OPERATORS ON THE FOURIER ALGEBRA WITH WEAKLY COMPACT EXTENSIONS

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Introduction. We let $G$ denote an infinite compact group, and $\hat{G}$ its dual. We use the notation of our book [3, Chapters 7 and 8]. Recall that $A(G)$ denotes the Fourier algebra of $G$ (an algebra of continuous functions on $G$), and $\mathcal{L}^\infty(\hat{G})$ denotes its dual space under the pairing $\langle f, \phi \rangle$, $(f \in A(G), \phi \in \mathcal{L}^\infty(\hat{G}))$. Further, note $\mathcal{L}^\infty(\hat{G})$ is identified with the $C^*$-algebra of bounded operators on $L^2(G)$ commuting with left translation. The module action of $A(G)$ on $\mathcal{L}^\infty(\hat{G})$ is defined by the following: for $f \in A(G), \phi \in \mathcal{L}^\infty(\hat{G}), f \cdot \phi \in \mathcal{L}^\infty(\hat{G})$ by

$$\langle g, f \cdot \phi \rangle = \langle fg, \phi \rangle,$$

$g \in A(G)$. Also $||f \cdot \phi||_\infty \leq ||f||_A ||\phi||_\infty$. (See [1] for the general setting.)

We have previously [4] studied the spaces $AP(G)$, respectively $W(G)$, consisting of those $\phi \in \mathcal{L}^\infty(\hat{G})$ for which the map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathcal{L}^\infty(\hat{G})$ is a compact, respectively weakly compact operator.

In this paper, we renorm $A(G)$ by continuously embedding $A(G)$ into the spaces $C(G), L^p(G)$ $(1 \leq p < \infty)$, and we characterize the space of those $\phi \in \mathcal{L}^\infty(\hat{G})$ for which the map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathcal{L}^\infty(\hat{G})$ extends to a weakly compact operator on $C(G), L^p(G)$ $(1 \leq p < \infty)$.

For $G$ abelian, $\hat{G}$ is a group and $A(G)$ is isomorphic to $l^1(\hat{G})$ by the Fourier transform $\mathcal{F}$. The module action of $A(G)$ on $l^\infty(\hat{G})$ is given by $f \cdot \phi = \mathcal{F}(\check{f}) \ast \phi, (\check{\phi}(x) = g(-x), x \in G, g$ a function on $G), (f \in A(G), \phi \in l^\infty(\hat{G}))$.

It follows that for $G$ abelian, the spaces $A(P(G))$, respectively $W(G)$ are the classical spaces of almost periodic, respectively weakly almost periodic, functions on $G$. A rewording of a result of Kluvánek [5] yields for $G$ abelian that the functions $\phi \in L^\infty(\hat{G})$, for which the map $f \mapsto f \cdot \phi = \mathcal{F}(\check{f}) \ast \phi$ from $A(G)$ to $L^\infty(\hat{G})$ extends to a weakly compact operator on $C(G), L^p(G)$ $(1 \leq p < \infty)$.

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the translation argument by a convolution argument involving $L^1(\hat{G})$, and then for the compact nonabelian case use the module action of $A(G)$ on $\mathcal{L}^\infty(\hat{G})$ instead of the convolution product.

Let $M(G)$ denote the measure algebra on $G$. For $\mu \in M(G)$, the Fourier-Stieltjes transform of $\mu$, $\hat{\mu}$ or $\mathcal{F} \mu$, is a matrix-valued function in $\mathcal{L}^\infty(\hat{G})$ defined for $\alpha \in \hat{G}$ by

$$\alpha \mapsto \hat{\mu}_\alpha = \int_G T_\alpha(x^{-1})d\mu(x), \quad (T_\alpha \in \alpha).$$

