set DE*, is not expressible in terms of $k, \wedge$, and c. (Let $E = \{0\} \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$.) See [10, Appendix I] for remarks on "derivative algebra."

One may also generalize Definition 2.1 by relaxing the postulates and/or replacing $\mathcal{P}(X)$ with non-Boolean lattices. Among the sizable research in this direction, we point out [6], which deals specifically with the 14-set theorem.

The present article is about the algebraic content of the 14-set theorem, but there are also investigations into its topological content. We have seen that a topological space X contains 14-sets when it is sufficiently rich, in particular when it contains a copy of the Euclidean line. One can ask for characterizations of topological spaces in which certain limitations occur, and a classification along these lines was given by Aull [1]. Several authors have given necessary and sufficient conditions for a specific subset $E \subseteq X$ to be a 14-set; Chapman [2] gave an exhaustive description of all possible degeneracies of the poset in Figure 1. Once again the reader is referred to [3] for precise statements of these and other results.

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