Seminar on Operator K-Theory Spring 2008

1/23 TK. Basics from chapter 1.

A C*-algebra (over C) is a complete normed algebra with \|ab\| \leq \|a\| \|b\| (hence a Banach algebra) equipped with an involution
\[ a \mapsto a^* \quad (\text{conjugate linear} \quad (ab)^* = b^* a^*) \]
with \( \|a^* a\| = \|a\|^2 \) for all \( a \in A \).

Examples:
1. \( C_c(X) \), \( X = \) loc. cpt. Haus, \( \mathbb{R} \)
    \[ \|f\| = \sup_{x \in X} |f(x)|. \]
2. \( B(H) \)
3. If \( H = \mathbb{C}^n \), \( B(H) = M_n \).

A is unital if \( A \) has \( 1 = 1_A \).

\(*\)-homomorphism \( \varphi : A \rightarrow B \). An algebra\n\(*\)-homomorphism \( \varphi \) with \( \varphi(a^*) = \varphi(a)^* \).
\( \varphi \) is unital if \( \varphi(1_A) = 1_B \).

Gelfand-Neimark Theorem: If \( A \) is a C*-alg, \( A \) a Hilbert space \( H \) and an isometric \(*\)-homomorphism \( \pi : A \rightarrow B(H) \). If \( A \) is separable, can take \( H \) to be separable.

(\( \pi \) is a isometric \(*\)-representation)
Let $f : A \to B$ be a positive linear functional.

**Idea:** Define a sesquilinear form

\[ b(x, y) = f(y^* x) . \]

\[ L = \{ x \in A : f(x^* x) = 0 \} \]

is a closed left ideal in $A$ (use Schwarz $\leq$).

Dotting inner product on $A/L_f$ by

\[ \langle [x], [y] \rangle = f(y^* x), \text{ where } [x] = x + L_f \]

complete this inner product to get Hilbert space $H_f$.

For $a \in A$, define $\pi_f(a) : A/L_f \to A/L_f$ by $\pi_f(a)[x] = [a x]$. Then get *-representation

\[ \pi_f : A \to B(\mathcal{H}_f), \quad \text{let } \pi : A \to B(\bigoplus_f \mathcal{H}_f) \]

\[ \pi(a) = \bigoplus_f \pi_f(a), \quad a \in A. \]

**Ideals, Quotients:** Ideals are closed.

2-sided, automatic: $a \in I \Rightarrow a^* \in I$ (Conway, p. 245)

\[ A/I = \{ a + I : a \in A \}, \quad \forall a + I \parallel = \inf \parallel a + x \parallel \quad x \in E \]

\[ = \text{dist} (a, I), \quad \pi(a) = a + I, \quad \pi : A \to A/I \]
Automorphisms: \( \| \phi(a) \| \leq \| a \| \) for all \( a \in A \). If \( \ker \phi = \{ 0 \} \) and \( \| \phi(a) \| = \| a \| \), we look at \( \ast - \) homomorphisms \( \phi : A \to B \).

\[ \phi(a^*) = \phi(a)^* \] \( \ker \phi \) is an ideal in \( A \).

\( \text{Im } \phi = \phi(A) = \text{Ran } \phi \),

\[ A \xrightarrow{\phi} B \]

\[ \pi \downarrow \]

\[ \phi_0 : \phi_0(\pi(a)) = \phi(a) \]

\( A/\ker \phi \) well-defined.

Exact sequence:

\[ \to A_n \xrightarrow{\phi_n} A_{n+1} \xrightarrow{\phi_{n+1}} \to \]

exact at \( A_n \) and \( \text{im } \phi_{n-1} = \ker \phi_n \).

Short exact:

\[ (1,1): \quad 0 \to I \xrightarrow{i} A \xrightarrow{\phi} B \to 0 \]

\( \text{Im } \phi = \ker \psi \), \( \phi \) is 1-1, 1-1 onto.

Example: \( I \subseteq A \) is ideal.

\[ 0 \to I \xrightarrow{i} A \xrightarrow{\pi} A/I \to 0 \]

Given \((1,1)\), \( \phi(I) = \ker \psi \) is an ideal in \( I \) and we have
Get commutative diagram.

\[ 0 \rightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \rightarrow 0 \]

How is \( \phi \) defined? \( \phi \) is onto, so \( \phi(\psi(a)) \)
\[ \text{det} \pi(a), \text{Clearly onto. If } \pi(a) = 0, \]
\[ \alpha = \phi(x) \text{ so } \pi(a) = \pi(\phi(x)) = 0, \]
\( \psi \) well defined.