For $\hat{\mu} \in \mathcal{L}^\infty(\hat{G})$ and $f \in A(G)$,

$$\langle f, \hat{\mu} \rangle = \int_G \hat{f}d\mu.$$

To see this consider:

$$\langle f, \hat{\mu} \rangle = \sum_{\alpha \in \hat{G}} n_\alpha \text{Tr}(\hat{f}_\alpha \hat{\mu}_\alpha)$$

$$= \sum_{\alpha \in \hat{G}} n_\alpha \sum_{i,j=1}^{n_\alpha} \hat{f}_{ij} \int_G T_{\alpha_{ij}}(x^{-1})d\mu(x)$$

$$= \sum_{\alpha \in \hat{G}} n_\alpha \int_G \text{Tr}(\hat{f}_\alpha T_{\alpha^{-1}}(x^{-1}))d\mu(x)$$

$$= \int_G \sum_{\alpha \in \hat{G}} n_\alpha \text{Tr}(T_{\alpha^{-1}} \hat{f}_\alpha)d\mu(x)$$

$$= \int_G \hat{f}d\mu.$$

Thus for $f \in A(G)$ and $\hat{\mu} \in \mathcal{L}^\infty(\hat{G})$ the map $f \mapsto f \cdot \hat{\mu}$ has the explicit form $f \mapsto (\hat{f}d\mu)^\wedge$.

1. **Embedding $A(G)$ into $C(G)$**.

**Theorem 1.** Let $\phi \in \mathcal{L}^\infty(\hat{G})$. The map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathcal{L}^\infty(\hat{G})$ extends to a weakly compact operator on $C(G)$ if and only if $\phi \in M(G)^\wedge$.

**Proof.** Let $\phi \in \mathcal{L}^\infty(\hat{G})$ be such that $\phi$ extends (uniquely, since $A(G)$ is uniformly dense in $C(G)$) to a weakly compact operator on $C(G)$. In particular, $\phi$ defines a bounded operator on $C(G)$ so there exists $M < \infty$ such that $\|f \cdot \phi\|_\infty \leq M\|f\|_\infty$. The linear functional $f \mapsto \langle \hat{f}, \phi \rangle$ on $A(G)$ is bounded in $C(G)$-norm since

$$|\langle \hat{f}, \phi \rangle| = |\langle 1, \hat{f} \cdot \phi \rangle| \leq ||1||_A \|\hat{f} \cdot \phi\|_\infty \leq M\|f\|_\infty.$$

Thus it extends by the Hahn-Banach theorem to all of $C(G)$, and by the Riesz representation theorem there exists $\mu \in M(G)$, $||\mu|| \leq M$, such that

$$\langle \hat{f}, \phi \rangle = \int_G \hat{f}d\mu = \langle \hat{f}, \hat{\mu} \rangle, \quad (f \in A(G)).$$

Hence $\phi = \hat{\mu}$. 

Conversely, let $\mu \in M(G)$. We may assume $\mu \geq 0$. Choose $p$ with $1 < p < \infty$. The map $f \mapsto f d\mu \mapsto (f d\mu)^*$ from $L^p(\mu) \to M(G) \to L^{\infty}(\hat{G})$ is weakly compact since $L^p(\mu)$ is reflexive [2, p. 483]. Thus the map

$$f \mapsto \tilde{f} \mapsto \tilde{f} d\mu \mapsto (\tilde{f} d\mu)^* = f \cdot \tilde{\mu}$$

from $C(G) \to L^p(\mu) \to M(G) \to L^{\infty}(\hat{G})$ is also weakly compact.

2. Embedding $A(G)$ into $L^p(G)$ ($1 < p < \infty$).

**Theorem 2.** Let $\phi \in L^{\infty}(\hat{G})$ and $1 < p < \infty$. The map $f \mapsto f \cdot \phi$ from $A(G)$ to $L^{\infty}(\hat{G})$ extends to a weakly compact (equivalently, bounded [2, p. 483]) operator on $L^p(G)$ if and only if $\phi \in \mathcal{F} L^q(G)$ ($1/p + 1/q = 1$).

**Proof.** Let $\phi \in L^{\infty}(\hat{G})$ be such that $f \mapsto f \cdot \phi$ extends (uniquely since $A(G)$ is $\|\cdot\|_p$-dense in $L^p(G)$) to a weakly compact (bounded) operator on $L^p(G)$. Since $L^p(G)^* = L^q(G)$, it follows as above that $\phi \in \mathcal{F} L^q(G)$.

