

Minimizing Squared Perpendicular Errors

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Abstract

The best fit line from minimizing the sum of the squared perpendicular error is given.

1 Introduction

With simple linear regression and data $\{(x_1, y_1), \dots, (x_n, y_n)\}$, we will minimize the sum of the squared vertical errors to find the "best-fit" line.

2 Minimizing Squared Perpendicular Errors

For a right triangle with legs $\{a, b\}$ and hypotenuse c with h the distance from the intersection of the legs to the hypotenuse. From similar triangles $a/h = c/b$, so $h^2 = (ab)^2/(a^2 + b^2)$. The sum of the squared perpendicular errors SSE_p for the fitted line $h(x) = \beta_0 + \beta_1 x$ has

$$SSE_p(\beta_0, \beta_1) = \sum_{i=1}^n \frac{(y_i - \beta_0 - \beta_1 x_i)^2 (x_i - (y_i - \beta_0)/\beta_1)^2}{(y_i - \beta_0 - \beta_1 x_i)^2 + (x_i - (y_i - \beta_0)/\beta_1)^2} \quad (1)$$

with

$$\frac{\partial}{\partial \beta_0} SSE_p(\beta_0, \beta_1) = \frac{2}{\beta_1^2 + 1} \left(\beta_1 \sum_{i=1}^n x_i + n\beta_0 - \sum_{i=1}^n y_i \right) \quad (2)$$

with root

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad (3)$$

as in the simple linear regression case

Set $S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2$, $S_{yy} = \sum_{i=1}^n (y_i - \bar{y})^2$, $S_{xy} = \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$. With β_0 from Equation 3,

$$SSE_p(\beta_1) = \frac{S_{yy} - 2\beta_1 S_{xy} + \beta_1^2 S_{xx}}{1 + \beta_1^2}, \quad (4)$$

and the equation for the slope β_1 is given by

$$\frac{\partial}{\partial \beta_1} SSE_p(\beta_1) = \frac{2(\beta_1^2 S_{xy} + \beta_1(S_{xx} - S_{yy}) - S_{xy})}{(1 + \beta_1^2)^2} \quad (5)$$

with roots (provided $S_{xy} \neq 0$)

$$\hat{\beta}_1 = \frac{S_{yy} - S_{xx} \pm \sqrt{(S_{yy} - S_{xx})^2 + 4S_{xy}^2}}{2S_{xy}}. \quad (6)$$

If $S_{xy} \neq 0$, then there are two roots from Equation 6, one positive and one negative. If $S_{xy} > 0$, then the minimum of $SSE_p(\beta_1)$ is the positive root, and if $S_{xy} < 0$, the minimum is the negative root. This follows from the fact that there are two critical points for $\frac{\partial}{\partial \beta_1} SSE_p(\beta_1)$ one to the left of zero and one to the right of zero, and that the value at zero of $\frac{\partial}{\partial \beta_1} SSE_p(0) = -2S_{xy}$.

If $S_{xy} = 0$, the minimum of $SSE_p(\beta_1)$ is found from Equation 4 and (1) is S_{yy} with $\beta_1 = 0$ for $S_{yy} < S_{xx}$; (2) is S_{xx} with $\beta_1 = \infty$ for $S_{xx} < S_{yy}$; and (3) is $S_{xx} = S_{yy}$ with any value of β_1 for $S_{xx} = S_{yy}$.

3 Example

For the data $\{(0,0), (1,1), (1,0), (0,0)\}$ with $\{\bar{x} = 1/2, \bar{y} = 1/4, S_{xx} = 1, S_{yy} = 3/4, S_{xy} = 1/2, \rho = \sqrt{3}/3\}$, the "best" fit straight line has $\hat{\beta}_1 = 0.78078$ and $\hat{\beta}_0 = -0.14039$.

4 Summary

We have investigated the fitted linear equation to data using as the criterion for fit the minimized sum of squares of the perpendicular errors.