(1.1) is split exact \( \forall \lambda \in B \rightarrow A \)
\[ \text{with } \epsilon \circ \lambda = \text{id}_B \]

\[ 0 \rightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \rightarrow 0 \]

Ex: \( A \oplus B \) has coordinatewise operations

\[ \epsilon_A : A \rightarrow A \oplus B \text{ i.e. } \epsilon_A(a) = (a, 0) \]
\[ \epsilon_B : B \rightarrow A \oplus B \text{ i.e. } \epsilon_B(b) = (0, b) \]
\[ \pi_A : A \oplus B \rightarrow A \text{ i.e. } \pi_A(a, b) = a \]
\[ \pi_B : A \oplus B \rightarrow B \text{ i.e. } \pi_B(a, b) = b \text{. Then} \]

\[ 0 \rightarrow A \xrightarrow{\epsilon_A} A \oplus B \xrightarrow{\pi_B} B \rightarrow 0 \text{ is \textit{split exact.}} \]
If $A$ does not have a $1$, you can adjoin one. Define

$$\bar{\Delta} = \delta(a, x); a \in A, x \in \mathbb{C}$$

Multiplication is defined by

$$(a, x) \cdot (b, y) = (ab + ab^* + y a, x y)$$

and $$(a, x)^* = (a^*, -x)$$

Then $\bar{\Delta}$ is a $*$--algebra with identity $1 = (0, 1) = 1_{\bar{\Delta}}$

$A \cong \{ (a, 0); a \in A \}$ as an ideal in $\bar{\Delta}$.

We freely use this identification, $a \mapsto (a, 0)$, and identify $(a, x)$ as $a + x1$

Then $||(b, y)||_A = ||b + y1||$

$$\leq \max \left\{ \sup_{a \in \mathcal{A}} ||ab + y a||, ||y|| \right\}$$

is the unique $C^*$--norm on $\bar{\Delta}$.

Note $||(b, 0)|| = \sup_{a \in \mathcal{A}} \frac{||ab||}{||a||}$, $a \in A, ||a||_1 = ||y||_1$

also, $b = a_0 = \frac{b^*}{||b||}$

$$||(b, 0)|| \geq ||a_0 b|| = \frac{||b^* b||}{||b||} = \frac{||b||^2}{||b||} = ||b||$$

so $||b|| = ||(b, 0)||$ and $A \mapsto (0, 0)$ is isometric.
Ex. Given $A$, form $\tilde{A} = \{a + \chi_1 \}_A; a \in A; \chi \in C$.

Consider $\pi: \tilde{A} \to C$; $\pi \left( a + \chi_1 \right) = \chi$.

If $\lambda: C \to \tilde{A}$; $\lambda(\chi) = \chi_1 A$.

Then $\pi \circ \lambda = \text{id}_C$, so

$$0 \to A \tilde{\to} \tilde{A} \tilde{\to} C \to 0$$

is split exact. But $\tilde{A} \equiv A \oplus C$.

$\iff A \text{ is unital. (Exercise 1.3)}$

Exercise 1.1. Given exact sequence

$$0 \to A \phi \to E \phi \to B \to 0$$

Then TFAE:

(i) $\exists$ a $*$-isomorphism $\Theta: E \to A \oplus B$.

so that the commutes:

$$0 \to A \phi \to E \phi \to B \to 0$$

$$\downarrow \quad \Theta \downarrow = \downarrow$$

$$0 \to A \phi \to A \oplus B \phi \to B \to 0$$

(ii) $\exists$ a $*$-homomorphism $\nu: E \to A$ with $\nu(1) = 1_B$. 
PF: Suppose (i) holds. Let $\nu = \pi_A \circ \Theta$.

Then $\nu \circ \phi = \pi_A \circ \Theta \circ \phi = \pi_A \circ \iota_A = \iota_A$.

Thus (i) $\Rightarrow$ (ii). Suppose (ii) holds. Set $\Theta = (\nu, +)$.