Conversely, let $h \in L^q(G)$. The map $f \mapsto f \cdot h$ from $A(G)$ to $L^{\infty}(\hat{G})$ is $L^p(G)$-bounded since

$$\|f \cdot h\|_\infty = \sup \{|\langle g, f \cdot h \rangle| : g \in A(G), \|g\|_A \leq 1\} = \sup \left\{ \left| \int g h \, dm_\alpha \right| : g \in A(G), \|g\|_A \leq 1 \right\} \leq \sup \{ \|fg\|_p \|h\|_q : g \in A(G), \|g\|_A \leq 1\} \leq \|f\|_p \|h\|_q.$$ 

3. Embedding $A(G)$ into $L^1(G)$.

**Theorem 3.** Let $G$ be an infinite compact group, and let $\phi \in L^{\infty}(\hat{G})$ with $\phi \neq 0$. The map $f \mapsto f \cdot \phi$ from $A(G)$ to $L^{\infty}(\hat{G})$ cannot be extended to a weakly compact operator on $L^1(G)$.

**Proof.** By way of contradiction, suppose $\phi \neq 0$ is such that $f \mapsto f \cdot \phi$ extends to a weakly compact operator on $L^1(G)$. Analogously to Theorem 2, we see that $\phi \in \mathcal{F} L^\infty(G)$. Let $\phi = \hat{h}, h \in L^\infty(G)$. Since $\|h\|_\infty \neq 0$, there exists a point $x \in G$ with the property that given a neighborhood $V$ of $x$, there exists a positive measurable subset $E$ of $V$ with $|h(x)| \geq \|h\|_\infty/2$ for all $x \in E$. Let $\{V_\lambda\}$ be a neighborhood basis for the point $x$. Define $g_\lambda = (1/(m_\alpha(E)h)) \chi_E$ where $E \subset V_\lambda$ is as above, and $\chi_A$ denotes the characteristic function of the set $A$. Note that $\|g_\lambda\|_1 \leq 2/\|h\|_\infty$. Thus by the weak compactness of the map $f \mapsto f \cdot \phi$, the set $\{g_\lambda \cdot \phi\}$ has a cluster point $T \in L^\infty(\hat{G})$. Let $\{W_\alpha\}$ be a weak neighborhood basis of $T$ in $L^\infty(\hat{G})$. For each $(V_\lambda, W_\alpha) \in \{V_\lambda\} \times \{W_\alpha\}$, there exists $g_{\lambda, \alpha} \in \{g_\lambda\}$ such that $\hat{g}_{\lambda, \alpha} \cdot \tilde{h} \in W_\alpha$ and support $g_{\lambda, \alpha} \subset V_\lambda$. Thus

$$\hat{g}_{\lambda, \alpha} \cdot \tilde{h} \xrightarrow{(\lambda, \alpha)} T$$

weakly, and hence also weak-*.
Now \( \{ \tilde{g}_{\lambda, a} \cdot \hat{k} \}_{(\lambda, a)} \) converges weak-* to \( \delta_{x^{-1}} \) for \( k \in A(G) \),

\[
\langle k, \tilde{g}_{\lambda, a} \cdot \hat{k} \rangle = \langle \tilde{g}_{\lambda, a}, \hat{k} \rangle = \int_{\hat{G}} \tilde{g}_{\lambda, a} \cdot \hat{k}(x) = k(x^{-1}).
\]

Thus \( T = \delta_{x^{-1}} \).

Recall that \( C^0(\hat{G}) \) denotes the subspace of \( \mathcal{L}^\infty(\hat{G}) \) consisting of those \( \phi \) for which the set \( \{ \alpha \in \hat{G} : ||\phi_\alpha||_\infty \geq \epsilon \} \) is finite for \( \epsilon > 0 \). However, \( \delta_{x^{-1}} \) is unitary, so \( \delta_{x^{-1}} \notin C^0(\hat{G}) \). Now \( \tilde{g}_{\lambda, a} \cdot \hat{k} \in L^1(G)^{\ast} \subset C^0(\hat{G}) \), which being strongly closed is also weakly closed. But then \( \delta_{x^{-1}} \) cannot be a weak cluster point of \( \{ \tilde{g}_{\lambda, a} \cdot \hat{k} \}_{(\lambda, a)} \), the required contradiction.