If $e \in E$ with $\Theta(e) = 0$, then $+e = 0$.

So $e = \phi(a)$ for some $a \in A$. Then $0 = \nu(e) = \nu(\phi(a)) = a$ and $a = 0$.

So $e = \phi(a) = 0$. Thus $\ker \phi = 0$.

Check onto, let $a \in A$. Then $\Theta(\phi(a)) = (\nu(\phi(a)), +\phi(a)) = (a, 0)$.

Thus $\Theta(E) \supseteq A \oplus 0$.

Let $b \in B$. Then $\exists e \in E$ with $+e = b$.

Then $\Theta(e) = (\nu(e), +e) = (\nu(e), b)$.

Thus $(0, b) = (\nu(e), b) - (\nu(e), 0)$.

$= \Theta(e) - \Theta(\phi(\nu(e)))$.

$\leq \Theta(\nu(E))$.

So $0 \oplus B \subseteq \Theta(E)$ and $\Theta$ isonto.
Spectral Theory. Let $A$ be unital. 
$GL(A) = \{ a \in A : a^{-1} \text{ exists} \}$

is an open set and $a \mapsto a^{-1}$ is a homeomorphism in $GL(A)$. 
$\text{sp}(a) = \{ \lambda \in \mathbb{C} : a - \lambda 1 \notin GL(A) \}$

is a non-empty compact set. 
$r(a) = \sup \{ |\lambda| : \lambda \in \text{sp}(a) \}$

$r(a) \leq ||a||$ and

$r(a) = \lim_{n \to \infty} ||a^n||^{1/n}$.

For non-unital $A$, think $A \subset \tilde{A}$, consider $\text{sp}(\tilde{a})$ with respect to $\tilde{A}$.

Gelfand. Let $A$ be a $C^*$-algebra. TFAE:

(i) $A$ is a $\tilde{A}$-bital

(ii) $A$ is *-isomorphic to $C_0(X)$

For some locally compact Hausdorff space $X$,

* $C_0(X)$ is unital $\iff X$ is compact.
$C_0(X) \sim \text{ separable } \iff X \text{ is separable}$

If $X, Y$ are locally compact, $TFAE$.

(a) $\varphi : C_0(X) \to C_0(Y)$ is a *-homomorphism.

(b) $f$ continuous $g : Y \to X$ with $\varphi(f) = f \circ g$ for all $f \in C_0(X)$.

Ideals in $C_0(X)$ look like

$$I_K = \{ f \in C_0(X) : f = 0 \text{ on } K \}$$

where $K$ is a closed set in $X$.

Note: The restriction map $I_K \to C_0(K^c)$

$$r_2 \begin{array}{c} \varphi \end{array} \begin{array}{c} \longrightarrow \end{array} \begin{array}{c} K^c \end{array}$$

is a *-isomorphism. There's a short exact sequence

$$0 \to C_0(K^c) \overset{\epsilon}{\longrightarrow} C_0(X) \overset{r_2}{\longrightarrow} C_0(K) \to 0$$

where $r_2(f) = f|_{K^c}$ and

$$\epsilon(f) = \begin{cases} f & \text{on } K^c \\ 0 & \text{on } K \end{cases}$$
\[
\text{Let } x \in \mathbb{C}, \quad x \mapsto f(x) = x^2 + 1
\]
\[
f(0) = 1, \quad f(1) = 2
\]
\[
\text{So } f^{-1}(1) = -1
\]
\[
\text{Hence, } \mathcal{L}(x) = f^{-1}(1) = -1
\]

\[
\text{Suppose } y \in C \cap \mathbb{C}, \quad y = x + 1
\]

\[
\text{where } \mathcal{L}(y) = (f(1), f(2)) = (2, 5)
\]

\[
\mathcal{L}(x) = (1, 2)
\]

\[
\text{Take } x = 0
\]

\[
\text{Then } x = \mathcal{L}(x) = 0
\]

\[
\mathcal{L}(x) = (0, 1)
\]

\[
\mathcal{L}(x) = (1, 0)
\]

\[
K = C \cap \mathbb{C}
\]

\[
\text{As } C(K) = \mathbb{C}(x), \quad \text{Take } C(K) = \mathbb{C}(x)
\]

\[
\text{Example 1.2: Take } X = \{1, 1.7\} (\text{compact})
\]

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