4. Abelian results for weakly compact extensions. Here \( G \) is a locally compact abelian (LCA) group. Theorem 1 has an LCA setting.

**Theorem 4.** Let \( \phi \in L^\infty(\hat{G}) \). The map \( f \mapsto \hat{f} \ast \phi \) from \( A(G) \) to \( L^\infty(\hat{G}) \) extends to a weakly compact operator on \( C^0(G) \) if and only if \( \phi \in M(G)^{\ast} \).

**Proof.** The proof is similar to Theorem 1.

5. Abelian results for compact extensions. In this section \( G \) is an LCA group.

**Theorem 5.** Let \( \phi \in L^\infty(\hat{G}) \). The map \( f \mapsto \hat{f} \ast \phi \) from \( A(G) \) to \( L^\infty(\hat{G}) \) extends to a compact operator on \( C^0(G) \) if and only if \( \phi \in M_d(G)^{\ast} \).

**Proof.** Let \( \mu \) be a discrete measure on \( G \), and let \( \mu_n \) be finitely supported measures with

\[
||\mu_n - \mu|| \to 0.
\]

Note that \( f \mapsto \hat{f} \ast \mu_n = (fd\mu_n)^{\ast} \) is a finite rank operator from \( C^0(G) \) to \( L^\infty(\hat{G}) \). Thus \( f \mapsto (fd\mu)^{\ast} \) from \( C^0(G) \) to \( L^\infty(\hat{G}) \) being a limit of compact operators is compact: note that

\[
||(fd\mu)^{\ast} - (fd\mu_n)^{\ast}||_\infty \leq ||f||_\infty \cdot ||\mu_n - \mu|| \quad (f \in C^0(G)).
\]

Conversely, suppose \( \phi \in L^\infty(\hat{G}) \) is such that \( f \mapsto \hat{f} \ast \phi \) extends to a compact (in particular, weakly compact) operator on \( C^0(G) \). Thus \( \phi \in M(G)^{\ast} \) by Section 4. Also we note that \( f \mapsto \hat{f} \ast \phi \) is a compact operator on \( A(G) \) since

\[
||f||_\infty \leq ||f||_A, \quad (f \in A(G)).
\]

Thus \( \phi \) is an almost periodic function on \( \hat{G} \) [4]; and thus \( \hat{\mu} \in M_d(G)^{\ast} \).

6. Compact extensions for compact groups. Here \( G \) is again an infinite compact group.

**Theorem 6.** Let \( G \) be an infinite compact nonabelian group of bounded representation type. The map \( f \mapsto f \cdot \phi \) from \( A(G) \) to \( \mathcal{L}^\infty(\hat{G}) \) extends to a compact operator on \( C(G) \) if and only if \( \phi \in M_d(G)^{\ast} \).
Proof. For $\phi \in M_d(G)^\wedge$, the proof is similar to the first part of Theorem 5. Conversely, let $f \mapsto f \cdot \phi$ from $A(G)$ to $L^\infty(\hat{G})$ extend to a compact operator on $C(G)$. By Theorem 1, $\phi \in M(G)^\wedge$. Write $\phi = \mu$. We must show that $\mu$ is a discrete measure.

For compact groups of bounded representation type, there exists an open abelian subgroup $H$ of finite index in $G$ [6]. Since discrete measures do induce compact operators on $C(G)$, we may suppose $\mu$ is purely continuous.

By translation, if necessary, we may assume $|\mu|_H \neq 0$. Observe that there is a natural imbedding of $A(H) \to A(G)$, (by extending $f \in A(H)$ to be 0 off $H$). Thus we have a continuous map $L^\infty(\hat{G}) \to l^\infty(\hat{H})$. Hence the map $C(H) \to L^\infty(\hat{G}) \to l^\infty(\hat{H})$ defined by

$$g \mapsto \mathcal{F}(gd\mu) \to \mathcal{F}_H(gd(\mu|H))$$

(where $\mathcal{F}_H$ denotes the Fourier-Stieltjes transform on $H$) is a compact operator which is a contradiction by Theorem 5.

References


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